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LOG BETTI COHOMOLOGY, LOG ÉTALE COHOMOLOGY, AND LOG DE RHAM COHOMOLOGY OF LOG SCHEMES OVER C

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§0. Introduction

The purpose of this paper is to extend the classical relationship between Betti cohomology, étale cohomology and de Rham cohomology for varieties over the complex number field $\boldsymbol{C}$ to the logarithmic geometry over $\boldsymbol{C}$ in the sense of Fontaine-Illusie.

We state the main results Theorem (0.2) and Theorem (0.5) of this paper.

(0.1). For varieties over $\boldsymbol{C}$, the above three cohomology theories are closely related. We have:

(1) **(étale vs. Betti)** Let X be a scheme locally of finite type over $\boldsymbol{C}$, and let F be a constructible sheaf of (torsion) abelian groups on the étale site $X_{\text{ét}}$. Then, we have

$$H^q(X_{\text{ét}}, F) \cong H^q(X_{\text{an}}, F_{\text{an}}) \quad \text{for any } q \in \boldsymbol{Z},$$

where X_{an} is the analytic space associated to X and F_{an} is the inverse image of F on X_{an}. (Cf. [AGrV] XVI 4.1. See also the proof of Theorem (2.6).)

(2) **(de Rham vs. Betti)** Let X be a smooth scheme over $\boldsymbol{C}$. Then, we have

$$\boldsymbol{H}^q(X, \Omega^{\cdot}_{X/\boldsymbol{C}}) \cong H^q(X_{\text{an}}, \boldsymbol{C}) \quad \text{for any } q \in \boldsymbol{Z},$$

where $\Omega^{\cdot}_{X/\boldsymbol{C}}$ is the de Rham complex of X ([Gr2]).

In this paper, we prove generalizations of these results to schemes over $\boldsymbol{C}$ endowed with logarithmic structures in the sense of Fontaine-Illusie.

Let X be an fs log scheme ([N1] (1.7)) over $\boldsymbol{C}$ whose underlying scheme $\overset{\circ}{X}$ is locally of finite type over $\boldsymbol{C}$. Then the analytic space X_{an} associated to $\overset{\circ}{X}$ is endowed with the inverse image of the log structure of X. For an analytic space Y over $\boldsymbol{C}$ endowed with an fs log structure (like X_{an}), we will define a topological space $Y^{\log}$ which is endowed with a continuous surjective map $\tau : Y^{\log} \to Y$ ((1.2)). We denote $(X_{\text{an}})^{\log}$ by $X_{\text{an}}^{\log}$. We prove:

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THEOREM (0.2). *Let X be an fs log scheme over $\boldsymbol{C}$ whose underlying scheme $\mathring{X}$ is locally of finite type over $\boldsymbol{C}$. Then*:

(1) **(log étale vs. log Betti)** *For any constructible (Definition (2.5.1)) sheaf of (torsion) abelian groups F on the logarithmic étale site $X_{\text{ét}}^{\log}$ ([N1] (2.2)), we have an isomorphism*

$$H^q(X_{\text{ét}}^{\log}, F) \cong H^q(X_{\text{an}}^{\log}, F_{\text{an}}^{\log}) \quad \text{for any } q \in \boldsymbol{Z},$$

where $F_{\text{an}}^{\log}$ is the inverse image of F on $X_{\text{an}}^{\log}$.

(2) **(log de Rham vs. log Betti)** *Assume that locally for the classical étale topology of $\mathring{X}$, there exist an fs monoid P (Definition (1.1.2)), an ideal Σ (Definition (4.1) (1)) of P, and a morphism*

$$f : X \to \operatorname{Spec}(\boldsymbol{C}[P]/(\Sigma))$$

of log schemes over $\boldsymbol{C}$ ($\boldsymbol{C}[P]$ denotes the semi-group ring of P over $\boldsymbol{C}$ and (Σ) denotes the ideal of this ring generated by Σ) such that the underlying morphism of schemes $\mathring{f}$ of f is smooth and such that the log structure of X is associated to $P \to \mathcal{O}_X$. Then we have a canonical isomorphism

$$\boldsymbol{H}^q(X, \omega_{X/\boldsymbol{C}}^{\cdot}) \cong H^q(X_{\text{an}}^{\log}, \boldsymbol{C}) \quad \text{for any } q \in \boldsymbol{Z},$$

where the left hand side is the "logarithmic de Rham cohomology" of X, i.e. the hyper-cohomology of the de Rham complex with log poles $\omega_{X/\boldsymbol{C}}^{\cdot}$ of X ([K] (1.7)).

Thus, if we call the cohomology group $H^q(X_{\text{an}}^{\log}, \boldsymbol{Q})$ the log Betti cohomology of X, Theorem (0.2) (1) shows that for a prime number p, the p-adic log étale cohomology $H^q(X_{\text{ét}}^{\log}, \boldsymbol{Q}_p) = \boldsymbol{Q} \otimes \varprojlim_n H^q(X_{\text{ét}}^{\log}, \boldsymbol{Z}/p^n\boldsymbol{Z})$ of X is isomorphic to $\boldsymbol{Q}_p \otimes$ the log Betti cohomology of X, and Theorem (0.2) (2) shows that under the assumption of Theorem (0.2) (2), the log de Rham cohomology is isomorphic to $\boldsymbol{C} \otimes$ the log Betti cohomology of X.

(0.3). A new aspect in this log version Theorem (0.2) is that $X_{\text{an}}^{\log}$ is no longer an analytic space over $\boldsymbol{C}$ if the log structure of X is non-trivial. It happens that $X_{\text{an}}^{\log}$ is homeomorphic to $\boldsymbol{S}^1$ (the circle) for some X in Theorem (0.2) (2). In general, for an fs log analytic space Y over $\boldsymbol{C}$, the canonical map $\tau : Y^{\log} \to Y$ is surjective and for $y \in Y, \tau^{-1}(y)$ is homeomorphic to the product of r copies of $\boldsymbol{S}^1$ where $r = \operatorname{rank}_{\boldsymbol{Z}}(M_{Y,y}^{\text{gp}}/\mathcal{O}_{Y,y}^*)$. For example, if Y is a Riemann surface endowed with the log structure corresponding to a finite subset E of Y, $Y^{\log}$ is obtained from Y by replacing each point of E by $\boldsymbol{S}^1$.

(0.4). We obtain the following "logarithmic Riemann-Hilbert correspondence". Let now X be an fs log analytic space over $\boldsymbol{C}$. Assume that there exist an open covering $(U_\lambda)_\lambda$ of X, fs monoids P_λ, and an ideal Σ_λ of P_λ for each λ, such that U_λ is isomorphic to an open analytic subspace of $\operatorname{Spec}(\boldsymbol{C}[P_\lambda]/(\Sigma_\lambda))_{\text{an}}$ endowed with the log structure associated to $P_\lambda \to \mathcal{O}_{U_\lambda}$.

Define the categories $D_{\text{nilp}}(X)$ and $L_{\text{unip}}(X^{\log})$ as follows. Let $D(X)$ be the category of vector bundles V on X endowed with an integrable connection with log poles

$$\nabla : V \to \omega^1_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V,$$

and let $L(X^{\log})$ be the category of local systems of finite dimensional $\boldsymbol{C}$-vector spaces on $X^{\log}$. Let $D_{\text{nilp}}(X)$ (resp. $L_{\text{unip}}(X^{\log})$) be the full subcategory of $D(X)$ (resp. $L(X^{\log})$) consisting of objects V (resp. L) satisfying the following condition locally on X: There exists a finite family of $\mathcal{O}_X$-subsheaves $(V_i)_{0 \le i \le n}$ of V satisfying $\nabla(V_i) \subset \omega^1_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V_i$ (resp. finite family of $\boldsymbol{C}$-subsheaves $(L_i)_{0 \le i \le n}$ of L) such that

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \quad (\text{resp. } 0 = L_0 \subset L_1 \subset \cdots \subset L_n = L),$$

and such that for each $1 \le i \le n$, V_i/V_{i-1} is a vector bundle and the connection induced on V_i/V_{i-1} does not have a pole (cf. Theorem (0.5) (1)) (resp. L_i/L_{i-1} is isomorphic to the inverse image of a local system of finite dimensional $\boldsymbol{C}$-vector spaces on X).

THEOREM (0.5). *Let the notation be as in* (0.4). *Then there exists an equivalence of categories*

$$\Phi : D_{\text{nilp}}(X) \xrightarrow{\sim} L_{\text{unip}}(X^{\log})$$

such that:

(1) *If V is an object of $D_{\text{nilp}}(X)$ whose connection does not have a pole, $\Phi(V)$ is the inverse image of* $\operatorname{Ker}(\nabla : V \to \omega^1_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V)$. *Here we say that the connection of V does not have a pole if the image of ∇ is contained in the image of $\Omega^1_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V \to \omega^1_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V$, with $\Omega^1_{X/\boldsymbol{C}}$ the usual sheaf of differential forms.*

(2) *We have canonical isomorphisms*

$$\omega^{\cdot}_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V \cong \mathrm{R}\tau_* \Phi(V) \quad \text{in the derived category } D(X, \boldsymbol{C})$$

for objects V of $D_{\text{nilp}}(X)$. In particular,

$$\boldsymbol{H}^q(X, \omega^{\cdot}_{X/\boldsymbol{C}} \otimes_{\mathcal{O}_X} V) \cong H^q(X^{\log}, \Phi(V)) \quad \text{for all } q,$$

and (*take $V = \mathcal{O}_X$ with $\nabla = d$, then $\Phi(V) = \boldsymbol{C}$*)

$$\boldsymbol{H}^q(X, \omega^{\cdot}_{X/\boldsymbol{C}}) \cong H^q(X^{\log}, \boldsymbol{C}).$$

The two categories $D_{\text{nilp}}(X)$ and $L_{\text{unip}}(X^{\log})$ are abelian categories because $L_{\text{unip}}(X^{\log})$ is clearly abelian. The functor Φ, being an equivalence of abelian categories, preserves exact sequences.

(0.6). The functor Φ is obtained as follows. Though $X^{\log}$ is not an analytic space in general, $X^{\log}$ is still endowed with a nice sheaf of rings $\mathcal{O}_X^{\log}$ ((3.2)) which is locally generated over $\tau^{-1}(\mathcal{O}_X)$ by logarithms of local sections of the log structure M_X of X. For an object V of $D(X)$, the connection ∇ of V and the

canonical derivation ((3.5))

$$0.6.1 \qquad d : \mathcal{O}_X^{\log} \to \omega_{X/C}^{1,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_{X/C}^1)$$

induces a connection

$$0.6.2 \qquad \nabla : V^{\log} \to \omega_{X/C}^{1,\log} \otimes_{\mathcal{O}_X^{\log}} V^{\log}$$

on $V^{\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(V)$. We define $\Phi(V)$ as the kernel of the map 0.6.2.

(0.7). The authors do not know if the equivalence in Theorem (0.5) is extended to larger categories including "perverse sheaves on $X^{\log}$". (The authors do not know the definitions of them). They have not yet obtained the functoriality of "Riemann-Hilbert correspondences" in Theorem (0.5).

(0.8). Finally in Remark (4.10), by using the results of this paper, we give a new construction of integral structures of some mixed Hodge structures considered by Steenbrink [S].

(0.9). A part of this work was done while one of the authors (K. Kato) was a visitor in Isaac Newton Institute whose hospitality is greatly appreciated. K. Fujiwara pointed out an error in Theorem (0.2) (1) in an earlier version. T. Kajiwara gave some advice on the presentation. The authors would like to thank them. The authors also express their sincere gratitude to the referee for various helpful advice. The second author would like to thank the first for his invitation to the collaboration.

1. The topological space $X^{\log}$

(1.1). Definition of fs log analytic spaces. See [Gr1] for the definition of analytic spaces. In particular the structure sheaf may have nilpotent elements. The definition of (fs) log analytic spaces is analogous to that of (fs) log schemes in [K] ([N1]) as follows:

DEFINITION (1.1.1). Let X be an analytic space (or ringed topos). A *pre-log structure* on X is a pair of a sheaf of monoids M on X and a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$. A pre-log structure (M, α) is said to be a *log structure* if $\alpha : \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. A *log analytic space* is an analytic space endowed with a log structure. For a log analytic space X, we denote by M_X the log structure of X.

A morphism of log analytic spaces, the log structure associated to a pre-log structure and a chart for a log structure are defined similarly as in [K]. For a survey, see also [I].

DEFINITION (1.1.2). A monoid M is called *saturated* if M is integral ([K] (2.2)) and satisfies the following condition;

if a is an element of M^{gp} and $a^n \in M$ for some $n \geqq 1$, then $a \in M$.

