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ON BOUNDARY BEHAVIOUR OF SYMPLECTOMORPHISMS

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Let $\Omega \subset \boldsymbol{C}^n$ be a strictly pseudoconvex domain, γ an admissible weight, and $K_\gamma(z,\zeta)$ the reproducing (or γ-Bergman) kernel for $L^2H(\Omega,\gamma)$, the space of square integrable functions, with respect to the measure $\gamma\,d\mu$, which are holomorphic in Ω ($d\mu$ is the Lebesgue measure in $\boldsymbol{R}^{2n}$), cf. e.g. Z. Pasternak-Winiarski [17]. Consider the complex tensor field:

$$H_\gamma = \sum_{1\le i,j\le n}\left(\frac{\partial^2}{\partial z_i\partial\bar z_j}\log K_\gamma(z,z)\right) dz_i\otimes d\bar z_j$$

and the corresponding real tangent $(0,2)$-tensor field g_γ given by:

$$g_\gamma = \operatorname{Re}\{H_\gamma|_{\chi(\Omega)\times\chi(\Omega)}\},$$

where $\chi(\Omega)$ is the $C^\infty(\Omega)$-module of all real tangent vector fields on Ω. Under suitable conditions (cf. section 2) g_γ is a Kählerian metric on Ω, hence $\omega_\gamma = -i\partial\bar\partial\log K_\gamma(z,z)$ is a symplectic structure (the Kähler 2-form of g_γ). One of the problems we take up in the present paper may be stated as follows. Let $F : \Omega\to\Omega$ be a symplectomorphism of (Ω,ω_γ) in itself, smooth up to the boundary. Does $F : \partial\Omega\to\partial\Omega$ preserve the contact structure of the boundary?

Our interest may be motivated as follows. If $F:\Omega\to\Omega$ is a biholomorphism then, by a celebrated result of C. Fefferman (cf. Theorem 1 in [4], p. 2) F is smooth up to the boundary, hence $F:\partial\Omega\to\partial\Omega$ is a CR diffeomorphism, and in particular a contact transformation. Also biholomorphisms are known to be isometries of the Bergman metric g_1 (cf. e.g. [7], p. 370) hence symplectomorphisms of (Ω,ω_1). On the other hand, one may weaken the assumption on F by requesting only that F be a C^∞ diffeomorphism and $F^*\omega_1=\omega_1$. Then, by a result of A. Korányi and H. M. Reimann [11], if F is smooth up to the boundary then $F:\partial\Omega\to\partial\Omega$ is a contact transformation.

The main ingredient in the proof of A. Korányi and H. M. Reimann's result is the fact that, when $\gamma\equiv 1$, a certain negative power of the Bergman kernel ($\rho(z) = K_1(z,z)^{-1/(n+1)}$) is a defining function of Ω (allowing one to relate the symplectic structure of Ω to the contact structure of its boundary). In turn, this is a consequence of C. Fefferman's asymptotic expansion of $K_1(z,\zeta)$ (cf. Theorem 2 in

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[4], p. 9). Therefore, should one extend A. Korányi and H. M. Reimann's ideas to weighted Bergman kernels and related structures, the first obstacle is whether a similar asymptotic expansion is known for $K_\gamma(z,\zeta)$. Indeed, this is available when $\Omega = \{\varphi < 0\}$ is a smoothly bounded strictly pseudoconvex domain and $\gamma = |\varphi|^m$, $m \in \{0,1,2,\dots\}$, by a result of M. M. Peloso [18] (cf. Theorem 1). Cf. also [19] for a study of the boundary behaviour of $K_\gamma(z,\zeta)$ when $\gamma = |\varphi|^\alpha$, $\alpha > -1$ (not necessarily an integer). However, each point of the curve $\alpha \mapsto |\varphi|^\alpha$ (in the Banach manifold $W(\Omega)$ of all weights on Ω) is isolated (cf. Theorem 2) hence our present knowledge of the asymptoyic properties of $K_\gamma(z,\zeta)$, as γ runs over $W(\Omega)$, is rather limited.

We apply Theorem 1 to study the boundary behaviour of a symplectomorphism of $(\Omega, \omega_{|\varphi|^m})$, $m \in \{1,2,\dots\}$ (cf. Theorem 3).

Using the analytic behaviour of $K_\gamma(z,\zeta)$ with respect to γ (cf. [16], p. 131) we prove an analogue of Fefferman's asymptotic formula for more general weights of the form: an essentially bounded function times a nonnegative integer power of the defining function (cf. Theorem 4).

In section 4 we show that the components of any symplectomorphism of a γ-Kobayashi domain Ω satisfy a Beltrami system (in the sense of [20]). If Ω is the Siegel domain, the tangential equations induced (on $\partial\Omega$) by this system turn out to be (cf. Proposition 2) the equations introduced in [10] in connection with the study of quasiconformal maps of strictly pseudoconvex CR manifolds (cf. also [9], [12]).

1. The Forelli-Rudin-Ligocka-Peloso asymptotic expansion formula

Let $\Omega \subset \mathbf{C}^n$ be an open set and $W(\Omega)$ the set of all *weights* on Ω (i.e. $\gamma \in W(\Omega)$ is a Lebesgue measurable function $\gamma : \Omega \to (0,\infty)$). For each $\gamma \in W(\Omega)$ let $L^2(\Omega,\gamma)$ be the Hilbert space of all functions $f : \Omega \to \mathbf{C}$ for which

$$\|f\|_\gamma = \left(\int_\Omega |f|^2 \gamma \, d\mu\right)^{1/2} < \infty.$$

Let $L^2H(\Omega,\gamma)$ be the set of all functions in $L^2(\Omega,\gamma)$ which are holomorphic in Ω. A weight $\gamma \in W(\Omega)$ is *admissible* (cf. [17]) if 1) $L^2H(\Omega,\gamma)$ is a closed subspace of $L^2(\Omega,\gamma)$, and 2) for any $z \in \Omega$ the evaluation functional $\delta_z : L^2H(\Omega,\gamma) \to \mathbf{C}$, $\delta_z(f) = f(z)$, is continuous. The set of all admissible weights on Ω is denoted by $AW(\Omega)$. If $\gamma \in AW(\Omega)$ then, by the Riesz representation theorem, there is a unique function $K_\gamma(z,\cdot)$ (called the *weighted Bergman kernel* of Ω, of weight γ, or the *γ-Bergman kernel* of Ω) so that $\overline{K_\gamma(z,\cdot)} \in L^2H(\Omega,\gamma)$ and

$$f(z) = \int_\Omega f(\zeta) K_\gamma(z,\zeta)\gamma(\zeta)\, d\mu(\zeta),$$

for any $f \in L^2H(\Omega,\gamma)$, $z \in \Omega$. For $\gamma = 1$ this is the ordinary Bergman kernel of Ω (cf. e.g. [2]).