We call a finitely generated, saturated monoid an *fs monoid.*

A log analytic space X is said to be *fs* if locally the log structure is associated to a pre-log structure of the form (P_X, α) where P_X is a constant sheaf of monoids for an fs monoid P.

In this § 1, let X be an fs log analytic space over $\boldsymbol{C}$.

(1.2). We define the topological space $X^{\log}$ as follows. In § 3, we will endow $X^{\log}$ with a structure of a ringed space.

As a set, we define $X^{\log}$ by

$$X^{\log} = \left\{ (x,h) \,\middle|\, x \in X, h \in \mathrm{Hom}(M_{X,x}^{\mathrm{gp}}, \boldsymbol{S}^1), h(f) = \frac{f(x)}{|f(x)|} \text{ for any } f \in \mathcal{O}_{X,x}^* \right\}.$$

We have an evident map $X^{\log} \to X; (x,h) \mapsto x$ which we will denote by τ.

The set $X^{\log}$ is defined also as follows: Let T be the analytic space $\mathrm{Spec}(\boldsymbol{C})$ endowed with the log structure M_T given by:

$$\Gamma(T, M_T) = \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1$$

where

$$\boldsymbol{R}_{\geq 0} = \{x \in \boldsymbol{R}; x \geq 0\}$$

$$\boldsymbol{S}^1 = \{x \in \boldsymbol{C}; |x| = 1\}$$

with the multiplicative semi-group laws, and where $M_T \to \mathcal{O}_T$ is given by

$$\alpha_T : \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1 \to \boldsymbol{C}; (x,y) \to xy.$$

Note that this log structure on T is not fs.

As a set, $X^{\log}$ is the set of all morphisms $T \to X$ of log analytic spaces over $\boldsymbol{C}$: We associate to $(x,h) \in X^{\log}$ the morphism $T \to X$ defined by the homomorphism $M_{X,x} \to \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1; a \mapsto (|(\alpha(a))(x)|, h(a))$.

We define the topology of $X^{\log}$ as follows.

(1.2.1). Assume there exists a chart $\beta : P \to M_X$ with P fs. When fixing such β, we can identify $X^{\log}$ with a closed subset of $X \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$ via the map

$$X^{\log} \hookrightarrow X \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1); \quad (x,h) \mapsto (x, h_P)$$

where h_P is the composite $P^{\mathrm{gp}} \to M_{X,x}^{\mathrm{gp}} \xrightarrow{h} \boldsymbol{S}^1$. The image is closed because $(x,\sigma) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$ is contained in the image if and only if for any $p \in P$, $(\beta(p))(x) = \sigma(p)|(\beta(p))(x)|$.

We endow $X^{\log}$ with the induced topology from $X \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$. Here the topology of $\mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$ is the evident one. This topology does not depend on the choice of a chart $P \to M_X$ because for another $\beta' = \beta \circ u$ with $u : P' \to P$, the map $\mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1) \to \mathrm{Hom}(P'^{\mathrm{gp}}, \boldsymbol{S}^1)$ is closed and continuous.

EXAMPLE (1.2.1.1). In the above we see that points of $X^{\log}$ are also given by pairs of points x of X and commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\beta} & \Gamma(X, \mathcal{O}_X) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle f \mapsto f(x)} \\ \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1 & \xrightarrow{\alpha_T} & \boldsymbol{C}, \end{array}$$

that is, pairs $(x, \mu) \in X \times \mathrm{Hom}(P, \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1)$ such that $(\beta(p))(x) = \alpha_T \circ \mu(p)$ for any $p \in P$. From this we have that for $X = (\mathrm{Spec}\, \boldsymbol{C}[\mathrm{P}])_{\mathrm{an}}$ with the canonical log structure, $X^{\log} = \mathrm{Hom}(P, \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1)$.

(1.2.2). By definition, locally on X, the assumption in (1.2.1) is satisfied. We define the topology of $X^{\log}$ locally on X according to (1.2.1). Then the local definitions glue together and give a well defined topology on $X^{\log}$.

(1.2.3). If the underlying analytic space $\overset{\circ}{X}$ of X is smooth over $\boldsymbol{C}$ and the log structure M_X is associated to a divisor D with normal crossings on $\overset{\circ}{X}$ (that is,

$$\begin{aligned} M_X = \{ & f \in \mathcal{O}_X; f \text{ is invertible outside } D\} \\ & \subset \mathcal{O}_X), \end{aligned}$$

$X^{\log}$ is the "real blowing up" of X along D (cf. [P], [M]).

(1.2.4). Y. Kawamata and Y. Namikawa [KN] independently constructed a real analytic manifold with corner X^* for a normal crossing variety X (endowed with a log structure in their sense) whose underlying topological space coincides with $X^{\log}$.

(1.2.5). A morphism of fs log analytic spaces $f : X \to Y$ induces a continuous map $f^{\log} : X^{\log} \to Y^{\log}$ by definition.

LEMMA (1.3). (1) *The map $\tau : X^{\log} \to X$ is continuous. Furthermore it is proper, that is, for any compact subset C of X, the subspace $\tau^{-1}(C)$ of $X^{\log}$ is compact.*

(2) *For $x \in X, \tau^{-1}(x)$ is homeomorphic to the product of r copies of $\boldsymbol{S}^1$ where r is the rank of $M^{\mathrm{gp}}_{X,x}/\mathcal{O}^*_{X,x}$.*

(3) *Let $f : X \to Y$ be a $\boldsymbol{C}$-morphism of fs log analytic spaces over $\boldsymbol{C}$. Assume $f^* M_Y \xrightarrow{\cong} M_X$. Then, the diagram of topological spaces*

$$\begin{array}{ccc} X^{\log} & \xrightarrow{f^{\log}} & Y^{\log} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian.

Proof. (1) The continuity is clear. The properness follows from the fact that locally on X, $X^{\log}$ is homeomorphic over X to a closed subset of $X \times (\boldsymbol{S}^1)^r$ for some $r \geq 0$ ((1.2.1)).

Next we prove (3). Since the problem is local on Y, we may suppose that there exists a chart $P \to M_Y$ with P fs. Then the diagram of topological spaces

$$\begin{array}{ccc} X^{\log} & \longrightarrow & Y^{\log} \\ \cap\downarrow & & \downarrow\cap \\ X \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1) & \longrightarrow & Y \times \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1) \end{array}$$

is cartesian so that the concerned diagram is also cartesian.

(2) By (3), we may assume that $X = \{x\}$. Take a chart $P := M_{X,x}/\mathcal{O}^*_{X,x} \to M_{X,x}$. Then $X^{\log} = \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$, which is the product of r copies of $\boldsymbol{S}^1$. □

(1.4). We define a sheaf $\mathscr{L}$ of abelian groups on $X^{\log}$.

Consider the exact sequence

$$0 \longrightarrow 2\pi i\boldsymbol{Z} \longrightarrow \mathrm{Cont}(\ , i\boldsymbol{R}) \xrightarrow{\exp} \mathrm{Cont}(\ , \boldsymbol{S}^1) \longrightarrow 0$$

where for a topological space A, $\mathrm{Cont}(\ , A)$ is the sheaf of continuous functions on $X^{\log}$ with values in A.

Let $c : \tau^{-1}(M_X^{\mathrm{gp}}) \to \mathrm{Cont}(\ , \boldsymbol{S}^1)$ be the homomorphism induced by the maps $M_X^{\mathrm{gp}}(U) \to \mathrm{Cont}(U^{\log}, \boldsymbol{S}^1); a \mapsto ((x,h) \mapsto h(a_x))$ for open sets U of X.

We define $\mathscr{L}$ to be the fiber product of $\mathrm{Cont}(\ , i\boldsymbol{R})$ and $\tau^{-1}(M_X^{\mathrm{gp}})$ over $\mathrm{Cont}(\ , \boldsymbol{S}^1)$, so that we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \mathscr{L} & \xrightarrow{\exp} & \tau^{-1}(M_X^{\mathrm{gp}}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow c & & \\ 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \mathrm{Cont}(\ , i\boldsymbol{R}) & \xrightarrow{\exp} & \mathrm{Cont}(\ , \boldsymbol{S}^1) & \longrightarrow & 0. \end{array}$$

We denote the projection $\mathscr{L} \to \tau^{-1}(M_X^{\mathrm{gp}})$ by exp, and we call $\mathscr{L}$ "the sheaf of logarithms of local sections of $\tau^{-1}(M_X^{\mathrm{gp}})$".

Since there is the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \tau^{-1}(\mathcal{O}_X) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}_X^*) & \longrightarrow & 0 \\ & & \| & & a\downarrow & & b\downarrow & & \\ 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \mathrm{Cont}(\ , i\boldsymbol{R}) & \xrightarrow{\exp} & \mathrm{Cont}(\ , \boldsymbol{S}^1) & \longrightarrow & 0 \end{array}$$

$a(f) = f - \mathrm{Re}(f)$ where Re is the real part,

$b(f) = c(\alpha^{-1}(f)) = |f|^{-1} f$ where $|\ |$ is the absolute value,

the two "exponential sequences" are put in the following commutative diagram

with exact rows

$$(1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \tau^{-1}(\mathcal{O}_X) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}_X^*) & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \downarrow \cap & & \\ 0 & \longrightarrow & 2\pi i\boldsymbol{Z} & \longrightarrow & \mathscr{L} & \xrightarrow{\exp} & \tau^{-1}(M_X^{\mathrm{gp}}) & \longrightarrow & 0. \end{array}$$

LEMMA (1.5). *For any sheaf F of abelian groups on X, we have a canonical isomorphism*

$$\mathrm{R}^q\tau_*\tau^*F \cong F_{(-q)} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \quad \textit{for all } q,$$

where $F_{(-q)}$ denotes the Tate twist $F \otimes_{\boldsymbol{Z}} (2\pi i)^{-q}\boldsymbol{Z}$.

Proof. First we note that $F \to \tau_*\tau^{-1}F$ is an isomorphism because of Lemma (1.3) (1) and the fact that each fiber of τ is connected by Lemma (1.3) (2). Since $\exp : \mathcal{O}_X = \tau_*\tau^{-1}(\mathcal{O}_X) \to \mathcal{O}_X^* = \tau_*\tau^{-1}(\mathcal{O}_X^*)$ is surjective, the commutative diagram (1.4) (1) gives a homomorphism $M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to \mathrm{R}^1\tau_*\boldsymbol{Z}_{(1)}$. By cup product, we have a canonical homomorphism from $F_{(-q)} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*)$ to $\mathrm{R}^q\tau_*\tau^*F$. To see that this homomorphism induces the desired isomorphism, by Lemma (1.3) (1) and by the proper base change theorem for locally compact spaces ([V] 1.2 or [Go] 4.11.1), we are reduced to the case where the underlying set of X is a point. In this case $X^{\log} \cong (\boldsymbol{S}^1)^r$ for some $r \geq 0$ by Lemma (1.3) (2), and Lemma (1.5) is reduced to the usual cohomology theory of $(\boldsymbol{S}^1)^r$. □

Remark (1.5.1). In (1.2.3), let $\overset{\circ}{j}$ be the open immersion $\overset{\circ}{X}\backslash D \hookrightarrow \overset{\circ}{X}$. Then it is well known that

$$1.5.1.1\qquad \mathrm{R}^q\overset{\circ}{j}_*\boldsymbol{Z} \cong \boldsymbol{Z}_{(-q)} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*).$$

This is related to Lemma (1.5) by the isomorphism $\mathrm{R}\tau_*\boldsymbol{Z} \cong \mathrm{R}\overset{\circ}{j}_*\boldsymbol{Z}$ which is shown for more general X as follows.

Let X be a log smooth fs log analytic space over $\boldsymbol{C}$. Let j denote the open immersion between log analytic spaces $U := \{x \in \overset{\circ}{X} \mid (M_X/\mathcal{O}_X^*)_x = 1\} \hookrightarrow X$. Then, by [O] 5.12, any point in $X^{\log}$ has a basis of neighbourhoods whose intersection with $U = U^{\log}$ is contractible. (Actually the theory of moment maps ([Ful] p. 81 Proposition) implies that $X^{\log}$ is a topological manifold with the boundary $X^{\log}\backslash U$.) Hence

$$1.5.1.2\qquad \boldsymbol{Z} \xrightarrow{\cong} \mathrm{R}j_*^{\log}\boldsymbol{Z}.$$

Applying $\mathrm{R}\tau_*$, we have $\mathrm{R}\tau_*\boldsymbol{Z} \cong \mathrm{R}\overset{\circ}{j}_*\boldsymbol{Z}$ as desired.

Note that 1.5.1.1 and 1.5.1.2 are compared with Grothendieck-Gabber's purity theorem in l-adic étale cohomology and with its log version (see [FK] (3.1), [N2] (2.0)) respectively.

2. Logarithmic étale cohomology

(2.1). The aim of this section is to prove Theorem (0.2) (1).

In this section, let X be an fs log scheme over $\boldsymbol{C}$ such that $\underline{X}$ is locally of finite type over $\boldsymbol{C}$. Let X_{an} be the analytic space associated to X endowed with the inverse image of the log structure of X. The following lemma implies that $Y \mapsto Y_{\mathrm{an}}^{\log}$ defines a continuous functor from the log étale site to the log Betti site, which preserves fiber products, and hence by [AGrV] IV 4.9.2 a morphism of topoi

$$\varepsilon : (X_{\mathrm{an}}^{\log})^{\sim} \to (X_{\text{ét}}^{\log})^{\sim}$$

$((\ )^{\sim}$ denotes the category of sheaves on a site) with ε_* given by $(\varepsilon_* F)(Y) = F(Y_{\mathrm{an}}^{\log})$ for a sheaf F on $X_{\mathrm{an}}^{\log}$ and an fs log scheme Y over X which is log étale and of Kummer type over X. See [N1] (2.2) for the definition of log étale sites.

LEMMA (2.2). *Let Y be an fs log scheme over X which is log étale and of Kummer type. Then the induced map $Y_{\mathrm{an}}^{\log} \to X_{\mathrm{an}}^{\log}$ is étale. (A continuous map $\pi : A \to B$ between topological spaces A, B is said to be étale if for any $a \in A$, there exists an open neighbourhood U of a such that $\pi(U)$ is open and the map $U \to \pi(U)$ induced by π is a homeomorphism.)*

Proof. Taking a chart, we reduce to the case that there is a cartesian diagram of fs log schemes

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Spec}(\boldsymbol{C}[Q]) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(\boldsymbol{C}[h])} \\ X & \longrightarrow & \mathrm{Spec}(\boldsymbol{C}[P]) \end{array}$$

with horizontal arrows strict where $h : P \to Q$ is a homomorphism of fs monoids of Kummer type, which means that h is injective and that there is a positive integer n such that Q^n is contained in $h(P)$, with P torsionfree. In virtue of Lemma (1.3) (3), $(\text{the above diagram})_{\mathrm{an}}^{\log}$ is also cartesian, so that we further assume $X = \mathrm{Spec}(\boldsymbol{C}[P])$. Then $X^{\log} = \mathrm{Hom}(P, \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1)$ and $Y^{\log} = \mathrm{Hom}(Q, \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1)$ by Example (1.2.1.1). It suffices to show that both $\mathrm{Hom}(Q, \boldsymbol{R}_{\geq 0}) \to \mathrm{Hom}(P, \boldsymbol{R}_{\geq 0})$ and $\mathrm{Hom}(Q^{\mathrm{gp}}, \boldsymbol{S}^1) \to \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{S}^1)$ are étale. But the fact that h is of Kummer type implies that the former is in fact a homeomorphism. The proof for the latter is reduced to the case that Q is also torsionfree, and to the case where $P^{\mathrm{gp}} = Q^{\mathrm{gp}} = \boldsymbol{Z}, h^{\mathrm{gp}}(a) = a^m$ for some $m \geq 1$ and any a, which is clear. □

To prove Theorem (0.2) (1), we first introduce the logarithmic version of the "Kummer exact sequence" as follows. To avoid confusions, we denote by $M_{X,\log}^{\mathrm{gp}}$ the sheaf associated to the presheaf $U \mapsto \Gamma(U, M_U^{\mathrm{gp}})$ on $X_{\text{ét}}^{\log}$, and denote simply by M_X^{gp} the sheaf M_X^{gp} on $X_{\text{ét}}$. (It can be proved that the above presheaf on $X_{\text{ét}}^{\log}$ is indeed a sheaf, but we do not need it here.)

PROPOSITION (2.3). *Let X be an fs log scheme and let n be an integer invertible on X. Then we have an exact sequence of sheaves on $X^{\log}_{\text{ét}}$*

$$0 \to (\boldsymbol{Z}/n\boldsymbol{Z})(1) \to M^{\text{gp}}_{X,\log} \xrightarrow{n} M^{\text{gp}}_{X,\log} \to 0.$$

Proof. Let $x_{(\log)} \to X$ be a log geometric point ([N1](2.5)) of X. We must show that $(M^{\text{gp}}_{X,\log})_{x_{(\log)}}$ is n-divisible. Take a local section $m \in M_X(U)$ with $U \in \text{Ob}\, X^{\log}_{\text{ét}}$. It suffices to show that m is locally n-divisible. Take a morphism of fs log schemes $U \to \text{Spec}(\boldsymbol{Z}[\boldsymbol{N}])$ such that the image of $1 \in \boldsymbol{N}$ in $M_X(U)$ coincides with m. Then $U' := U \times_{\text{Spec}(\boldsymbol{Z}[\boldsymbol{N}])} \text{Spec}(\boldsymbol{Z}[\boldsymbol{N}^{1/n}])$ is log étale over U of Kummer type, and the projection $U' \to U$ is surjective. Clearly m is n-divisible on U'. □

In Proposition (2.3), denote by f the forgetting-log morphism $(X^{\log}_{\text{ét}})^{\sim} \to (\overset{\circ}{X}_{\text{ét}})^{\sim}$. Then the connecting map $M^{\text{gp}}_X \to \text{R}^1 f_*(\boldsymbol{Z}/n\boldsymbol{Z})(1)$ factors as

$$M^{\text{gp}}_X/\mathcal{O}^*_X \to \text{R}^1 f_*(\boldsymbol{Z}/n\boldsymbol{Z})(1).$$

Next let F be an abelian étale sheaf on ($\overset{\circ}{X}$, trivial log structure) such that $F = \bigcup_{n:\text{invertible on } X} \text{Ker}\,(n : F \to F)$. Then the previous homomorphism induces by cup product a homomorphism $\iota : F(-q) \otimes_{\boldsymbol{Z}} \bigotimes^q (M^{\text{gp}}_X/\mathcal{O}^*_X) \to \text{R}^q f_* f^* F$ for any q. Here $(-q)$ means the Tate twist.

THEOREM (2.4). *Let the notation and assumptions be as above. Then ι induces an isomorphism*

$$(*) \qquad F(-q) \otimes_{\boldsymbol{Z}} \bigwedge^q (M^{\text{gp}}_X/\mathcal{O}^*_X) \to \text{R}^q f_* f^* F \quad \text{for any } q.$$

Proof. To prove that ι induces $(*)$ and that $(*)$ is an isomorphism, we may assume that $\overset{\circ}{X}$ is the spectrum of a strict henselian local ring by [N1] (4.2), and it is sufficient to prove that the stalk of the map ι at the closed point x of $\overset{\circ}{X}$ factors into the isomorphism. Let

$$I = \text{Hom}((M^{\text{gp}}_X/\mathcal{O}^*_X)_x, \hat{\boldsymbol{Z}}'(1))$$

where $\hat{\boldsymbol{Z}}'(1) = \varprojlim_n (\boldsymbol{Z}/n\boldsymbol{Z})(1)$ in which n ranges over all integers invertible on X. Let $x_{(\log)}$ be a logarithmic geometric point lying over x. Then, by [N1] (4.1), we have $(\text{R}^q f_* f^* F)_x = H^q(X^{\log}_{\text{ét}}, f^* F) = H^q(I, (f^* F)_{x(\log)}) = H^q(I, F_x)$, where I acts on F_x trivially.

Further

$$H^q(I, F_x) \cong F_x \otimes_{\boldsymbol{Z}} \bigwedge^q (\text{Hom}(I, \hat{\boldsymbol{Z}}'))$$

$$= F_x(-q) \otimes_{\boldsymbol{Z}} \bigwedge^q (M^{\text{gp}}_X/\mathcal{O}^*_X)_x.$$

It is easily checked that the composite of these isomorphisms is compatible with the stalk of the map ι. □

DEFINITION (2.5.1). Let X be an fs log scheme and A a ring. A sheaf F of groups (resp. A-modules) on $X_{\text{ét}}^{\log}$ is said to be *constructible* if for any open affine $U \subset \mathring{X}$, there exists a finite decomposition $(U_i)_{i \in I}$ of U consisting of constructible reduced subschemes such that the inverse image of F to U_i is locally constant whose local values are finite (resp. A-modules of finite presentation), where the log structure of U_i is the restricted one from X.

LEMMA (2.5.2). *Let X be an fs log scheme and A a ring. Let F be a constructible sheaf of A-modules. Assume that $\mathring{X}$ is of equi-characteristic. Then F is log étale locally the inverse image of a constructible sheaf (in the sense of* [AGrV] IX) *of A-modules on the classical étale site by the forgetting-log morphism.*

Proof. We may and will assume that X is quasi-compact and quasi-separated and that there is a chart $X \to \operatorname{Spec}(\boldsymbol{Z}P)$ for some fs monoid P with $P^{\times} = 1$. By [N1] (3.3) 8, any constructible sheaf of A-modules on X is the cokernel of a homomorphism $A_{V,X} \to A_{U,X}$ for some $U, V \in \operatorname{Ob} X_{\text{ét}}^{\log}$ such that both $U \to X$ and $V \to X$ are of finite presentation. On the other hand by the assumption, for any quasi-compact $U \in \operatorname{Ob} X_{\text{ét}}^{\log}$, there is an integer $n \geq 1$ invertible on X such that $U \times_{\operatorname{Spec}(\boldsymbol{Z}P)} \operatorname{Spec}(\boldsymbol{Z}P^{1/n}) \to X' := X \times_{\operatorname{Spec}(\boldsymbol{Z}P)} \operatorname{Spec}(\boldsymbol{Z}P^{1/n})$ is strict. Note here that $X' \to X$ is log étale. Thus we may assume that F is the cokernel of a homomorphism $\varepsilon^* A_{V,\mathring{X}} \to \varepsilon^* A_{U,\mathring{X}}$ for some $U, V \in \operatorname{Ob} \mathring{X}_{\text{ét}}$ such that both $U \to \mathring{X}$ and $V \to \mathring{X}$ are of finite presentation. Here ε is the forgetting-log morphism $X_{\text{ét}}^{\log} \to \mathring{X}_{\text{ét}}$. Since ε^* is full, F is the inverse image of the cokernel of a homomorphism $A_{V,\mathring{X}} \to A_{U,\mathring{X}}$, which is constructible. □

Theorem (0.2) (1) follows from

THEOREM (2.6). *Let X and $\varepsilon : (X_{\text{an}}^{\log})^{\sim} \to (X_{\text{ét}}^{\log})^{\sim}$ be as in* (2.1). *Then, we have*

$$F \xrightarrow{\cong} \mathrm{R}\varepsilon_* \varepsilon^* F$$

for any constructible torsion sheaf F on $X_{\text{ét}}^{\log}$.

Proof. We may assume that $\mathring{X}$ is quasi-compact. Then F is a $\boldsymbol{Z}/n\boldsymbol{Z}$-Module for some n. Applying Lemma (2.5.2), we may assume that $F = f^* G$ for some constructible torsion sheaf G on $\mathring{X}_{\text{ét}}$, since the problem is log étale local on X.

Now consider the commutative diagram

$$\begin{array}{ccc} (X_{\text{an}}^{\log})^{\sim} & \xrightarrow{\varepsilon} & (X_{\text{ét}}^{\log})^{\sim} \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle f} \\ (\mathring{X}_{\text{an}})^{\sim} & \xrightarrow{g} & (\mathring{X}_{\text{ét}})^{\sim}. \end{array}$$

It is sufficient to prove that

$$2.6.1 \qquad \mathrm{R}f_*f^*G \to \mathrm{R}f_*\mathrm{R}\varepsilon_*\varepsilon^*f^*G$$

is an isomorphism (the log étale localization of the isomorphism 2.6.1 gives Theorem (2.6) because for each $x \in X$, we have $(\ )_{x(\log)} = \varinjlim_{X'} f_{X'*}(\ |_{X'})_{\overline{x'}}$ where $_{x(\log)} \to X$ is a log geometric point lying over x, X' runs over the category of X-morphisms $_{x(\log)} \to X'$ with $X' \in \mathrm{Ob}\, X_{\text{ét}}^{\log}$, x' is the image of $_{x(\log)} \to X'$ and $f_{X'}$ is f for X').