Let Ω be a smoothly bounded strictly pseudoconvex domain $\Omega = \{z \in \boldsymbol{C}^n : \varphi(z) < 0\}$ where φ is such that the Levi form L_φ satisfies

$$L_\varphi(w)\xi \geq C_1|\xi|^2, \quad \xi \in \boldsymbol{C}^n,$$

for $\varphi(w) < \delta_0$, $\delta_0 > 0$, and C_1 depending only on Ω. Set

$$\Psi(\zeta, z) = (F(\zeta, z) - \varphi(z))\chi(|\zeta - z|) + (1 - \chi(|\zeta - z|))|\zeta - z|^2 \tag{1}$$

where

$$F(\zeta, z) = -\sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(z)(\zeta_j - z_j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j z_k}(z)(\zeta_j - z_j)(\zeta_k - z_k)$$

and χ is a C^∞ cut-off function of the real variable t, with $\chi(t) = 1$ for $|t| < \varepsilon_0/2$ and $\chi(t) = 0$ for $|t| \geq 3\varepsilon_0/4$. We may state the following

THEOREM 1 (*Forelli-Rudin-Ligocka-Peloso*[1]). *For any nonnegative integer* $m \in \{0, 1, 2, \ldots\}$, $|\varphi|^m \in AW(\Omega)$. *Let* $K_m(\zeta, z)$ *be the* $|\varphi|^m$*-Bergman kernel for* $L^2H(\Omega, |\varphi|^m)$. *Then*

$$K_m(\zeta, z) = c_\Omega |\nabla\varphi(z)|^2 \cdot \det L_\varphi(z) \cdot \Psi(\zeta, z)^{-(n+1+m)} + E(\zeta, z) \tag{2}$$

where $E \in C^\infty(\bar{\Omega} \times \bar{\Omega} - \Delta)$, Δ *is the diagonal of* $\partial\Omega \times \partial\Omega$, *and* E *satisfies the estimate*

$$|E(\zeta, z)| \leq c'_\Omega |\Psi(\zeta, z)|^{-(n+1+m)+1/2} \cdot |\log|\Psi(\zeta, z)||.$$

This extends C. Fefferman's asymptotic expansion formula for the Bergman kernel of a strictly pseudoconvex domain (cf. [4] for $m = 0$) to the case of $|\varphi|^m$-Bergman kernels, $m \in \{1, 2, \ldots\}$ (cf. Lemma 2.2 in [18], p. 229). Part of the proof (relating $K_m(\zeta, z)$ to the ordinary Bergman kernel of the domain $\{(z, \xi) \in \boldsymbol{C}^n \times \boldsymbol{C}^m : \varphi(z) + |\xi|^2 < 0\}$) actually works for an arbitrary (admissible) weight. Indeed, one has the following

LEMMA 1. *Let* $m \in \{1, 2, \ldots\}$ *and* $\gamma \in AW(\Omega)$. *Let* $K_{\Omega_m}((z, \xi), (w, \eta))$ *be the Bergman kernel of the domain* $\Omega_m = \{(z, \xi) \in \Omega \times \boldsymbol{C}^m : |\xi|^{2m} < \gamma(z)\}$. *Then*

$$K_\gamma(z, w) = \frac{\omega_{2m-1}}{2m} K_{\Omega_m}((z, 0), (w, 0)). \tag{3}$$

Proof. For simplicity set $K(z, w) = K_{\Omega_m}((z, 0), (w, 0))$. Also, for fixed $z, w \in \Omega$, we set $u(\eta) = K_{\Omega_m}((z, 0), (w, \eta))$. As K_{Ω_m} is anti-holomorphic in η, u is

[1] We learned Theorem 1 from [18]. However, M. M. Peloso claims Theorem 1 is implicit in [14], while E. Ligocka employs an older idea by F Forelli and W. Rudin [6].

harmonic. Hence

$$u(0) = \frac{2m}{\omega_{2m-1}} \gamma(w)^{-1} \int_{B(0,\gamma(w)^{1/(2m)})} u(\eta)\, d\mu(\eta),$$

where ω_s is the 'area' of the sphere $S^s \subset \boldsymbol{R}^{s+1}$ ($(w,\eta) \in \Omega_m$ yields $\eta \in B(0,\gamma(w)^{1/(2m)})$). Therefore

$$K(z,w)\gamma(w) = \frac{2m}{\omega_{2m-1}} \int_{|\eta|^{2m}<\gamma(w)} K_{\Omega_m}((z,0),(w,\eta))\, d\mu(\eta). \tag{4}$$

For each $f \in L^2H(\Omega,\gamma)$ set $\tilde{f}(z,\xi) = f(z)$. Clearly $\tilde{f}$ is holomorphic in Ω_m. Also

$$\begin{aligned}
\|\tilde{f}\|^2_{L^2(\Omega_m)} &= \int_{\Omega_m} |\tilde{f}(z,\xi)|^2\, d\mu(z,\xi) \\
&= \int_\Omega |f(z)|^2 \left(\int_{|\xi|^{2m}<\gamma(z)} d\mu(\xi) \right) d\mu(z) \\
&= \frac{\omega_{2m-1}}{2m} \int_\Omega |f(z)|^2 \gamma(z)\, d\mu(z) = \frac{\omega_{2m-1}}{2m} \|f\|^2_\gamma < \infty
\end{aligned}$$

i.e. $\tilde{f} \in L^2(\Omega_m)$. As K_{Ω_m} reproduces the L^2 holomorphic functions on Ω_m, one has (by (4))

$$\begin{aligned}
f(z) = \tilde{f}(z,0) &= \int_{\Omega_m} K_{\Omega_m}((z,0),(w,\eta))\tilde{f}(w,\eta)\, d\mu(w,\eta) \\
&= \int_\Omega f(w) \left(\int_{|\eta|^2<\gamma(w)} K_{\Omega_m}((z,0),(w,\eta))\, d\mu(\eta) \right) d\mu(w) \\
&= \frac{\omega_{2m-1}}{2m} \int_\Omega f(w) K(z,w)\gamma(w)\, d\mu(w),
\end{aligned}$$

i.e. $(\omega_{2m-1}/2m)K(z,w)$ reproduces the functions in $L^2H(\Omega,\gamma)$. As u is anti-holomorphic, $|u|^2$ is subharmonic. Hence

$$|u(0)|^2 \leq \frac{1}{\mathrm{Vol}(B(0,\gamma(w)^{1/(2m)}))} \int_{B(0,\gamma(w)^{1/(2m)})} |u(\eta)|^2 d\mu(\eta)$$

or

$$|K(z,w)|^2 \leq \frac{2m}{\omega_{2m-1}} \gamma(w)^{-1} \int_{|\eta|^2<\gamma(w)^{1/(2m)}} |K_{\Omega_m}((z,0),(w,\eta))|^2 d\mu(\eta).$$

Finally, we may integrate against $w \in \Omega$ so that to get

$$\begin{aligned}
&\int_\Omega |K(z,w)|^2 \gamma(w)\, d\mu(w) \\
&\qquad \leq \frac{2m}{\omega_{2m-1}} \int_{\Omega_m} |K_{\Omega_m}((z,0),(w,\eta))|^2 d\mu(w,\eta) < \infty
\end{aligned}$$

i.e. $K(z,\cdot)\in L^2(\Omega,\gamma)$. Then (3) follows from the uniqueness statement in the Riesz representation theorem.

When $\gamma=|\varphi|^m$, $m\in\{1,2,\dots\}$, the domain Ω_m is strictly pseudoconvex and (2) follows from Lemma 1 and from Fefferman's asymptotic expansion formula for K_{Ω_m}, i.e.

$$K_{\Omega_m}((z,\xi),(w,\eta))$$
$$=\text{const.}|\nabla\varphi_1(w,\eta)|\cdot\det L_{\varphi_1}(w,\eta)\cdot\Psi((z,\xi),(w,\eta))^{-(n+m+1)}+E((z,\xi),(w,\eta)),$$

for some $E\in C^\infty(\bar{\Omega}_m\times\bar{\Omega}_m-\Delta_1)$ satisfying the estimate

$$|E((z,\xi),(w,\eta))|\le\text{const.}|\Psi((z,\xi),(w,\eta))|^{-(n+m+1)+1/2}\cdot|\log|\Psi((z,\xi),(w,\eta))||$$

where Ψ is defined as in (1), with the obvious modifications, while $\varphi_1(z,\xi)=\varphi(z)+|\xi|^2$ and Δ_1 is the diagonal of $\partial\Omega_m\times\partial\Omega_m$ (as $\partial\Omega\times\{0\}\subset\partial\Omega_m$, Δ imbeds in Δ_1).