We know by Lemma (1.5) and Theorem (2.4) that the base change map

$$g^*\mathrm{R}f_*K \to \mathrm{R}\tau_*\varepsilon^*K \quad (\text{where } K = f^*G)$$

is an isomorphism because it is easy to see that it is compatible with the composite $\mathscr{H}^q(g^*\mathrm{R}f_*f^*G) \overset{(2.4)}{=} g^*G_{(-q)} \otimes \bigwedge^q(M_{X_{\text{an}}}^{\text{gp}}/\mathcal{O}_{X_{\text{an}}}^*) \overset{(1.5)}{=} \mathscr{H}^q(\mathrm{R}\tau_*\tau^*g^*G)$ where $_{(-q)}$ means the Tate twist.

On the other hand, by the classical comparison theorem for the étale cohomology, the adjunction map $\mathrm{id} \to \mathrm{R}g_*g^*$ is an isomorphism, which follows easily from [AGrV] XVI 4.1. Hence

$$\begin{aligned} \mathrm{R}f_*f^*G &\xrightarrow{\text{adj}} \mathrm{R}g_*g^*\mathrm{R}f_*f^*G \\ &\longrightarrow \mathrm{R}g_*\mathrm{R}\tau_*\varepsilon^*f^*G \\ &= \mathrm{R}f_*\mathrm{R}\varepsilon_*\varepsilon^*f^*G \end{aligned}$$

is an isomorphism. □

Remark (2.7). Let $f : X \to Y$ be a morphism of fs log schemes over $\boldsymbol{C}$ such that $\mathring{X}$ and $\mathring{Y}$ are of finite type over $\boldsymbol{C}$. Consider the diagram

$$\begin{array}{ccc} (X_{\text{an}}^{\log})^{\sim} & \xrightarrow{f} & (Y_{\text{an}}^{\log})^{\sim} \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ (X_{\text{ét}}^{\log})^{\sim} & \xrightarrow{f} & (Y_{\text{ét}}^{\log})^{\sim}. \end{array}$$

Then it is not true that

$$2.7.1 \qquad \varepsilon^*\mathrm{R}f_*F \xrightarrow{\cong} \mathrm{R}f_*\varepsilon^*F$$

for constructible torsion abelian sheaves F on $X_{\text{ét}}^{\log}$, though this isomorphism should be a log version of [AGrV] XVI 4.1 (Theorem 0.2 (1) says that the 2.7.1 is true in the case $Y = \mathrm{Spec}(\boldsymbol{C})$ with the trivial log structure). A counterexample of 2.7.1 is the following. Let $X = \mathrm{Spec}(\boldsymbol{C})$ (resp. $Y = \mathrm{Spec}(\boldsymbol{C})$) endowed with the log structure associated to

$$\boldsymbol{N} \to \boldsymbol{C} \quad (\text{resp. } \boldsymbol{N}^2 \to \boldsymbol{C})$$

which sends x to 0 if $x \neq 0$ and sends 0 to 1. Let $f : X \to Y$ be the morphism

induced by $\boldsymbol{N}^2 \to \boldsymbol{N}; (x, y) \mapsto x + y$. Then $X_{\text{an}}^{\log} = \boldsymbol{S}^1$, $Y_{\text{an}}^{\log} = \boldsymbol{S}^1 \times \boldsymbol{S}^1$, and $X_{\text{an}}^{\log} \to Y_{\text{an}}^{\log}$ is the diagonal embedding. Let $n \neq 0$, and consider the constant sheaf $\boldsymbol{Z}/n\boldsymbol{Z}$ on $X_{\text{ét}}^{\log}$. Then the support of $\varepsilon^* f_*(\boldsymbol{Z}/n\boldsymbol{Z})$ is the whole space. On the other hand, $f_*\varepsilon^*(\boldsymbol{Z}/n\boldsymbol{Z})$ is the zero extension of $\boldsymbol{Z}/n\boldsymbol{Z}$ on the diagonal of $\boldsymbol{S}^1 \times \boldsymbol{S}^1$.

3. The sheaf $\mathcal{O}_X^{\log}$

Let X be an fs log analytic space over $\boldsymbol{C}$, and let $\tau : X^{\log} \to X$ be the canonical morphism.

(3.1). We define the sheaf $\mathcal{O}_X^{\log}$ of rings on $X^{\log}$. Roughly speaking, $\mathcal{O}_X^{\log}$ is a $\tau^{-1}(\mathcal{O}_X)$-algebra generated by "logarithms" of local sections of $\tau^{-1}(M_X^{\text{gp}})$. The precise definition of $\mathcal{O}_X^{\log}$ is as follows.

(3.2). Recall that we defined in (1.4) a sheaf $\mathscr{L}$ of abelian groups on $X^{\log}$ and a homomorphism of sheaves of abelian groups $h : \tau^{-1}(\mathcal{O}_X) \to \mathscr{L}$ which sit in the exact sequence

$$0 \longrightarrow \tau^{-1}(\mathcal{O}_X) \xrightarrow{h} \mathscr{L} \xrightarrow{\exp} \tau^{-1}(M_X^{\text{gp}}/\mathcal{O}_X^*) \longrightarrow 0. \tag{3.2.1}$$

Consider commutative $\tau^{-1}(\mathcal{O}_X)$-algebras $\mathscr{A}$ on $X^{\log}$ endowed with a homomorphism $\mathscr{L} \to \mathscr{A}$ of sheaves of abelian groups which commutes with h. We define $\mathcal{O}_X^{\log}$ to be the universal one among such $\mathscr{A}$. More explicitly, $\mathcal{O}_X^{\log}$ is defined by

$$\mathcal{O}_X^{\log} = (\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \operatorname{Sym}_{\boldsymbol{Z}}(\mathscr{L}))/\mathfrak{a}$$

where $\operatorname{Sym}_{\boldsymbol{Z}}(\mathscr{L})$ is the symmetric algebra of $\mathscr{L}$ over $\boldsymbol{Z}$ and $\mathfrak{a}$ is the ideal of $\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \operatorname{Sym}_{\boldsymbol{Z}}(\mathscr{L})$ generated locally by local sections of the form

$$f \otimes \underline{1} - 1 \otimes h(f) \quad \text{for } f \text{ a local section of } \tau^{-1}(\mathcal{O}_X).$$

Here $\underline{1}$ means the $1 \in \boldsymbol{Z} = \operatorname{Sym}^0(\mathscr{L})$, whereas $h(f)$ belongs to $\mathscr{L} = \operatorname{Sym}^1(\mathscr{L})$.

LEMMA (3.3). *Let $x \in X$, y a point of $X^{\log}$ with image x in X, and let $(t_i)_{1 \le i \le n}$ be a family of elements of the stalk $\mathscr{L}_y$ whose image under* exp *is a $\boldsymbol{Z}$-basis of $(M_X^{\text{gp}}/\mathcal{O}_X^*)_x$. Then, as an $\mathcal{O}_{X,x}$-algebra, $\mathcal{O}_{X,y}^{\log}$ is isomorphic to the polynomial ring $\mathcal{O}_{X,x}[T_1, \ldots, T_n]$ in n variables by*

$$\mathcal{O}_{X,x}[T_1, \ldots, T_n] \to \mathcal{O}_{X,y}^{\log}; T_i \mapsto t_i.$$

Proof. By 3.2.1, we have an isomorphism

$$\tau^{-1}(\mathcal{O}_X)_y \oplus \boldsymbol{Z}^{\oplus n} \xrightarrow{\cong} \mathscr{L}_y; \quad (f, (m_i)_{1 \le i \le n}) \mapsto f + \sum_{i=1}^{n} m_i t_i.$$

This shows

$$\mathcal{O}_{X,y}^{\log} \cong \tau^{-1}(\mathcal{O}_X)_y \otimes_{\boldsymbol{Z}} \mathrm{Sym}_{\boldsymbol{Z}}(\boldsymbol{Z}^{\oplus n})$$

$$= \tau^{-1}(\mathcal{O}_X)_y[T_1, \ldots, T_n]. \qquad \square$$

LEMMA (3.4). *For $r \in \boldsymbol{Z}$, let $\mathrm{fil}_r(\mathcal{O}_X^{\log})$ be the image of $\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \bigoplus_{i=0}^{r} \mathrm{Sym}_{\boldsymbol{Z}}^i(\mathscr{L})$ in $\mathcal{O}_X^{\log}$, where $\mathrm{Sym}_{\boldsymbol{Z}}^i(\ )$ denotes the i-th symmetric power over $\boldsymbol{Z}$. Then*

$$\mathrm{fil}_0(\mathcal{O}_X^{\log}) = \tau^{-1}(\mathcal{O}_X),$$

and the canonical map

$$\tau^{-1}(M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \cong \mathscr{L}/\tau^{-1}(\mathcal{O}_X) \subset \mathrm{fil}_1(\mathcal{O}_X^{\log})/\tau^{-1}(\mathcal{O}_X) \quad (3.2.1)$$

induces an isomorphism

$$\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \tau^{-1}(\mathrm{Sym}_{\boldsymbol{Z}}^r(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)) \xrightarrow{\cong} \mathrm{fil}_r(\mathcal{O}_X^{\log})/\mathrm{fil}_{r-1}(\mathcal{O}_X^{\log})$$

for any $r \geq 0$.

Proof. This is checked on the stalks as follows. Let x and y be as in Lemma (3.3). Then via the isomorphism in Lemma (3.3), the inverse image of $\mathrm{fil}_r(\mathcal{O}_X^{\log})_y$ in $\tau^{-1}(\mathcal{O}_X)_y[T_1, \ldots, T_n]$ coincides with the $\tau^{-1}(\mathcal{O}_X)_y$-submodule consisting of polynomials of degree $\leqq r$. $\square$

(3.5). Now we consider differential forms on $X^{\log}$. Let $\Omega_X^1 = \Omega_{X/\boldsymbol{C}}^1$ be the sheaf of differential forms on the underlying analytic space $\overset{\circ}{X}$ of X (that is, Ω_X^1 is the conormal sheaf of $\overset{\circ}{X}$ embedded in $\overset{\circ}{X} \times \overset{\circ}{X}$ diagonally), and let $\omega_X^1 = \omega_{X/\boldsymbol{C}}^1$ be the sheaf of differential forms with log poles on X (that is,

$$\omega_X^1 = (\Omega_X^1 \oplus (\mathcal{O}_X \otimes_{\boldsymbol{Z}} M_X^{\mathrm{gp}}))/N$$

where N is the $\mathcal{O}_X$-subsheaf of the direct sum generated locally by local sections of the form $(-d\alpha(x), \alpha(x) \otimes x)$ with $x \in M_X$). The map $M_X^{\mathrm{gp}} \to \omega_X^1$; $x \mapsto (0, 1 \otimes x) \bmod N$ is denoted by $d\log$, and its restriction to $\mathcal{O}_X^*$ coincides with $f \mapsto f^{-1}df$.

Let Ω_X^q (resp. ω_X^q) be the q-th exterior power of Ω_X^1 (resp. ω_X^1) over $\mathcal{O}_X$. Let

$$\omega_X^{q,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\omega_X^q).$$

By the definition of $\mathcal{O}_X^{\log}$ as a quotient of $\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \mathrm{Sym}_{\boldsymbol{Z}}(\mathscr{L})$, we see that there exists a unique derivation

$$d : \mathcal{O}_X^{\log} \to \omega_X^{1,\log}$$

which extends $\tau^{-1}(\mathcal{O}_X) \xrightarrow{d} \tau^{-1}(\Omega_X^1) \to \tau^{-1}(\omega_X^1)$ and which satisfies

$$dx = d\log(\exp(x)) \quad \text{for } x \in \mathscr{L},$$

and that this d is extended to

$$d : \omega_X^{q,\log} \to \omega_X^{q+1,\log}; \quad x \otimes y \mapsto dx \wedge y + x \otimes dy$$

$(x \in \mathcal{O}_X^{\log}, y \in \tau^{-1}(\omega_X^q))$ satisfying $d \circ d = 0$.