Let $L^\infty_{\mathbf{R}}(\Omega)$ be the Banach space (algebra) of all real valued Lebesgue measurable, essentially bounded functions on $\Omega=\{\varphi<0\}$, with the norm $\|g\|_\infty=\text{esssup}_{z\in\Omega}|g(z)|$, $g\in L^\infty_{\mathbf{R}}(\Omega)$. By a result of Z. Pasternak-Winiarski (cf. Proposition 2.3 in [16], p. 116) $W(\Omega)$ is a Banach manifold modelled on $L^\infty_{\mathbf{R}}(\Omega)$, and $AW(\Omega)$ is an open subset of $W(\Omega)$. Note that the Fefferman like asymptotic expansion of a weighted Bergman kernel is known (cf. Theorem 1 above) only for the points of the curve $C:(-1,\infty)\to W(\Omega)$, $C(\alpha)=|\varphi|^\alpha\in AW(\Omega)$, $\alpha>-1$, corresponding to the integer values of the parameter. Of course, it is desirable to extend Theorem 1 to all $\gamma\in AW(\Omega)$. As a measure of the amount of job left unsolved we may state the following

THEOREM 2. *Let $\Omega=\{\varphi<0\}$ be a domain in $\mathbf{C}^n$. The curve $C:(-1,\infty)\to W(\Omega)$, $C(\alpha)=|\varphi|^\alpha$, $\alpha>-1$, is discontinuous and each point of C is an isolated point.*

Set

$$U(\Omega)=\{g\in L^\infty_{\mathbf{R}}(\Omega):\text{essinf}_{z\in\Omega}g(z)>0\}$$

(an open subset of $L^\infty_{\mathbf{R}}(\Omega)$). Given $\mu\in W(\Omega)$ let $\Phi_\mu:U(\Omega)\to W(\Omega)$ be defined by $(\Phi_\mu g)(z)=g(z)\mu(z)$, $g\in U(\Omega)$, $z\in\Omega$, and set $U(\Omega,\mu)=\Phi_\mu(U(\Omega))$. By Proposition 2.3 in [16], p. 116, the family

$$\{\Phi_\mu(A):\mu\in W(\Omega),\ A\subseteq U(\Omega),\ A\ \textit{open}\}$$

is a basis of open sets for the topology of $W(\Omega)$. At this point, we may prove Theorem 2. Given $\alpha_0>-1$, C is continuous in α_0 if and only if for any open subset $A\subseteq U(\Omega)$ with $1\in A$, there is $\delta_A>0$ so that $|\varphi|^{\alpha-\alpha_0}\in A$ for any $|\alpha-\alpha_0|<\delta_A$. Note that for each $u:\bar{\Omega}\to[0,\infty)$, if $u\in C^0(\bar{\Omega})$ and $u|_{\partial\Omega}=0$ then $\text{essinf}_\Omega u\le 0$ (indeed, if $\text{essinf}_\Omega u>0$ then

$$u(z)\ge L \tag{5}$$

for some $L > 0$. A priori (5) holds a.e. in Ω, yet $\{u < L\}$ is open, hence empty. Therefore (5) holds everywhere in Ω and, for $z \to \partial\Omega$, it gives $L \leq 0$, a contradiction).

LEMMA 2. *Let* $\alpha_0 > -1$, $\delta > 0$ *and* A *an open subset of* $U(\Omega)$ *with* $1 \in A$. *Then* $|\varphi|^{\alpha-\alpha_0} \in A$ *if and only if* $\alpha = \alpha_0$.

Proof. If $\alpha > \alpha_0$ then (by the observation above) $|\varphi|^{\alpha-\alpha_0}|_{\partial\Omega} = 0$ yields $|\varphi|^{\alpha-\alpha_0} \notin U(\Omega)$. If in turn $\alpha < \alpha_0$ then $\lim_{z\to\partial\Omega}|\varphi(z)|^{\alpha-\alpha_0} = \infty$ hence $|\varphi|^{\alpha-\alpha_0} \notin L^\infty_{\boldsymbol{R}}(\Omega)$, just by observing that, for each $v : \Omega \to [0,\infty)$, if $v \in C^0(\Omega)$ and $\lim_{z\to\partial\Omega}v(z) = \infty$ then $\mathrm{esssup}_\Omega v = \infty$.

Finally $U(\Omega, |\varphi|^{\alpha_0})$ is an open neighborhood of $|\varphi|^{\alpha_0}$ yet (by Lemma 2) it contains no other point of C.

2. Symplectomorphisms of γ-Kobayashi domains

Let $\Omega = \{\varphi < 0\}$ be a domain and $\gamma \in AW(\Omega)$ and admissible weight. By a result in [17] one has the representation

$$K_\gamma(\zeta, z) = \sum_k \phi_k(\zeta)\overline{\phi_k(z)}$$

for any complete orthonormal system $\{\phi_k\}$ in $L^2H(\Omega,\gamma)$. Hence $K_\gamma(z,z) > 0$ for any $z \in \Omega$, provided that A) for each $z \in \Omega$ there is $f \in L^2H(\Omega,\gamma)$ with $f(z) \neq 0$. If the weight $\gamma = (1+h)|\varphi|^m$ (with $h \in L^\infty_{\boldsymbol{R}}(\Omega)$, $\|h\|_\infty < 1/2$, $m \in \{1,2,\ldots\}$) satisfies condition A) then it makes sense to consider the function

$$\rho_{h,m}(z) = K_{(1+h)|\varphi|^m}(z,z)^{-1/(n+1+m)}, \quad z \in \Omega,$$

and (by Theorem 4)

$$\rho_{h,m}(z) \leq |\varphi(z)|\{\Phi(z) + C[|\varphi(z)|^{1/2}|\log|\varphi(z)|| + (1+F(z))^2]\}^{-1/(n+1+m)}$$

for some $\Phi \in C^\infty(\bar{\Omega})$ so that $\Phi(z) \neq 0$ near $\partial\Omega$. Hence $\rho_{h,m}(z) \to 0$ as $z \to \partial\Omega$. As the boundary behaviour of $[K^{(k)}_{1,|\varphi|^m}h^{(k)}](z,w)$, $k \geq 1$ (cf. notations in section 3) is not known, one may not conclude that $\rho_{h,m}(z)$ is a defining function for Ω. However, as a corollary of Theorem 1 one has

$$K_m(z,z) = \Phi(z)|\varphi(z)|^{-(n+1+m)} + \tilde{\Phi}(z)\log|\varphi(z)|,$$

for some Φ, $\tilde{\Phi} \in C^\infty(\bar{\Omega})$, $\Phi(z) \neq 0$ near $\partial\Omega$, hence $\rho_m = \rho_{0,m} \in C^\infty(\bar{\Omega})$ and $\nabla\rho_m \neq 0$ on $\partial\Omega$, i.e. ρ_m can be used as a defining function for Ω ($\Omega = \{\rho_m > 0\}$).

Let $\Omega_n = \{\zeta \in \boldsymbol{C}^n : \varphi_n(\zeta) < 0\}$ be the Siegel domain, where $\varphi_n(\zeta) = |\zeta'|^2 - \mathrm{Im}(\zeta_1)$, and for each $\zeta = (\zeta_1,\ldots,\zeta_n)$ one sets $\zeta' = (\zeta_2,\ldots,\zeta_n)$. Let $K_\alpha(\zeta,z)$ be the $|\varphi_n|^\alpha$-Bergman kernel for $L^2H(\Omega_n, |\varphi_n|^\alpha)$, $\alpha > -1$. As Ω_n is unbounded and α not necessarily an integer, neither Theorem 1 nor its proof apply, yet $\rho_\alpha(\zeta) = K_\alpha(\zeta,\zeta)^{-1/(n+1+\alpha)}$ is a (well defined) defining function for Ω_n. Indeed (cf. [1]) K_α may be explicitely computed as

$$K_\alpha(\zeta, z) = \frac{2^{n-1+\alpha}c_{n,\alpha}}{[i(\bar{z}_1 - \zeta_1) - 2\langle\zeta', z'\rangle]^{n+1+\alpha}}$$

$$c_{n,\alpha} = \pi^{-n}(\alpha+1)\cdots(\alpha+n)$$

hence $\rho_\alpha(\zeta) = C\varphi_n(\zeta)$, for some constant C depending only on n and α.