LEMMA (3.6). *Let Y be an fs log analytic space and let X be a closed analytic subspace of Y defined by an ideal I of $\mathcal{O}_Y$. Endow X with the inverse image of the log structure of Y.*

(1) *We have an exact sequence*

$$I/I^2 \xrightarrow{d} \omega_Y^1/I\omega_Y^1 \to \omega_X^1 \to 0.$$

(2) *Assume that for any $y \in Y$, the ideal I_y of $\mathcal{O}_{Y,y}$ is generated by the images of some elements of M_y. Then, $\omega_Y^1/I\omega_Y^1 \xrightarrow{\cong} \omega_X^1$.*

Proof. (1) follows by standard arguments. To prove (2), it is sufficient to show that the map d in (1) is the zero map. Indeed for $f \in M_{Y,y}$ such that $\alpha(f) \in I_y, d\alpha(f) = \alpha(f)d\log(f) \in I_y\omega_{Y,y}^1$. □

PROPOSITION (3.7). *Let X be an fs log analytic space satisfying the assumption of Theorem* (0.5). *Then, the $\mathcal{O}_X$-module ω_X^1 is locally free of finite rank.*

Proof. We may assume $X = \mathrm{Spec}(\boldsymbol{C}[P]/(\Sigma))_{\mathrm{an}}$ for an ideal Σ (Definition (4.1) (1)) of an fs monoid P. Let $Y = \mathrm{Spec}(\boldsymbol{C}[P])_{\mathrm{an}}$. Then

$$\mathcal{O}_Y \otimes_{\boldsymbol{Z}} P^{\mathrm{gp}} \xrightarrow{\cong} \omega_Y^1; \quad x \otimes y \mapsto xd\log(y)$$

and hence the $\mathcal{O}_Y$-module ω_Y^1 is free of finite rank. Now we apply Lemma (3.6) (2) for these X, Y, and we have $\omega_X^1 = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \omega_Y^1$. □

THEOREM (3.8) **(logarithmic Poincaré lemma).** *Let X be an fs log analytic space satisfying the assumption of Theorem* (0.5). *Then,*

$$\boldsymbol{C} \to \omega_X^{\cdot,\log}$$

is a quasi-isomorphism.

This theorem will be proved in (4.7).

4. Logarithmic Riemann-Hilbert correspondence

The aim of this section is to prove Theorem (0.2) (2), Theorem (0.5) and Theorem (3.8).

We begin with preliminaries on monoids.

DEFINITION (4.1). Let P be a monoid.

(1) A subset Σ of P is called an *ideal* of P if $ax \in \Sigma$ for any $a \in P$ and any $x \in \Sigma$.

(2) An ideal $\mathfrak{p}$ of P is called a *prime ideal* of P if the complement $P \setminus \mathfrak{p}$ is a submonoid of P.

(3) An ideal $\mathfrak{q}$ of P is called a *primary ideal* of P if $\mathfrak{q} \neq P$ and if one of the following two conditions (i) and (ii) is satisfied by each element a of P:

(i) $a^n \in \mathfrak{q}$ for some $n \geq 1$.

(ii) $\{x \in P; ax \in \mathfrak{q}\} = \mathfrak{q}$.

Any prime ideal is primary.

LEMMA (4.2). *Let P be a monoid and let $\mathfrak{q}$ be a primary ideal of P.*

(1) *The set $\mathfrak{p} = \{a \in P; a^n \in \mathfrak{q}$ for some $n \geq 1\}$ is a prime ideal of P, and $P \setminus \mathfrak{p}$ coincides with the set of elements a of P satisfying the condition* (3) (ii) *in Definition* (4.1).

(2) *Assume P is finitely generated. Then there exists an element a of P such that*

$$\{x \in P; ax \in \mathfrak{q}\} = \mathfrak{p}.$$

Proof. The proof of (1) is easy and left to the reader. We prove (2). For $a \in P \setminus \mathfrak{q}$, let $\Sigma_a = \{x \in P; ax \in \mathfrak{q}\}$. Then Σ_a is an ideal of P. Since the ring $\boldsymbol{Z}[P]$ is Noetherian, there exists a maximal element in the set of these ideals Σ_a. Assume Σ_a is maximal. We prove $\Sigma_a = \mathfrak{p}$. We show first $\Sigma_a \subset \mathfrak{p}$. If $x \in \Sigma_a$ does not belong to $\mathfrak{p}$, $ax \in \mathfrak{q}$ implies (by (1)) $a \in \mathfrak{q}$ which is a contradiction. We next show $\mathfrak{p} \subset \Sigma_a$. Let $x \in \mathfrak{p}$. Since $a \notin \mathfrak{q}$ and some power of x belongs to $\mathfrak{q}$, we can find the largest integer $n \geq 0$ such that $b \underset{\text{def}}{=} ax^n$ does not belong to $\mathfrak{q}$. Since $\Sigma_a \subset \Sigma_b$ and Σ_a is maximal, we have $\Sigma_a = \Sigma_b$. Since $x \in \Sigma_b$ we have $x \in \Sigma_a$. □

LEMMA (4.3). *Let P be a finitely generated monoid and let $\mathfrak{q}$ be an ideal of P satisfying the following conditions* (i) *and* (ii).

(i) $P \neq \mathfrak{q}$.

(ii) *If Σ and Σ' are ideals of P such that $\mathfrak{q} = \Sigma \cap \Sigma'$, then $\mathfrak{q} = \Sigma$ or $\mathfrak{q} = \Sigma'$.*

Then $\mathfrak{q}$ is a primary ideal.

(Just as in the theory of commutative rings, we can deduce from Lemma (4.3) that any ideal of P has a "primary decomposition".)

Proof. Let $a \in P$. For $n \geq 1$, let $\Sigma_n = \{x \in P; a^n x \in \mathfrak{q}\}$. Then

$$\mathfrak{q} \subset \Sigma_1 \subset \Sigma_2 \subset \cdots\cdots.$$

Since $\boldsymbol{Z}[P]$ is a Noetherian ring, there exists $n \geq 1$ such that $\Sigma_n = \Sigma_{n+1} = \Sigma_{n+2} = \cdots$. Let $\Sigma' = a^n P \cup \mathfrak{q}$. (Note that the union of two ideals is an ideal.) Then, $\mathfrak{q} = \Sigma_1 \cap \Sigma'$. In fact, if $x \in \Sigma_1 \cap \Sigma'$ and $x \notin \mathfrak{q}$, then $x = a^n y$ for some $y \in P$ (because $x \in \Sigma'$) and $a^{n+1} y = ax \in \mathfrak{q}$ (because $x \in \Sigma_1$). Hence $y \in \Sigma_{n+1} = \Sigma_n$ and this shows $x \in \mathfrak{q}$, a contradiction. By (ii), $\mathfrak{q} = \Sigma_1 \cap \Sigma'$ implies $\mathfrak{q} = \Sigma_1$ or $\mathfrak{q} = \Sigma'$.

If $\mathfrak{q} = \Sigma_1$, a has the property (ii) in Definition (4.1). If $\mathfrak{q} = \Sigma'$, a has the property (i) in Definition (4.1). □

LEMMA (4.4). *Let X be an fs log scheme (resp. an fs log analytic space) over $\boldsymbol{C}$. Assume we are given a chart $P \to M_X$ with P an fs monoid. Let $\mathfrak{p}$ be a prime ideal of P which is sent to $0 \in \mathcal{O}_X$ under $P \to M_X \to \mathcal{O}_X$. Let X' be the fs log scheme (resp. fs log analytic space) whose underlying scheme (resp. analytic space) is the same as X but whose log structure is associated to $P\backslash\mathfrak{p} \to \mathcal{O}_{X'}$. For $q, r \in \boldsymbol{Z}$, let $\mathrm{Fil}_r\omega_X^q$ be the $\mathcal{O}_X$-subsheaf of $\omega_X^q = \omega_{X/\boldsymbol{C}}^q$ defined by*

$$\mathrm{Fil}_r\omega_X^q = \begin{cases} 0 & \text{if } r < 0 \\ \mathrm{Image}(\omega_X^r \otimes \omega_{X'}^{q-r} \to \omega_X^q) & \text{if } 0 \le r \le q \\ \omega_X^q & \text{if } q \le r. \end{cases}$$

Let V be a vector bundle on X' endowed with an integrable connection with log poles $\nabla : V \to \omega_{X'}^1 \otimes_{\mathcal{O}_X} V$. Then:

(1) *$\mathrm{Fil}_r\omega_X^\cdot \otimes_{\mathcal{O}_X} V$ is a subcomplex of $\omega_X^\cdot \otimes_{\mathcal{O}_X} V$.*

(2) *For any $r \in \boldsymbol{Z}$, we have an isomorphism of complexes*

$$\overset{r}{\bigwedge}(P^{\mathrm{gp}}/(P\backslash\mathfrak{p})^{\mathrm{gp}}) \otimes_{\boldsymbol{Z}} \omega_{X'}^\cdot \otimes_{\mathcal{O}_X} V[-r] \xrightarrow{\cong} (\mathrm{Fil}_r\omega_X^\cdot \otimes_{\mathcal{O}_X} V)/(\mathrm{Fil}_{r-1}\omega_X^\cdot \otimes_{\mathcal{O}_X} V)$$

whose degree q part is given by

$$(y_1 \wedge \cdots \wedge y_r) \wedge x \mapsto d\log(y_1) \wedge \cdots \wedge d\log(y_r) \wedge x$$

$(x \in \omega_{X'}^{q-r} \otimes_{\mathcal{O}_X} V, y_1, \ldots, y_r \in P^{\mathrm{gp}})$, where the differential of the left hand side is (identity on $\bigwedge^r \cdots$) $\otimes$ (∇ on $\omega_{X'}^\cdot \otimes V$).

(3) *Let $a \in P^{\mathrm{gp}}$ and assume a does not belong to $(P\backslash\mathfrak{p})^{\mathrm{gp}}$. Then the complex $(\omega_X^q)_{q \in \boldsymbol{Z}}$ with the differential*

$$d_a : \omega_X^q \to \omega_X^{q+1}; \quad x \mapsto d\log(a) \wedge x + dx$$

is acyclic.

Proof. The proof of (1) is straightforward.

(2) follows from the exact sequence

$$0 \to \omega_{X'}^1 \to \omega_X^1 \to \mathcal{O}_X \otimes_{\boldsymbol{Z}} P^{\mathrm{gp}}/(P\backslash\mathfrak{p})^{\mathrm{gp}} \to 0 \tag{4.4.1}$$

which is obtained as follows. When we regard P and $P\backslash\mathfrak{p}$ as constant sheaves on X, the inverse image $\mathscr{S}$ of $\mathcal{O}_X^*$ in P under $P \to \mathcal{O}_X$ is contained in $P\backslash\mathfrak{p}$ since $\mathfrak{p}$ is sent to $0 \in \mathcal{O}_X$. Hence

$$(P\backslash\mathfrak{p})^{\mathrm{gp}}/\mathscr{S}^{\mathrm{gp}} \xrightarrow{\cong} M_{X'}^{\mathrm{gp}}/\mathcal{O}_X^*, \quad P^{\mathrm{gp}}/\mathscr{S}^{\mathrm{gp}} \xrightarrow{\cong} M_X^{\mathrm{gp}}/\mathcal{O}_X^*,$$

and hence we have the exact sequence

$$0 \to M_{X'}^{\mathrm{gp}} \to M_X^{\mathrm{gp}} \to P^{\mathrm{gp}}/(P\backslash\mathfrak{p})^{\mathrm{gp}} \to 0. \tag{4.4.2}$$

On the other hand,

$$\omega^1_X = (\Omega^1_X \oplus (\mathcal{O}_X \otimes_{\boldsymbol{Z}} M_X^{\mathrm{gp}}))/N, \quad \omega^1_{X'} = (\Omega^1_X \oplus (\mathcal{O}_X \otimes_{\boldsymbol{Z}} M_{X'}^{\mathrm{gp}}))/N'$$

where N is as in (3.5) and N' is defined similarly. We show $N = N'$. Indeed, if x is a local section of M_X, locally $x = au, a \in P, u \in \mathcal{O}_X^*$. If $a \in P\backslash\mathfrak{p}$, $(-d\alpha(x), \alpha(x) \otimes x)$ belongs to N'. If $a \in \mathfrak{p}, \alpha(x) = 0$ and hence $(-d\alpha(x), \alpha(x) \otimes x)$ also belongs to N'. Hence $N = N'$, and this shows

4.4.3 $$0 \to \omega^1_{X'} \to \omega^1_X \to \mathcal{O}_X \otimes_{\boldsymbol{Z}} M_X^{\mathrm{gp}}/M_{X'}^{\mathrm{gp}} \to 0 \quad \text{(exact)}.$$

Now 4.4.1 is obtained from 4.4.2 and 4.4.3.

We prove (3). We have $d_a(\mathrm{Fil}_r\omega^q_X) \subset \mathrm{Fil}_{r+1}\omega^{q+1}_X$. It is sufficient to prove that for each $r \in \boldsymbol{Z}$, the complex $(\mathrm{Fil}_{r+q}\omega^q_X/\mathrm{Fil}_{r+q-1}\omega^q_X)_{q\in\boldsymbol{Z}}$ with the differential induced by d_a is acyclic. But by (2), this complex is isomorphic to the complex $((\bigwedge^{r+q}_{\boldsymbol{Q}} H) \otimes_{\boldsymbol{Q}} \omega^{-r}_{X'})_{q\in\boldsymbol{Z}}$ with the differential $x \otimes y \mapsto (a \wedge x) \otimes y$ where $H = (P^{\mathrm{gp}}/(P\backslash\mathfrak{p})^{\mathrm{gp}}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. □

LEMMA (4.5). *Let P be an fs monoid and let $X = \mathrm{Spec}(\boldsymbol{C}[P])_{\mathrm{an}}$ with the log structure associated to the canonical map $P \to \mathcal{O}_X$. Let $x \in X$ and define a prime ideal $\mathfrak{b}$ of P to be the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ under $P \to \mathcal{O}_{X,x}$. Let V be an object of $D(X)$ (0.4) whose connection does not have a pole. For any ideal $\mathfrak{a}$ of P, let $X(\mathfrak{a}) = \mathrm{Spec}(\boldsymbol{C}[P]/(\mathfrak{a}))_{\mathrm{an}}$, $V(\mathfrak{a}) = \mathcal{O}_{X(\mathfrak{a})} \otimes_{\mathcal{O}_X} V$, and $V(\mathfrak{a})^{\nabla=0} = \mathrm{Ker}(V(\mathfrak{a}) \to \omega^1_{X(\mathfrak{a})} \otimes_{\mathcal{O}_{X(\mathfrak{a})}} V(\mathfrak{a}))$. Denote the complex $\omega^{\cdot}_X \otimes_{\mathcal{O}_X} V$ by C. Then:*

(1) *The restriction of $V(\mathfrak{b})^{\nabla=0}$ to some open neighbourhood of x in $X(\mathfrak{b})$ is a local system, and $\mathcal{O}_{X(\mathfrak{b}),x} \otimes_{\boldsymbol{C}} V(\mathfrak{b})^{\nabla=0}_x \xrightarrow{\cong} V(\mathfrak{b})_x$.*

(2) *For any q, the map*

$$\bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_x \otimes_{\boldsymbol{Z}} V(\mathfrak{b})^{\nabla=0}_x \to \mathscr{H}^q(C/\mathfrak{b}C)_x,$$

which is induced by $d\log : M_X^{\mathrm{gp}} \to \omega^1_X$, is bijective.

(3) *For any ideal $\mathfrak{a}$ of P such that $\mathfrak{a} \subset \mathfrak{b}$, the stalk at x of the canonical map of complexes*

4.5.1 $$C/\mathfrak{a}C \to C/\mathfrak{b}C$$

is a quasi-isomorphism.

In the above, $\mathfrak{a}C$ denotes the subcomplex of C whose degree q part is defined to be the $\mathcal{O}_X$-subsheaf of $\omega^q_X \otimes_{\mathcal{O}_X} V$ generated by $a\omega^q_X \otimes_{\mathcal{O}_X} V$ with $a \in \mathfrak{a}$. The definition of $\mathfrak{b}C$ is similar.

Proof. Note that the complex $C/\mathfrak{a}C$ is isomorphic to the de Rham complex $\omega^{\cdot}_{X(\mathfrak{a})} \otimes_{\mathcal{O}_{X(\mathfrak{a})}} V(\mathfrak{a})$ of $V(\mathfrak{a})$ by Lemma (3.6) (2).

We prove (1) and (2). The underlying analytic space of $X(\mathfrak{b})$ is $\mathrm{Spec}(\boldsymbol{C}[P\backslash\mathfrak{b}])_{\mathrm{an}}$, and x belongs to the non-singular open analytic subspace $\mathrm{Spec}(\boldsymbol{C}[(P\backslash\mathfrak{b})^{\mathrm{gp}}])_{\mathrm{an}}$ of it. By the well known theory of integrable connections on vector bundles on non-singular analytic spaces ([D1]I, 2.17), the restriction of $V(\mathfrak{b})^{\nabla=0}$ to the open neighbourhood $\mathrm{Spec}(\boldsymbol{C}[(P\backslash\mathfrak{b})^{\mathrm{gp}}])_{\mathrm{an}}$ of x in $X(\mathfrak{b})$ is a local

system, and the stalk at x of $\mathcal{O}_{X(\mathfrak{b})} \otimes_{\boldsymbol{C}} V(\mathfrak{b})^{\nabla=0} \to V(\mathfrak{b})$ (resp. $V(\mathfrak{b})^{\nabla=0} \to \Omega^{\bullet}_{X(\mathfrak{b})} \otimes_{\mathcal{O}_{X(\mathfrak{b})}} V(\mathfrak{b})$) is an isomorphism (resp. a quasi-isomorphism). By Lemma (4.4) (which we apply by taking $X(\mathfrak{b}), P, \mathfrak{b}$, as $X, P, \mathfrak{p}$ and considering $\mathscr{H}^q$ of both sides of Lemma (4.4) (2)), $\mathscr{H}^q((\mathrm{Fil}_r\omega^{\bullet}_{X(\mathfrak{b})} \otimes V(\mathfrak{b}))/(\mathrm{Fil}_{r-1}\omega^{\bullet}_{X(\mathfrak{b})} \otimes V(\mathfrak{b})))_x$ is isomorphic to $\bigwedge^r P^{\mathrm{gp}}/(P\backslash\mathfrak{b})^{\mathrm{gp}} \otimes_{\boldsymbol{Z}} \mathscr{H}^{q-r}(\Omega^{\bullet}_{X(\mathfrak{b})} \otimes V(\mathfrak{b}))_x$, which is isomorphic to $\bigwedge^q P^{\mathrm{gp}}/(P\backslash\mathfrak{b})^{\mathrm{gp}} \otimes_{\boldsymbol{Z}} V(\mathfrak{b})_x^{\nabla=0}$ if $q = r$ and is zero if $q \neq r$. From this we have that the stalk at x of

$$\bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes_{\boldsymbol{Z}} V(\mathfrak{b})^{\nabla=0} \to \mathscr{H}^q(C/\mathfrak{b}C)$$

is an isomorphism.

We prove (3) in four steps.

STEP 1. We show that to prove Lemma (4.5) (3), we may assume $\mathfrak{a}$ is a prime ideal. Assume there exists $\mathfrak{a}$ for which 4.5.1 is not a quasi-isomorphism. Since $\boldsymbol{Z}[P]$ is a Noetherian ring, the set of such $\mathfrak{a}$ has a maximal element $\mathfrak{q}$. We show that $\mathfrak{q}$ is a prime ideal.

We first show that $\mathfrak{q}$ satisfies the conditions in Lemma (4.3). Indeed, assume Σ and Σ' are ideals of P such that $\mathfrak{q} = \Sigma \cap \Sigma'$. Then, we have an exact sequence

$$0 \to C/\mathfrak{q}C \to C/\Sigma C \oplus C/\Sigma' C \to C/\Sigma'' C \to 0$$

where $\Sigma'' = \Sigma \cup \Sigma'$. Since $\mathfrak{q}$ has the maximal property, if $\mathfrak{q} \neq \Sigma, \mathfrak{q} \neq \Sigma'$, then $C/\Sigma C \to C/\mathfrak{b}C, C/\Sigma' C \to C/\mathfrak{b}C, C/\Sigma'' C \to C/\mathfrak{b}C$ are quasi-isomorphisms. This shows that $C/\mathfrak{q}C \to C/\mathfrak{b}C$ is a quasi-isomorphism, a contradiction. Hence by Lemma (4.3), $\mathfrak{q}$ is a primary ideal. Let $\mathfrak{p} = \{a \in P; a^n \in \mathfrak{q} \text{ for some } n \geq 1\}$. By Lemma (4.2) (2), there exists $a \in P$ such that $\{x \in P; ax \in \mathfrak{q}\} = \mathfrak{p}$. If a does not belong to $\mathfrak{p}$, then $\mathfrak{q} = \{x \in P; ax \in \mathfrak{q}\} = \mathfrak{p}$, that is, $\mathfrak{q}$ is a prime ideal. Now assume $a \in \mathfrak{p}$. Consider the exact sequence

$$0 \to \mathfrak{q}'C/\mathfrak{q}C \to C/\mathfrak{q}C \to C/\mathfrak{q}'C \to 0,$$

where C is as above and $\mathfrak{q}' = \mathfrak{q} \cup Pa$. Since a does not belong to $\mathfrak{q}, \mathfrak{q}' \supsetneqq \mathfrak{q}$ and hence $C/\mathfrak{q}'C$ is quasi-isomorphic to $C/\mathfrak{b}C$. On the other hand, we show below that $\mathfrak{q}'C/\mathfrak{q}C$ is acyclic. This shows $C/\mathfrak{q}C \to C/\mathfrak{b}C$ is a quasi-isomorphism, a contradiction. Now the proof of the acyclicity of $\mathfrak{q}'C/\mathfrak{q}C$ is as follows. The complex $\mathfrak{q}'C/\mathfrak{q}C$ is isomorphic to the complex $(\omega^q_{X(\mathfrak{p})} \otimes_{\mathcal{O}_{X(\mathfrak{p})}} V(\mathfrak{p}))_{q \in \boldsymbol{Z}}$ with the differential $x \mapsto d\log(a) \wedge x + \nabla(x)$. (In fact the isomorphism is induced by $\omega^q_X/\mathfrak{p}\omega^q_X \otimes V \to \mathfrak{q}'C^q/\mathfrak{q}C^q; x \mapsto ax$.) Now the acyclicity follows from Lemma (4.4) (3). Thus we are reduced to the case $\mathfrak{a}$ is a prime ideal.

STEP 2. We show that to prove Lemma (4.5) (3) for the pair $(P, \mathfrak{a})$, it is enough to prove Lemma (4.5) (3) for the pairs $(P\backslash\mathfrak{p}, \phi)$ for prime ideals $\mathfrak{p} \subset \mathfrak{b}$ of P. Here ϕ denotes the empty ideal of $P\backslash\mathfrak{p}$.

By Step 1, we may assume $\mathfrak{a}$ is a prime ideal of P. Let $P' = P\backslash\mathfrak{a}, X' = \mathrm{Spec}(\boldsymbol{C}[P'])_{\mathrm{an}}$ with the log structure associated to $P' \to \mathcal{O}_{X'}$. Then the underlying analytic space of $X(\mathfrak{a})$ (resp. $X(\mathfrak{b})$) coincides with that of $X' = X'(\phi)$ (resp.

that of $X'(\mathfrak{b}')$ where $\mathfrak{b}' = P' \cap \mathfrak{b}$). Assume that we have shown

4.5.2 $$\omega^{\cdot}_{X'(\mathfrak{a})} \otimes_{\mathcal{O}_{X'(\mathfrak{a})}} V(\mathfrak{a}) \to \omega^{\cdot}_{X'(\mathfrak{b})} \otimes_{\mathcal{O}_{X'(\mathfrak{b})}} V(\mathfrak{b})$$

is a quasi-isomorphism. Then for any $r \in \boldsymbol{Z}$,

4.5.3$_r$ $$\mathrm{gr}_r(\omega^{\cdot}_{X(\mathfrak{a})} \otimes_{\mathcal{O}_{X(\mathfrak{a})}} V(\mathfrak{a})) \to \mathrm{gr}_r(\omega^{\cdot}_{X(\mathfrak{b})} \otimes_{\mathcal{O}_{X(\mathfrak{b})}} V(\mathfrak{b}))$$

is a quasi-isomorphism because

$$(4.5.3)_r = \bigwedge^r P^{\mathrm{gp}}/(P\backslash\mathfrak{a})^{\mathrm{gp}} \otimes_{\boldsymbol{Z}} (4.5.2)[-r]$$

by Lemma (4.4) (2). This will show that

$$\omega^{\cdot}_{X(\mathfrak{a})} \otimes_{\mathcal{O}_{X(\mathfrak{a})}} V(\mathfrak{a}) \to \omega^{\cdot}_{X(\mathfrak{b})} \otimes_{\mathcal{O}_{X(\mathfrak{b})}} V(\mathfrak{b})$$

is a quasi-isomorphism. Thus, to prove Lemma (4.5) (3), we may replace $P, \mathfrak{a}, \mathfrak{b}$ by $P', \phi, \mathfrak{b}'$.