Let $\Omega \subset \boldsymbol{C}^n$ be a domain and $\gamma \in AW(\Omega)$. In general g_γ is not definite, or even nondegenerate. For instance, if Ω is bounded and $\gamma \in L^1(\Omega)$ then g_γ is a Kählerian metric on Ω (cf. [3]) yet the arguments in [3] break down for the case of an unbounded domain. We call Ω a *γ-Kobayashi domain* if (Ω, γ) satisfies condition A) and additionally B) for any $z \in \Omega$ and any $Z \in T^{1,0}(\Omega)_z$, $Z \neq 0$, there is $f \in L^2H(\Omega,\gamma)$ so that $f(z) = 0$ and $Z(f) \neq 0$ (our A)–B) correspond to the conditions (A.1)–(A.2) in [8], pp. 271–272, hence the adopted terminology). Here $T^{1,0}(\Omega)$ is the holomorphic tangent bundle over Ω. The unit ball in $\boldsymbol{C}^n$ is a 1-Kobayashi domain. The Siegel domain Ω_n is an (unbounded) $|\varphi_n|^\alpha$-Kobayashi domain for any $\alpha > -1$ (cf. Lemmae 4 and 5 in [1]). By a result in [15], A)–B) imply that g_γ is a Kählerian metric on Ω, hence (Ω, ω_γ) is a symplectic manifold.

From now on, it is understood that Ω is a strictly pseudoconvex domain satisfying all hypothesis of Theorem 1. We may state:

THEOREM 3. *Let $m \in \{0,1,2,\dots\}$ and $\Omega = \{\varphi < 0\}$ a $|\varphi|^m$-Kobayashi domain. Let F be a symplectomorphism of (Ω, ω_m), i.e. a C^∞ diffeomorphism $F : \Omega \to \Omega$ with $F^*\omega_m = \omega_m$. If F is smooth up to the boundary then $F : \partial\Omega \to \partial\Omega$ is a contact transformation.*

Here ω_m is short for $\omega_{|\varphi|^m}$. For $\gamma = 1$ and $m = 0$ Theorem 3 is the result by A. Korányi and H. M. Reimann quoted in the introduction. The proof is imitative of that of Proposition 1 in [11], p. 1121. We need some notation. Let $\mathscr{F}$ be the foliation of U (a one-sided neighborhood of the boundary of Ω) by level sets of ρ_m (so that $\rho_m^{-1}(0) = \partial\Omega$). Each leaf $M_c = \rho_m^{-1}(c)$ is a strictly pseudoconvex CR manifold with the CR structure $T_{1,0}(M_c) = [T(M_c) \otimes \boldsymbol{C}] \cap T^{1,0}(U)$. Let $T_{1,0}(\mathscr{F})$ be the subbundle of $T(U) \otimes \boldsymbol{C}$ whose portion over M_c is $T_{1,0}(M_c)$. As Ω is strictly pseudoconvex, there is a uniquely defined complex vector field ξ of type (1,0) on U which is orthogonal to $T_{1,0}(\mathscr{F})$ with respect to $\partial\bar{\partial}\rho_m$ and for which $\partial\rho_m(\xi) = 1$ (cf. e.g. [13], p. 163). Define $r : U \to \boldsymbol{R}$ by setting $r = 2(\partial\bar{\partial}\rho_m)(\xi, \bar{\xi})$ so that ξ and r are characterized by

$$\xi \rfloor \partial\bar{\partial}\rho_m = r\bar{\partial}\rho_m, \quad \partial\rho_m(\xi) = 1. \tag{6}$$

Let $\theta_m = i(\bar{\partial} - \partial)\rho_m/2$ and $N = 2\,\mathrm{Re}(\xi)$. Then $(d\rho_m)N = 2$ and $\theta_m(N) = 0$.

Note that

$$\omega_m = i(n+1+m)\left(\frac{\partial\bar{\partial}\rho_m}{\rho_m} - \frac{\partial\rho_m \wedge \bar{\partial}\rho_m}{\rho_m^2}\right). \tag{7}$$

Set $H(\mathscr{F}) = \mathrm{Re}\{T_{1,0}(\mathscr{F}) \oplus \overline{T_{1,0}(\mathscr{F})}\}$ (so that the portion of $H(\mathscr{F})$ over a leaf M_c is the maximally complex, or Levi, distribution of M_c). Then (by (7))

$$\omega_m(X,N) = 0,$$

for any $X \in H(\mathscr{F})$. On the other hand, we may write (7) as

$$\omega_m = (n+1+m)\left(\frac{d\theta_m}{\rho_m} - \frac{d\rho m \wedge \theta_m}{\rho_m^2}\right)$$

hence (by $F^*\omega_m = \omega_m$)

$$0 = \omega_m((dF)X,(dF)N) = (n+1+m)\rho_m^{-1}d\theta_m((dF)X,(dF)N)$$
$$-(n+1+m)\rho_m^{-2}(d\rho_m \wedge \theta_m)((dF)X,(dF)N)$$

for any $X \in H(\mathscr{F})$. As F is smooth up to the boundary,

$$(d\theta_m)((dF)X,(dF)N)$$

stays finite near $\partial\Omega$. Hence, in the limit

$$(d\rho_m)((dF)X)\theta_m((dF)N) - (d\rho_m)((dF)N)\theta_m((dF)X)$$

vanishes on $\partial\Omega$. If X lies in $H(\partial\Omega)$, the maximal complex distribution of $\partial\Omega$ as a CR manifold, then $(dF)X \in T(\partial\Omega)$ hence $(d\rho_m)((dF)X) = 0$. Finally $(d\rho_m)((dF)N) \neq 0$ (as F is a diffeomorphism and $d\rho_m \neq 0$ on $\partial\Omega$) hence $\theta_m((dF)X) = 0$ for any $X \in H(\partial\Omega)$. q.e.d.

Let ω_α be short for $\omega_{|\varphi_n|^\alpha}$, $\alpha > -1$. Although Ω_n is unbounded and α not necessarily an integer, Theorem 3 remains true for a symplectomorphism F of $(\Omega_n, \omega_\alpha)$, i.e. if F is smooth up to $\partial\Omega_n$ then the restriction of F to $\partial\Omega_n$ is a contact transformation (the proof is a verbatim transcription of the proof of Theorem 3, where ρ_m is replaced by ρ_α).

3. The effect of the analytic behaviour of weighted Bergman kernels

Let U be an open subset of a normed space $\mathscr{X}$ and let $\mathscr{Y}$ be a topological vector space. Together with [16], one says that a map $f : U \to \mathscr{Y}$ is *analytic* on U if for any $x \in U$ there is a ball $B \subset \mathscr{X}$ of center $0 \in \mathscr{X}$ so that $x + B \subset U$ and

$$f(x+h) = f(x) + \sum_{k=1}^{\infty} a_k(h,\ldots,h) \tag{8}$$

for any $h \in B$, where $a_k : \mathscr{X}^k \to \mathscr{Y}$ is a continuous k-linear function, $k \in \{1,2,\ldots\}$, and the series in (8) converges uniformly on B.

Let $HA(\Omega)$ be the vector space of all real analytic functions $F : \Omega \times \Omega \to \boldsymbol{C}$ which are holomorphic with respect to the first n variables and anti-holomorphic with respect to the last n variables. Set

$$\|F\|_X = \sup_{(z,\zeta)\in X^2} |F(z,\zeta)|$$

for $F \in HA(\Omega)$, $X \subset \Omega$. The family of seminorms

$$\{\|\cdot\|_X : X \subset \Omega, X \textit{ compact}\}$$

makes $HA(\Omega)$ into a Fréchet space. By a result of Z. Pasternak-Winiarski (cf. Theorem 5.1 in [16], p. 131) the map $U(\Omega) \to HA(\Omega)$, $g \mapsto K_{g\gamma}$, is analytic on $U(\Omega)$ for any $\gamma \in AW(\Omega)$.