STEP 3. We prove Lemma (4.5) in the case $P = \boldsymbol{N}^r$ for some $r \geq 0$. In this case, for any prime ideal $\mathfrak{p}$ of P, $P\backslash\mathfrak{p}$ is isomorphic to $\boldsymbol{N}^s$ for some $s \leq r$. Thus we may assume $P = \boldsymbol{N}^r$ and $\mathfrak{a} = \phi$ by Step 2. We have $X = \boldsymbol{C}^r$ as an analytic space and $\omega^{\cdot}_X$ is the well known de Rham complex with log poles [D2]. In this case, since V has no pole, V is locally isomorphic to $\mathcal{O}_X^n$ with the connection d, and hence we are reduced to [D2] 3.1.8.2.

STEP 4. Now we complete the proof of Lemma (4.5) (3). By Step 2, we may assume $\mathfrak{a} = \phi$. For a non-empty ideal I of P, we have the toric variety $Y = (B_I(\mathrm{Spec}(\boldsymbol{C}[P])))_{\mathrm{an}}$ as in [KKMS]I, Theorem 10 which has a natural log structure ([K](3.7)(1)). Let f denote the morphism $Y \to X$. We have

$$\omega^q_Y \cong \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\omega^q_X)$$

and

$$\mathcal{O}_X \xrightarrow{\cong} \mathrm{R}f_*\mathcal{O}_Y$$

([KKMS]I, Corollary 1 c) to Theorem 12 and GAGA ([Gr3] XII Théorème 4.2)). These isomorphisms give an isomorphism in the derived category

$$C \xrightarrow{\cong} \mathrm{R}f_*(C_Y) \quad \text{where } C = \omega^{\cdot}_X \otimes_{\mathcal{O}_X} V, \quad C_Y = \omega^{\cdot}_Y \otimes_{\mathcal{O}_Y} f^*V.$$

By [KKMS]I, Theorem 11, there exists I such that for any $y \in Y$, $(M_Y/\mathcal{O}_Y^*)_y$ is isomorphic to $\boldsymbol{N}^{r(y)}$ for some $r(y) \geq 0$. The left vertical arrow in the following commutative diagram is an isomorphism.

$$\begin{array}{ccc} C & \longrightarrow & C/\mathfrak{b}C \\ \cong\downarrow & & \downarrow \\ \mathrm{R}f_*(C_Y) & \longrightarrow & \mathrm{R}f_*(C_Y/\mathfrak{b}C_Y) \end{array}$$

Since Lemma (4.5) (3) is already proved in the case $P = \boldsymbol{N}^r, r \geqq 0$, the lower horizontal arrow is an isomorphism at x, too. Hence $\mathscr{H}^q(C)_x \to \mathscr{H}^q(C/\mathfrak{b}C)_x$ is

injective. Now we consider $\mathscr{H}^0$. By Lemma (4.5) (1), $\mathscr{H}^0(C/\mathfrak{b}C)_x \xrightarrow{\cong} V_x/\mathfrak{m}_x V_x$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$. Take a point $y \in Y$ lying over x. Then we have maps $\mathscr{H}^0(C/\mathfrak{b}C)_x \to \mathscr{H}^0(\mathrm{R}f_*(C_Y/\mathfrak{b}C_Y))_x \to \mathscr{H}^0(C_Y/\mathfrak{b}C_Y)_y \to (f^*V)_y/\mathfrak{m}_y(f^*V)_y$. Since $V_x/\mathfrak{m}_x V_x \xrightarrow{\cong} (f^*V)_y/\mathfrak{m}_y(f^*V_y)$, we have that $\mathscr{H}^0(C/\mathfrak{b}C)_x \to \mathscr{H}^0(\mathrm{R}f_*(C_Y/\mathfrak{b}C_Y))_x$ is injective. By the above commutative diagram, this shows $\mathscr{H}^0(C)_x \xrightarrow{\cong} \mathscr{H}^0(C/\mathfrak{b}C)_x$. From this, we obtain $\mathscr{O}_{X,x} \otimes_{\boldsymbol{C}} \mathscr{H}^0(C)_x \xrightarrow{\cong} V_x$. This shows that $\mathscr{H}^0(C)$ is a local system on X and $\mathscr{O}_X \otimes_{\boldsymbol{C}} \mathscr{H}^0(C) \xrightarrow{\cong} V$. Hence $C = \omega_X^{\cdot} \otimes_{\boldsymbol{C}} \mathscr{H}^0(C)$. Now the composite

$$\bigwedge^q (M_X^{\mathrm{gp}}/\mathscr{O}_X^*)_x \otimes_{\boldsymbol{Z}} \mathscr{H}^0(C)_x \to \mathscr{H}^q(C)_x \to \mathscr{H}^q(C/\mathfrak{b}C)_x$$

is an isomorphism by Lemma (4.5) (2). Hence $\mathscr{H}^q(C)_x \to \mathscr{H}^q(C/\mathfrak{b}C)_x$ is surjective, and it is bijective for we already know it is injective. □

PROPOSITION (4.6). *Let X be an fs log analytic space satisfying the assumption of Theorem* (0.5). *Then*:

(1) (*Generalization of* [D2] 3.1.8.2) *For all $q \in \boldsymbol{Z}$, we have isomorphisms*

$$\boldsymbol{C} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathscr{O}_X^*) \xrightarrow{\cong} \mathscr{H}^q(\omega_X^{\cdot})$$

induced by $d\log : M_X^{\mathrm{gp}} \to \omega_X^1$.

(2) *Let V be a vector bundle on X endowed with an integrable connection with log poles $\nabla : V \to \omega_X^1 \otimes_{\mathscr{O}_X} V$ which has no pole. Then the kernel $V^{\nabla=0}$ of ∇ is a local system on X and $\mathscr{O}_X \otimes_{\boldsymbol{C}} (V^{\nabla=0}) \xrightarrow{\cong} V$.*

Proof. The question being local, we may assume that

$$X = \mathrm{Spec}(\boldsymbol{C}[P]/\mathfrak{a}\boldsymbol{C}[P])_{\mathrm{an}}$$

where P is an fs monoid and $\mathfrak{a}$ is an ideal of P. Let $x \in X$, and let $\mathfrak{b} \subset P$ be the inverse image of the maximal ideal of $\mathscr{O}_{X,x}$.

We prove (1). By Lemma (4.5) (3),

$$\mathscr{H}^q(\omega_X^{\cdot})_x \cong \mathscr{H}^q(\omega_X^{\cdot}/\mathfrak{b}\omega_X^{\cdot})_x.$$

By Lemma (4.5) (2),

$$\mathscr{H}^q(\omega_X^{\cdot}/\mathfrak{b}\omega_X^{\cdot})_x \cong \boldsymbol{C} \otimes_{\boldsymbol{Z}} \bigwedge^q M_{X,x}^{\mathrm{gp}}/\mathscr{O}_{X,x}^*.$$

We prove (2). By Lemma (4.5) (3),

$$V_x^{\nabla=0} = \mathscr{H}^0(\omega_X^{\cdot} \otimes_{\mathscr{O}_X} V)_x \cong \mathscr{H}^0(\omega_X^{\cdot} \otimes_{\mathscr{O}_X} V/\mathfrak{b}V)_x = (V/\mathfrak{b}V)_x^{\nabla=0}.$$

By Lemma (4.5) (1),

$$\mathscr{O}_{X,x}/\mathfrak{b}\mathscr{O}_{X,x} \otimes_{\boldsymbol{C}} (V/\mathfrak{b}V)_x^{\nabla=0} \xrightarrow{\cong} (V/\mathfrak{b}V)_x.$$

These isomorphisms show that $\mathrm{mod}\,\mathfrak{b}\mathscr{O}_{X,x}$ of the $\mathscr{O}_{X,x}$-homomorphism

$$4.6.1 \qquad \mathscr{O}_{X,x} \otimes_{\boldsymbol{C}} V_x^{\nabla=0} \to V_x$$

is an isomorphism. Since $\mathcal{O}_{X,x} \otimes_{\boldsymbol{C}} V_x^{\nabla=0}$ and V_x are free $\mathcal{O}_{X,x}$-modules, this implies that 4.6.1 is bijective. Hence $\mathcal{O}_X \otimes_{\boldsymbol{C}} V^{\nabla=0} \xrightarrow{\cong} V$. This shows that $V^{\nabla=0}$ is a local system. □

(4.7). Now we prove the "log Poincaré lemma" Theorem (3.8). Fix $x \in X$ and $y \in X^{\log}$ such that x is the image of y in X. Let $(t_i)_{1 \leq i \leq n}$ be a finite family of elements of $\mathscr{L}_y$ whose image under exp is a $\boldsymbol{Z}$-basis of $(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_x$. Let $R = \boldsymbol{C}[T_1, \ldots, T_n]$ with T_i independent indeterminates, and let $R \to \mathcal{O}_{X,y}^{\log}$ be the $\boldsymbol{C}$-homomorphism which sends T_i to t_i. Since $\boldsymbol{C} \to \Omega_{R/\boldsymbol{C}}^{\cdot}$ is a quasi-isomorphism as is well known, it is enough to show that $\Omega_{R/\boldsymbol{C}}^{\cdot} \to \omega_{X,y}^{\cdot,\log}$ is a quasi-isomorphism. For $r \in \boldsymbol{Z}$, let $\mathrm{Fil}_r(\Omega_{R/\boldsymbol{C}}^{\cdot})$ be the subcomplex of $\Omega_{R/\boldsymbol{C}}^{\cdot}$ whose degree q part is the $\boldsymbol{C}$-submodule of $\Omega_{R/\boldsymbol{C}}^q$ generated by elements of the form $f\eta$ with f a polynomial of degree $\leq r$ and $\eta \in \Omega_{R/\boldsymbol{C}}^q$. On the other hand, let $\mathrm{Fil}_r(\omega_X^{\cdot,\log})$ be the subcomplex of $\omega_X^{\cdot,\log}$ whose degree q part is the image of $\mathrm{fil}_r(\mathcal{O}_X^{\log}) \otimes \omega_{X/\boldsymbol{C}}^q$ (Lemma (3.4)). Then by Lemma (3.4),

$$\mathrm{Fil}_r(\omega_X^{\cdot,\log})/\mathrm{Fil}_{r-1}(\omega_X^{\cdot,\log}) \cong \tau^{-1}(\mathrm{Sym}_{\boldsymbol{Z}}^r(M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes_{\boldsymbol{Z}} \omega_X^{\cdot}),$$

and by Proposition (4.6) (1), $\mathscr{H}^q$ of the right hand side is isomorphic to $\tau^{-1}(\mathrm{Sym}_{\boldsymbol{Z}}^r(M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*)) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$. On the other hand, $\mathrm{Fil}_r(\Omega_{R/\boldsymbol{C}}^{\cdot})/\mathrm{Fil}_{r-1}(\Omega_{R/\boldsymbol{C}}^{\cdot})$ is the complex $q \mapsto \mathrm{Sym}_{\boldsymbol{Z}}^r(\bigoplus_{i=1}^n \boldsymbol{Z}T_i) \otimes_{\boldsymbol{Z}} (\bigwedge^q \bigoplus_{i=1}^n \boldsymbol{Z}T_i) \otimes_{\boldsymbol{Z}} \boldsymbol{C} \cong \mathrm{Sym}_{\boldsymbol{Z}}^r(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_x \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_x \otimes_{\boldsymbol{Z}} \boldsymbol{C}$ with zero differential. Hence we have that

$$\mathrm{Fil}_r(\Omega_{R/\boldsymbol{C}}^{\cdot})/\mathrm{Fil}_{r-1}(\Omega_{R/\boldsymbol{C}}^{\cdot}) \to \mathrm{Fil}_r(\omega_{X,y}^{\cdot,\log})/\mathrm{Fil}_{r-1}(\omega_{X,y}^{\cdot,\log})$$

is a quasi-isomorphism for any r. This shows that $\Omega_{R/\boldsymbol{C}}^{\cdot} \to \omega_{X,y}^{\cdot,\log}$ is a quasi-isomorphism.

(4.8). We prove Theorem (0.5).

For an object V of $D_{\mathrm{nilp}}(X)$, let $V^{\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(V)$ and let $\mathrm{DR}(V)$ (resp. $\mathrm{DR}(V^{\log})$) be the de Rham complex $\omega_X^{\cdot} \otimes_{\mathcal{O}_X} V$ (resp. $\omega_X^{\cdot,\log} \otimes_{\mathcal{O}_X^{\log}} V^{\log}$). Let

$$\Phi(V) = \mathscr{H}^0(\mathrm{DR}(V^{\log})).$$

We show:

4.8.1 $\Phi(V)$ is a local system of finite dimensional $\boldsymbol{C}$-vector spaces on $X^{\log}$.

4.8.2 $\Phi(V) \to \mathrm{DR}(V^{\log})$ is a quasi-isomorphism.

4.8.3 $\mathcal{O}_X^{\log} \otimes_{\boldsymbol{C}} \Phi(V) \xrightarrow{\cong} V^{\log}$.

We first show $\mathscr{H}^q(\mathrm{DR}(V^{\log})) = 0$ for any $q \neq 0$. Let $D_0(X)$ be the full subcategory of $D_{\mathrm{nilp}}(X)$ consisting of objects whose connections do not have poles. Then the proof of $\mathscr{H}^q(\mathrm{DR}(V^{\log})) = 0$ for $q \neq 0$ is reduced to the case where V belongs to $D_0(V)$. In this case, by Proposition (4.6) (2), $V = \mathcal{O}_X \otimes_{\boldsymbol{C}} F$ for a local system F of finite dimensional $\boldsymbol{C}$-vector spaces on X, and hence we are reduced to the case $V = \mathcal{O}_X, \nabla = d$. In this case we are reduced to the log Poincaré lemma Theorem (3.8). Thus $\mathscr{H}^q(\mathrm{DR}(V^{\log})) = 0$ for $q \neq 0$ and 4.8.2 has proved.

For an exact sequence $0 \to V' \to V \to V'' \to 0$ of objects of $D_{\mathrm{nilp}}(X), 0 \to \Phi(V') \to \Phi(V) \to \Phi(V'') \to 0$ is exact since $\mathscr{H}^1(\mathrm{DR}((V')^{\log})) = 0$. By this, 4.8.1 and 4.8.3 are reduced to the case V belongs to $D_0(X)$, and hence by Proposition (4.6) (2), to the case $V = \mathscr{O}_X$ with $\nabla = d$. In this case $\Phi(V) = \boldsymbol{C}$ by the log Poincaré lemma, so 4.8.1 and 4.8.3 follow.

By 4.8.3, we have

4.8.4 Φ preserves tensor products and duals.

Now we prove

4.8.5 $\mathrm{DR}(V) \to \mathrm{R}\tau_*\mathrm{DR}(V^{\log}) \cong \mathrm{R}\tau_*\Phi(V)$ is an isomorphism for any object V of $D_{\mathrm{nilp}}(X)$.

This is reduced to the case where V belongs to $D_0(X)$, and hence to the case $V = \mathscr{O}_X$ and $\nabla = d$. In this case, $\mathrm{DR}(V) = \omega_X^{\cdot}$ and $\Phi(V) = \boldsymbol{C}$. For any $q \in \boldsymbol{Z}$, the composite $\boldsymbol{C} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathscr{O}_X^*) \xrightarrow{\cong} \mathscr{H}^q(\omega_X^{\cdot}) \to \mathrm{R}^q\tau_*\boldsymbol{C} \cong \boldsymbol{C} \otimes_{\boldsymbol{Z}} \bigwedge^q (M_X^{\mathrm{gp}}/\mathscr{O}_X^*)$ is the identity.

4.8.6 Φ is fully faithful.

Proof. For objects V_1 and V_2 of $D_{\mathrm{nilp}}(X)$,

$$\mathrm{Hom}(V_1, V_2) \cong \boldsymbol{H}^0(X, \mathrm{DR}(V_1^* \otimes V_2)) \quad (V_1^* \text{ is the dual of } V_1)$$

(because $\mathrm{Hom}(V_1, V_2)$ is identified with the $\mathscr{O}_X$-linear maps $\mathscr{O}_X \to V_1^* \otimes V_2$ which are compatible with d),

$$\mathrm{Hom}(\Phi(V_1), \Phi(V_2)) \cong H^0(X^{\log}, \Phi(V_1^* \otimes V_2)) \text{ by 4.8.4}$$

$$\cong \boldsymbol{H}^0(X, \mathrm{R}\tau_*\Phi(V_1^* \otimes V_2)).$$

Hence 4.8.6 is reduced to 4.8.5. □

4.8.7 Φ is essentially surjective.

Proof. Let L be an object of $L_{\mathrm{unip}}(X^{\log})$. To show that L comes from $D_{\mathrm{nilp}}(X)$, since we may work locally on X, we may assume that there exist $\boldsymbol{C}$-subsheaves L_i $(0 \le i \le n)$ of L such that $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ and such that for $1 \le i \le n, L_i/L_{i-1}$ is the inverse image of a local system on X. We prove that L comes from $D_{\mathrm{nilp}}(X)$ by induction on n. The case $n = 1$ is clear and we may assume $n \ge 2$. Then, by induction, $L_1 \cong \Phi(V_2), L/L_1 \cong \Phi(V_1)$ for some objects V_1, V_2 of $D_{\mathrm{nilp}}(X)$. Since L corresponds to an element of $\mathrm{Ext}^1(L/L_1, L_1)$, it is sufficient to show that $\mathrm{Ext}^1(V_1, V_2) \to \mathrm{Ext}^1(\Phi(V_1), \Phi(V_2))$ is surjective. But

$$\mathrm{Ext}^1(V_1, V_2) \cong \boldsymbol{H}^1(X, \mathrm{DR}(V_1^* \otimes V_2)),$$

$$\mathrm{Ext}^1(\Phi(V_1), \Phi(V_2)) \cong H^1(X^{\log}, \Phi(V_1^* \otimes V_2)) \cong \boldsymbol{H}^1(X, \mathrm{R}\tau_*\Phi(V_1^* \otimes V_2))$$

and hence we are reduced to 4.8.5. □

(4.9). We prove Theorem (0.2) (2). Let $g : X_{\mathrm{an}} \to X$ be the canonical morphism. By Theorem (0.5) (2) for X_{an}, it remains to prove

$$\omega_X^{\cdot} \xrightarrow{\cong} \mathrm{R}g_*\omega_{X_{\mathrm{an}}}^{\cdot}. \tag{4.9.1}$$

We may assume that there exists a morphism $f : X \to \mathrm{Spec}(\boldsymbol{C}[P]/(\Sigma))$ for an fs monoid P and an ideal Σ of P, such that the log structure of X is associated to $P \to \mathcal{O}_X$ and such that the underlying morphism of f is smooth.

We reduce the proof of 4.9.1 to the case $\Sigma = \phi$, by the methods in Step 1 and Step 2 in the proof of Lemma (4.5) (3). In fact if 4.9.1 does not hold, there exists a maximal element in the set of ideals $\mathfrak{a} \supset \Sigma$ for which 4.9.1 for $X(\mathfrak{a}) = X \times_{\mathrm{Spec}(\boldsymbol{C}[P]/(\Sigma))} \mathrm{Spec}(\boldsymbol{C}[P]/(\mathfrak{a}))$ does not hold. By the same argument as in Step 1 in the proof of Lemma (4.5) (3), we see that such maximal element is a prime ideal. By replacing the log structure of $X(\mathfrak{a})$ by the one associated to $P \setminus \mathfrak{a} \to \mathcal{O}_{X(\mathfrak{a})}$, as in Step 2 in the proof of Lemma (4.5) (3), we are reduced to the case $\Sigma = \phi$. Then, as in Step 4 there, we take $Y = (B_I(\mathrm{Spec}\,\boldsymbol{C}[P]))_{\mathrm{an}} \to X$ for some non-empty ideal I of P such that for any $y \in Y$, $(M_Y/\mathcal{O}_Y^*)_y \cong \boldsymbol{N}^{r(y)}$ for some $r(y) \geq 0$. Then $\omega_X^{\cdot} \xrightarrow{\cong} \mathrm{R}h_*\omega_Y^{\cdot}$ and $\omega_{X_{\mathrm{an}}}^{\cdot} \xrightarrow{\cong} \mathrm{R}(h_{\mathrm{an}})_*\omega_{Y_{\mathrm{an}}}^{\cdot}$. Hence we may assume that $P = \boldsymbol{N}^r$ for some $r \geq 0$ and $\Sigma = \phi$. In this case, $\omega_X^{\cdot} \xrightarrow{\cong} \mathrm{R}g_*\omega_{X_{\mathrm{an}}}^{\cdot}$ is well known ([Gr2]).

Remark (4.10). By using the space $X^{\log}$ in this paper, we obtain a new construction of integral structures of some mixed Hodge structures considered by Steenbrink [S].

First, as in [S] §4, let D be a complete complex algebraic variety with a log structure satisfying the following condition: Locally on D, there exists a complex smooth variety X and a reduced divisor D' on X with normal crossings such that D is isomorphic to D' endowed with the pull back of the log structure of X associated to D' ((1.2.3)). Then we have the integral structure

$$H^m(D_{\mathrm{an}}^{\log}, \boldsymbol{Z})$$

in $H^m(D_{\mathrm{an}}^{\log}, \boldsymbol{Z}) \otimes \boldsymbol{C} \cong H^m(D, \omega_D^{\cdot})$ (Theorem (0.2) (2)).

Next, as in [S] §5, let D be as above and assume we are given a global section t of M_D satisfying the following condition: For any smooth point x of D, the image of t in $M_{D,x}/\mathcal{O}_{D,x}^* \cong \boldsymbol{N}$ is 1. Let S be $\mathrm{Spec}(\boldsymbol{C})$ endowed with the log structure associated to

$$\boldsymbol{N} \to \boldsymbol{C}; \quad n \mapsto 0^n$$

(with the convention $0^0 = 1$). Then we have a morphism

$$f : D \to S$$

of log schemes which sends $1 \in \boldsymbol{N}$ to t. Consider the subsheaf

$$\boldsymbol{Z}\left[\frac{1}{2\pi i}\log(t)\right] \subset \mathcal{O}_{D_{\mathrm{an}}}^{\log}$$

where $\log(t)$ is a local section of $\mathscr{L}$ on $D_{\mathrm{an}}^{\log}$ whose image under $\exp : \mathscr{L} \to$

$\tau^{-1}(M_{D_{\mathrm{an}}}^{\mathrm{gp}})$ is t. The sheaf $\boldsymbol{Z}[(1/2\pi i)\log(t)]$ is independent on the choice of $\log(t)$. We have the integral structure

$$H^m\left(D_{\mathrm{an}}^{\log}, \boldsymbol{Z}\left[\frac{1}{2\pi i}\log(t)\right]\right)$$

in

$$H^m\left(D_{\mathrm{an}}^{\log}, \boldsymbol{Z}\left[\frac{1}{2\pi i}\log(t)\right]\right) \otimes \boldsymbol{C} = H^m(D, \omega_{D/S}^{\cdot}).$$

Here the last equality is obtained as follows. For $r \geq 0$, let $L_r = \bigoplus_{i=0}^{r} \boldsymbol{C}\log(t)^i \subset \boldsymbol{C}[\log(t)]$. Then L_r is an object of $L_{\mathrm{unip}}(X^{\log})$ and the object V_r of $D_{\mathrm{nilp}}(X)$ corresponding to L_r is given by $V_r = \bigoplus_{i=0}^{r} \mathcal{O}_{D_{\mathrm{an}}} e_i$ with $\nabla e_i = d\log(t) \otimes e_{i-1}$ $(1 \leqq i \leqq r), \nabla e_0 = 0$. (In fact, let $e_i = (1/i!)(\log(t) \otimes 1 - 1 \otimes \log(t))^i \in \mathcal{O}_{D_{\mathrm{an}}}^{\log} \otimes L_r$.) It can be shown that $\varinjlim_r V_r \otimes \omega_{D_{\mathrm{an}}}^{\cdot}$ is quasi-isomorphic to $(\omega_{D/S}^{\cdot})_{\mathrm{an}}$ by $e_i \mapsto 0$ $(1 \leqq i \leqq r), e_0 \mapsto 1$.

From this we have

$$\begin{aligned} H^m(D_{\mathrm{an}}^{\log}, \boldsymbol{C}[\log(t)]) &= H^m(D_{\mathrm{an}}^{\log}, \varinjlim_r L_r) \\ &= H^m(D_{\mathrm{an}}, (\omega_{D/S}^{\cdot})_{\mathrm{an}}) \cong H^m(D, \omega_{D/S}^{\cdot}) \end{aligned}$$

where the last isomorphism is by the compactness of D (GAGA).

Remark (4.11). T. Fujisawa [Fuj] studies integral structures of $\omega_{X/Y}^{\cdot}$ for log smooth morphisms $X \to Y$ of fine log analytic spaces. (He does not use the topological spaces $X^{\log}, Y^{\log}$ of this paper.)

Remark (4.12). Y. Kawamata and Y. Namikawa [KN] also define an integral structure of $\omega_{D/S}^{\cdot}$ for such D/S as in Remark (4.10). They use a fiber of $f^{\log}$ regarded as a semi-analytic space.

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