THEOREM 4. *Let $\Omega = \{\varphi < 0\}$ be a smoothly bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$ so that $L_\varphi(w)\xi \geq \text{const.}|\xi|^2$, $\xi \in \boldsymbol{C}^n$, for $\varphi(w) < \delta_0$, $\delta_0 > 0$. Then for any $h \in B(0, 1/2) \subset L^\infty_{\boldsymbol{R}}(\Omega)$ there is $E_h \in C^\infty(\Omega \times \Omega)$ so that*

$$K_{(1+h)|\varphi|^m}(z,w) = c_\Omega |\nabla\varphi(w)|^2 \cdot \det L_\varphi(w) \cdot \Psi(z,w)^{-(n+1+m)} + E_h(z,w) \tag{9}$$

and E_h satisfies the estimate

$$\begin{aligned}(E_h(z,w)| \leq C \cdot \{&|\Psi(z,w)|^{-(n+1+m)+1/2}|\log|\Psi(z,w)||\\ &+ |\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}(1 + F(z) + F(w) + F(z)F(w))\}\end{aligned} \tag{10}$$

where $F(z) = |\varphi(z)|^{3/2} + |\varphi(z)|^{1/2}|\log|\varphi(z)||$ and C is a constant depending only on Ω and $m \geq 1$, $m > n-1$.

The proof of Theorem 4 relies on (2) and on the analyticity of the weighted Bergman kernel as a map $AW(\Omega) \to HA(\Omega)$, $\gamma \mapsto K_\gamma$. Set

$$\begin{aligned}&[K^{(k)}_{g,\gamma}(h_1,\dots,h_k)](z,w)\\ &= \int_\Omega K_{g\gamma}(u_1,w)h_1(u_1)\gamma(u_1)\,d\mu(u_1)\\ &\quad\cdot \int_\Omega K_{g\gamma}(u_2,u_1)h_2(u_2)\gamma(u_2)\,d\mu(u_2)\dots\\ &\quad\cdot \int_\Omega K_{g\gamma}(u_k,u_{k-1})h_k(u_k)K_{g\gamma}(z,u_k)\gamma(u_k)\,d\mu(u_k)\end{aligned}$$

for $\gamma \in AW(\Omega)$, $g \in U(\Omega)$, $h_1,\dots,h_k \in L^\infty_{\boldsymbol{R}}(\Omega)$, $k \geq 1$. Then

$$K^{(k)}_{g,\gamma}(h_1,\dots,h_k) \in HA(\Omega)$$

(cf. Lemma 5.1 in [16], p. 129). By (2) and by (5.5) in Theorem 5.1 of [16], p. 131, it follows that (9) holds good with

$$E_h = E + \sum_{k=1}^{\infty} (-1)^k K^{(k)}_{1,|\varphi|^m} h^{(k)}$$

where $h^{(k)} = (h, \dots, h)$ (k components), $E \in C^\infty(\bar{\Omega} \times \bar{\Omega} - \Delta)$ satisfies the estimate in Theorem 1, and the series is uniformly convergent on $B(0, 1/2) = \{h \in L^\infty_{\boldsymbol{R}}(\Omega) : \|h\|_\infty < 1/2\}$ with respect to any seminorm $\|\cdot\|_X$ on $HA(\Omega)$, with X an arbitrary compact subset of Ω. It remains that we prove the estimate (10). Let $k \geq 3$ (the cases $k = 1$ and $k = 2$ are looked at later on). Then (by (2))

$$\begin{aligned}
&[K^{(k)}_{1,|\varphi|^m} h^{(k)}](z, w) \\
&\quad = \int \{c_\Omega |\nabla\varphi(u_1)|^2 \cdot \det L_\varphi(u_1) \cdot \overline{\Psi(w, u_1)}^{-(n+1+m)} + \overline{E(w, u_1)}\} \\
&\qquad \cdot \{c_\Omega |\nabla\varphi(u_k)|^2 \cdot \det L_\varphi(u_k) \cdot \Psi(z, u_k)^{-(n+1+m)} + E(z, u_k)\} \\
&\qquad \cdot h(u_1) h(u_k) [K^{(k-2)}_{1,|\varphi|^m} h^{(k-2)}](u_k, u_1) |\varphi(u_1)|^m |\varphi(u_k)|^m \, d\mu(u_1) \, d\mu(u_k)
\end{aligned}$$

hence

$$\begin{aligned}
&[K^{(k)}_{1,|\varphi|^m} h^{(k)}](z, w) \\
&\quad = c_\Omega^2 I_1(z, w) + c_\Omega (I_2(z, w) + I_3(z, w)) + I_4(z, w),
\end{aligned}$$

where

$$\begin{aligned}
I_j(z, w) = &\int_\Omega G_j(u_1, w) \left(\int_\Omega [K^{(k-2)}_{1,|\varphi|^m} h^{(k-2)}](u_k, u_1) H_j(z, u_k) |\varphi(u_k)|^m \, d\mu(u_k) \right) \\
&\cdot |\varphi(u_1)|^m \, d\mu(u_1),
\end{aligned}$$

for $1 \leq j \leq 4$ and G_j, H_j are given by

$$\begin{aligned}
G_1(u_1, w) &= G_2(u_1, w) = |\nabla\varphi(u_1)|^2 \cdot \det L_\varphi(u_1) \cdot \overline{\Psi(w, u_1)}^{-(n+1+m)} h(u_1) \\
G_3(u_1, w) &= G_4(u_1, w) = \overline{E(w, u_1)} h(u_1) \\
H_1(z, u_k) &= H_3(z, u_k) = |\nabla\varphi(u_k)|^2 \cdot \det L_\varphi(u_k) \cdot \Psi(z, u_k)^{-(n+1+m)} h(u_k) \\
H_2(z, u_k) &= H_4(z, u_k) = E(z, u_k) h(u_k).
\end{aligned}$$

By a result in [16], p. 131, we have

$$\|[K^{(k-2)}_{1,|\varphi|^m} h^{(k-2)}](\cdot, u_1)\|_m \leq \|h\|_\infty^{k-2} \|K_m(\cdot, u_1)\|_m$$

where $\|\cdot\|_m$ is short for $\|\cdot\|_{|\varphi|^m}$. Then we may perform the estimates

$$
\begin{aligned}
|I_j(z,w)| \le & \int_\Omega |G_j(u_1,w)| \\
& \cdot \left(\int_\Omega |[K_{1,|\varphi|^m}^{(k-2)} h^{(k-2)}](u_k,u_1)|^2 |\varphi(u_k)|^m \, d\mu(u_k) \right)^{1/2} \\
& \cdot \left(\int_\Omega |H_j(z,u_k)|^2 \cdot |\varphi(u_k)|^m \, d\mu(u_k) \right)^{1/2} |\varphi(u_1)|^m \, d\mu(u_1) \\
= & \int_\Omega |G_j(u_1,w)| \cdot \| [K_{1,|\varphi|^m}^{(k-2)} h^{(k-2)}](\cdot,u_1) \|_m \cdot \|H_j(z,\cdot)\|_m |\varphi(u_1)|^m \, d\mu(u_1) \\
\le & \|h\|_\infty^{k-2} \|H_j(z,\cdot)\|_m \cdot \int_\Omega |G_j(u_1,w)| \cdot \|K_m(\cdot,u_1)\|_m |\varphi(u_1)|^m \, d\mu(u_1).
\end{aligned}
$$

Yet

$$\|K_m(\cdot,u_1)\|_m \le \text{const.} |\varphi(u_1)|^{-(n+1+m)/2}$$

by Lemma 2.8 in [18], p. 233. Hence

$$
\begin{aligned}
\text{(11)} \qquad |I_j(z,w)| \le & \ \text{const.} \|h\|_\infty^{k-2} \|H_j(z,\cdot)\|_m \\
& \cdot \int_\Omega |G_j(u_1,w)| \cdot |\varphi(u_1)|^{-(n+1-m)/2} \, d\mu(u_1).
\end{aligned}
$$

We look at the case $j = 1$. To this end, set

$$J_{v,a}(z) = \int_\Omega \frac{|\varphi(w)|^v \, d\mu(w)}{|\Psi(z,w)|^{n+1+v+a}}$$

for $v > -1$ and $a \in \boldsymbol{R}$. By Lemma 2.7 in [18], p. 232, one has

$$
J_{v,a}(z) \le \begin{cases} \text{const.} & \text{if } a < 0 \\ |\log|\varphi(z)|| & \text{if } a = 0 \\ |\varphi(z)|^{-a} & \text{if } a > 0. \end{cases}
$$

Then

$$
\begin{aligned}
\|H_1(z,\cdot)\|_m^2 &= \int_\Omega |H_1(z,u_k)|^2 |\varphi(u_k)|^m \, d\mu(u_k) \\
&\le \text{const.} \|h\|_\infty^2 \int_\Omega |\Psi(z,u_k)|^{-2(n+1+m)} |\varphi(u_k)^m| \, d\mu(u_k) \\
&= \text{const.} \|h\|_\infty^2 J_{m,n+1+m}(z)
\end{aligned}
$$

so that

$$\|H_1(z,\cdot)\|_m \le \text{const.} \|h\|_\infty |\varphi(z)|^{-(n+1+m)/2}.$$

Then (by (11))

$$\begin{aligned}|I_1(z,w)| &\le \text{const.}\|h\|_\infty^{k-1}|\varphi(z)|^{-(n+1+m)/2}\\ &\quad\cdot\int_\Omega |G_1(u_1,w)|\,|\varphi(u_1)|^{-(n+1+m)/2}\,d\mu(u_1)\\ &\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}\int_\Omega \frac{|\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1)}{|\Psi(w,u_1)|^{n+1+m}}\\ &= \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}J_{-(n+1-m)/2,\,(n+1+m)/2}(w).\end{aligned}$$

We may conclude that

$$|I_1(z,w)| \le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}. \tag{12}$$

Next (for $j=2$)

$$\begin{aligned}\|H_2(z,\cdot)\|_m^2 &= \int_\Omega |E(z,u_k)|^2|h(u_k)|^2|\varphi(u_k)|^m\,d\mu(u_k)\\ &\le \text{const.}\|h\|_\infty^2\int_\Omega |\Psi(z,u_k)|^{-2(n+1+m)+1}|\log|\Psi(z,u_k)||^2|\varphi(u_k)|^m\,d\mu(u_k).\end{aligned}$$

This integral may be written as a sum $\int_{\{|\Psi(z,u_k)|\ge 1\}}+\int_{\{|\Psi(z,u_k)|<1\}}$. In the first integral $\log|\Psi(z,u_k)|\le|\Psi(z,u_k)|$ while for the second (cf. [18], p. 229)

$$\begin{aligned}|\Psi(z,u_k)| &\ge \text{const.}(|\varphi(z)|+|\varphi(u_k)|+|z-u_k|^2+|\operatorname{Im}\Psi(z,u_k)|)\\ &\ge \text{const.}|\varphi(z)|\end{aligned}$$

yields $|\log|\Psi(z,u_k)||\le\text{const.}|\log|\varphi(z)||$. Hence

$$\begin{aligned}\|H_2(z,\cdot)\|_m^2 &\le \text{const.}\|h\|_\infty^2\\ &\quad\cdot\Big(\int_\Omega |\Psi(z,u_k)|^{-2(n+m)+1}|\varphi(u_k)|^m\,d\mu(u_k)\\ &\quad+\text{const.}|\log|\varphi(z)||^2\int_\Omega |\Psi(z,u_k)|^{-2(n+m)-1}|\varphi(u_k)|^m\,d\mu(u_k)\Big)\\ &= \text{const.}\|h\|_\infty^2(J_{m,n-2+m}(z)+|\log|\varphi(z))|^2J_{m,n+m}(z))\end{aligned}$$

i.e.

$$\|H_2(z,\cdot)\|_m \le \text{const.}\|h\|_\infty|\varphi(z)|^{-(n+m)/2}(|\varphi(z)|+|\log|\varphi(z)||).$$

Then (by (11))

$$
\begin{aligned}
|I_2(z,w)| &\le \text{const.}\|h\|_\infty^{k-1}|\varphi(z)|^{-(n+m)/2}(|\varphi(z)| + |\log|\varphi(z)||) \\
&\quad \cdot \int_\Omega |G_2(u_1,w)|\,|\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1) \\
&\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+m)/2}(|\varphi(z)| \\
&\quad + |\log|\varphi(z)||)\int_\Omega \frac{|\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1)}{|\Psi(w,u_1)|^{n+1+m}} \\
&= \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+m)/2}(|\varphi(z) + |\log|\varphi(z)||)J_{-(n+1-m)/2,(n+1+m)/2}(w)
\end{aligned}
$$

i.e.

$$
(13) \qquad |I_2(z,w)| \le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}F(z).
$$

Next (as $H_1 = H_3$)

$$
\begin{aligned}
|I_3(z,w)| &\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2} \\
&\quad \cdot \int_\Omega |E(w,u_1)| \cdot |\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1) \\
&\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}\int_\Omega |\Psi(w,u_1)|^{-(n+1+m)+1/2} \\
&\quad \cdot |\log|\Psi(w,u_1)|| \cdot |\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1) \\
&\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2} \\
&\quad \cdot \left\{ \int_\Omega |\Psi(w,u_1)|^{-(n+m)+1/2} \cdot |\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1) \right. \\
&\quad \left. + \text{const.}|\log|\varphi(w)||\int_\Omega |\Psi(w,u_1)|^{-(n+m)-1/2}|\varphi(u_1)|^{-(n+1-m)/2}\,d\mu(u_1)\right\} \\
&= \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2} \\
&\quad \cdot \{J_{-(n+1-m)/2,(n-2+m)/2}(w) + |\log|\varphi(w)||J_{-(n+1-m)/2,(n+m)/2}(w)\} \\
&\le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2} \\
&\quad \cdot \{|\varphi(w)|^{-(n-2+m)/2} + |\log|\varphi(w)|| \cdot |\varphi(w)|^{-(n+m)/2}\}
\end{aligned}
$$

i.e.

$$
(14) \qquad |I_3(z,w)| \le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}F(w).
$$

Finally (as $H_2 = H_4$ and $G_3 = G_4$)

$$
(15) \qquad |I_4(z,w)| \le \text{const.}\|h\|_\infty^{k}|\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}F(z)F(w).
$$

The estimates (12)–(15) lead to

$$(16)\quad |[K^{(k)}_{1,|\varphi|^m}h^{(k)}](z,w)| \le \text{const.}\|h\|^k_\infty|\varphi(z)|^{-(n+1+m)/2} \cdot |\varphi(w)|^{-(n+1+m)/2}(1+F(z)+F(w)+F(z)F(w)).$$

To deal with $K^{(1)}_{1,|\varphi|^m}$ we firstly note that

$$[K^{(1)}_{1,|\varphi|^m}h^{(1)}](z,w) = c_\Omega^2 J_1(z,w) + c_\Omega(J_2(z,w)+J_3(z,w)) + J_4(z,w)$$

where

$$J_1(z,w) = \int_\Omega |\nabla\varphi(u_1)|^4 |\det L_\varphi(u_1)|^2 \Psi(z,u_1)^{-(n+1+m)} \cdot \overline{\Psi(w,u_1)}^{-(n+1+m)} h(u_1)|\varphi(u_1)|^m\, d\mu(u_1)$$

$$J_2(z,w) = \int_\Omega |\nabla\varphi(u_1)|^2 \cdot \overline{\det L_\varphi(u_1)} \cdot E(z,u_1) \cdot \overline{\Psi(w,u_1)}^{-(n+1+m)} h(u_1)|\varphi(u_1)|^m\, d\mu(u_1)$$

$$J_3(z,w) = \int_\Omega |\nabla\varphi(u_1)|^2 \cdot \det\, L_\varphi(u_1) \cdot \Psi(z,u_1)^{-(n+1+m)} \cdot \overline{E(w,u_1)} h(u_1)|\varphi(u_1)|^m\, d\mu(u_1)$$

$$J_4(z,w) = \int_\Omega E(z,u_1)\overline{E(w,u_1)} h(u_1)|\varphi(u_1)|^m\, d\mu(u_1).$$

Then

$$\begin{aligned}
&|J_1(z,w)| \\
&\le \text{const.}\|h\|_\infty \int_\Omega |\Psi(z,u_1)|^{-(n+1+m)}|\Psi(w,u_1)|^{-(n+1+m)}|\varphi(u_1)|\, d\mu(u_1) \\
&\le \text{const.}\|h\|_\infty \cdot \left(\int_\Omega |\Psi(z,u_1)|^{-2(n+1+m)}|\varphi(u_1)|^m\, d\mu(u_1)\right)^{1/2} \\
&\quad \cdot \left(\int_\Omega |\Psi(w,u_1)|^{-2(n+1+m)}|\varphi(u_1)|^m\, d\mu(u_1)\right)^{1/2} \\
&= \text{const.}\|h\|_\infty J_{m,n+1+m}(z)^{1/2} J_{m,n+1+m}(w)^{1/2}
\end{aligned}$$

i.e.

$$(17)\qquad |J_1(z,w)| \le \text{const.}\|h\|_\infty|\varphi(z)|^{-(n+1+m)/2}|\varphi(w)|^{-(n+1+m)/2}.$$

Next

$$
\begin{aligned}
&|J_2(z,w)| \\
&\quad \le \text{const.}\|h\|_\infty \int_\Omega |E(z,u_1)| \cdot |\Psi(w,u_1)|^{-(n+1+m)} |\varphi(u_1)|^m \, d\mu(u_1) \\
&\quad \le \text{const.}\|h\|_\infty \left(\int_\Omega |E(z,u_1)|^2 |\varphi(u_1)|^m \, d\mu(u_1) \right)^{1/2} \\
&\qquad \cdot \left(\int_\Omega |\Psi(w,u_1)|^{-2(n+1+m)} |\varphi(u_1)|^m \, d\mu(u_1) \right)^{1/2} \\
&\quad = \text{const.}\|h\|_\infty \cdot \|E(z,\cdot)\|_m J_{m,n+1+m}(w)^{1/2} \\
&\quad \le \text{const.}\|h\|_\infty \cdot \|E(z,\cdot)\|_m |\varphi(w)|^{-(n+1+m)/2}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&|E(z,\cdot)\|_m^2 \\
&\quad \le \text{const.} \int_\Omega |\Psi(z,u_1)|^{-2(n+1+m)+1} \cdot |\log|\Psi(z,u_1)||^2 |\varphi(u_1)|^m \, d\mu(u_1) \\
&\quad \le \text{const.} \left(\int_\Omega |\Psi(z,u_1)|^{-2(n+1+m)+3} |\varphi(u_1)|^m \, d\mu(u_1) \right. \\
&\qquad \left. + |\log|\varphi(z)||^2 \int_\Omega |\Psi(z,u_1)|^{-2(n+1+m)+1} |\varphi(u_1)|^m \, d\mu(u_1) \right) \\
&\quad = \text{const.} (J_{m,n-2+m}(z) + |\log|\varphi(z)||^2 J_{n,n+m}(z)) \\
&\quad \le \text{const.} |\varphi(z)|^{-(n+m)} (|\varphi(z)|^2 + |\log|\varphi(z)||^2)
\end{aligned}
$$

i.e.

$$
\|E(z,\cdot)\|_m \le |\varphi(z)|^{-(n+1+m)/2} F(z).
$$

We conclude that

$$
\text{(18)} \qquad |J_2(z,w)| \le \text{const.}\|h\|_\infty |\varphi(z)|^{-(n+1+m)/2} |\varphi(w)|^{-(n+1+m)/2} F(z).
$$

Similarly

$$
\text{(19)} \qquad |J_3(z,w)| \le \text{const.}\|h\|_\infty |\varphi(z)|^{-(n+1+m)/2} |\varphi(w)|^{-(n+1+m)/2} F(w).
$$

Finally

$$
|J_4(z,w)| \le \text{const.}\|h\|_\infty \|E(z,\cdot)\|_m \cdot \|E(w,\cdot)\|_m
$$

i.e.

$$
\text{(20)} \qquad |J_4(z,w)| \le \text{const.}\|h\|_\infty |\varphi(z)|^{-(n+1+m)/2} |\varphi(w)|^{-(n+1+m)/2} F(z) F(w).
$$

By taking into account the estimates (17)–(20) it follows that (16) holds good for

$k = 1$ as well. To deal with $K^{(2)}_{1,|\varphi|^m}h^{(2)}$ one firstly uses the Schwarz inequality and Lemma 2.8 in [18], p. 233, so that to obtain

$$|[K^{(2)}_{1,|\varphi|^m}h^{(2)}](z,w)| \leq \text{const.}\|h\|^2_\infty |\varphi(z)|^{-(n+1+m)/2} \int_\Omega |K_m(u_1,w)| \cdot |\varphi(u_1)|^{-(n+1-m)/2}\, d\mu(u_1).$$

On the other hand

$$\begin{aligned}
&\int_\Omega |K_m(u_1,w)| \cdot |\varphi(u_1)|^{-(n+1-m)/2}\, d\mu(u_1) \\
&\leq \text{const.}\Bigg(\int_\Omega |\Psi(w,u_1)|^{-(n+1+m)} |\varphi(u_1)|^{-(n+1-m)/2}\, d\mu(u_1) \\
&\quad + \int_\Omega |\Psi(w,u_1)|^{-(n+1+m)+1/2} |\log|\Psi(w,u_1)|| \cdot |\varphi(u_1)|^{-(n+1-m)/2}\, d\mu(u_1)\Bigg) \\
&= \text{const.}(J_{-(n+1-m)/2,(n+1+m)/2}(w) + J_{-(n+1-m)/2,(n-2+m)/2}(w) \\
&\quad + |\log|\varphi(w)|| J_{-(n+1-m)/2,(n+m)/2}(w)) \\
&\leq \text{const.}(|\varphi(w)|^{-(n+1+m)/2} + |\varphi(w)|^{-(n-2+m)/2} \\
&\quad + |\log|\varphi(w)|| \cdot |\varphi(w)|^{-(n+m)/2}) = \text{const.}|\varphi(w)|^{-(n+1+m)/2}(1+F(w))
\end{aligned}$$

hence we may conclude that (16) holds for $k = 2$ as well. At this point (16) furnishes

$$\begin{aligned}
&\sum_{k=1}^{\infty} |[K^{(k)}_{1,|\varphi|^m}h^{(k)}](z,w)| \\
&\quad \leq \text{const.}\frac{1}{1-\|h\|_\infty} |\varphi(z)|^{-(n+1+m)/2} |\varphi(w)|^{-(n+1+m)/2} \\
&\quad\quad \cdot (1 + F(z) + F(w) + F(z)F(w))
\end{aligned}$$

which, together with the estimate in Theorem 1, yields (10).

4. The complex dilatation of a symplectomorphism and the Beltrami equations

Let $\Omega \subset \mathbf{C}^n$ be a γ-Kobayashi domain, for some $\gamma \in AW(\Omega)$. Let F be a symplectomorphism of (Ω, ω_γ) in itself. We have

Lemma 3. *For any* $z \in \Omega$ *and any* $Z \in T^{1,0}(\Omega)_z$, $Z \neq 0$, *one has* $(d_z F)\bar{Z} \notin T^{1,0}(\Omega)_{F(z)}$.

The proof is by contradiction. Assume that $(d_z F)\bar{Z} \in T^{1,0}(\Omega)_{F(z)}$ for some $Z \in T^{1,0}(\Omega)_z$, $Z \neq 0$, and some $z \in \Omega$. As F is a diffeomorphism $(d_z F)\bar{Z} \neq 0$. Hence

$$0 < \|(d_z F)\bar{Z}\|^2 = g_{\gamma,F(z)}((d_z F)\bar{Z}, (d_z F)Z)$$
$$= -i\omega_{\gamma,z}(\bar{Z}, Z) = -\|Z\|^2,$$

a contradiction.

Let $T^{1,0}(\Omega)_F$ consist of all $Z \in T(\Omega) \otimes \boldsymbol{C}$ with $(dF)Z \in T^{1,0}(\Omega)$.

LEMMA 4. *For any symplectomorphism F of (Ω, ω_γ) there is a $\boldsymbol{C}$-antilinear bundle map* $\mathrm{dil}(F) : T^{1,0}(\Omega) \to T^{1,0}(\Omega)$ *so that*

$$T^{1,0}(\Omega)_F = \{Z - \overline{\mathrm{dil}(F)Z} : Z \in T^{1,0}(\Omega)\}.$$

To prove Lemma 4, let $\pi_{0,1} : T(\Omega) \otimes \boldsymbol{C} \to T^{0,1}(\Omega)$ be the natural projection. Then

$$T^{1,0}(\Omega)_F = \mathrm{Ker}(\pi_{0,1} \circ (dF)).$$

Let $(z^1, \ldots, z^n)$ be the natural complex coordinates on $\boldsymbol{C}^n$. Set

$$F^j_k = \frac{\partial F^j}{\partial z^k}, \quad F^j_{\bar{k}} = \frac{\partial F^j}{\partial \bar{z}^k},$$

etc.. Then $\det(F^{\bar{k}}_{\bar{j}}) \neq 0$ everywhere on Ω. Indeed, if $\det(F^{\bar{k}}_{\bar{j}}(z_0)) = 0$ at some $z_0 \in \Omega$ then $\sum_k F^{\bar{j}}_{\bar{k}}(z_0)\overline{\zeta^k} = 0$, $1 \le j \le n$, for some $(\zeta^1, \ldots, \zeta^n) \in \boldsymbol{C}^n - \{0\}$. Set $Z = \sum_j \zeta_j (\partial/\partial z^j)_{z_0} \in T^{1,0}(\Omega)_{z_0}$. Then $Z \neq 0$ and

$$(d_{z_0} F)\bar{Z} = \sum_{j,k} \overline{\zeta^k} F^j_{\bar{k}}(z_0) \left(\frac{\partial}{\partial z^j}\right)_{F(z_0)} \in T^{1,0}(\Omega)_{F(z_0)},$$

a contradiction (by Lemma 3). Let $\mathrm{dil}(F) : T^{1,0}(\Omega) \to T^{1,0}(\Omega)$ be given by $\mathrm{dil}(F)(\partial/\partial z^j) = \sum_k \mathrm{dil}(F)^k_j \partial/\partial z^k$ (followed by $\boldsymbol{C}$-antilinear extension) where

$$F^\ell_{\bar{j}} = \sum_k \mathrm{dil}(F)^k_j F^\ell_k. \tag{21}$$

Finally, note that $\partial/\partial z^j - \overline{\mathrm{dil}(F)\partial/\partial z^j} \in \mathrm{Ker}(\pi_{0,1} \circ (dF))$. q.e.d.

The bundle map $\mathrm{dil}(F)$ is referred to as the *complex dilatation* (of the symplectomorphism F).

PROPOSITION 1. *Let F be a symplectomorphism of (Ω, ω_γ) and $\mathrm{dil}(F)$ its complex dilatation. Then*

$$\omega_\gamma(Z, \overline{\mathrm{dil}(F)W}) + \omega_\gamma(\overline{\mathrm{dil}(F)Z}, W) = 0,$$

for any $Z, W \in T^{1,0}(\Omega)$. Also, $\mathrm{dil}(F) = 0$ if and only if F is holomorphic.

Indeed, if $Z \in T^{1,0}(\Omega)$ then $(dF)(Z - \overline{\mathrm{dil}(F)Z}) \in T^{1,0}(\Omega)$. Therefore, as ω_γ vanishes on complex vector of the same type,

$$\begin{aligned} 0 &= \omega_\gamma((dF)(Z - \overline{\mathrm{dil}(F)Z}), (dF)(W - \overline{\mathrm{dil}(F)W})) \\ &= \omega_\gamma(Z - \overline{\mathrm{dil}(F)Z}, W - \overline{\mathrm{dil}(F)W}) = -\omega_\gamma(Z, \overline{\mathrm{dil}(F)W}) - \omega_\gamma(\overline{\mathrm{dil}(F)Z}, W) \end{aligned}$$

for any $Z, W \in T^{1,0}(\Omega)$.

By (21), each component F^J of the symplectomorphism F satisfies the first order PDE (with variable coefficients)

$$\frac{\partial f}{\partial \bar{z}^J} = \sum_k d^k_{\bar{j}} \frac{\partial f}{\partial z^k} \tag{22}$$

where $d^k_{\bar{j}} = \mathrm{dil}(F)^k_{\bar{j}}$. We refer to (22) as the *Beltrami equations* (cf. e.g. [20]). On the other hand, with any contact transformation $F : M \to N$ between two strictly pseudoconvex CR manifolds M and N one may associate (cf. [10], p. 61) a complex dilatation $\mu : T_{1,0}(M) \to T_{1,0}(M)$ and whenever $M = \boldsymbol{H}_{n-1}$ (the Heisenberg group) and N is a real hypersurface in $\boldsymbol{C}^n$ (carrying the standard CR structure induced from the complex structure of $\boldsymbol{C}^n$), the components F^J of F satisfy the PDE

$$L_{\bar{\alpha}} f = \sum_{\beta=1}^{n-1} \mu^\beta_{\bar{\alpha}} L_\beta f \tag{23}$$

where $L_{\bar{\alpha}} = \partial/\partial \bar{z}^\alpha - i z^\alpha \partial/\partial t$ are the *Lewy operators* (cf. e.g. [5], p. 435–436) on $\boldsymbol{H}_{n-1}$, and $\mu L_\alpha = \sum_\beta \mu^\beta_{\bar{\alpha}} L_\beta$. We refer to (23) as the *tangential Beltrami equations.*

Consider the Siegel domain $\Omega_n = \{\varphi_n < 0\}$ and let $F = (F^1, \dots, F^n)$ be a symplectomorphism of $(\Omega_n, \omega_\alpha)$ in itself. Let $\mathscr{F}_n$ be the foliation of $\boldsymbol{C}^n$ by level sets of φ_n. If F is smooth up to $\partial\Omega_n$ then μ (the complex dilatation of F) restricted to $T_{1,0}(\mathscr{F}_n)$ converges to the complex dilatation of the boundary contact transformation (the proof is a word by word repetition of the proof of Proposition 2 in [11], p. 1122). Also, if $\phi : \boldsymbol{H}_{n-1} \to \partial\Omega_n$ is the CR isomorphism $\phi(z,t) = (t + i|z|^2, z)$, then each $F^J \circ \phi$ satisfies the tangential Beltrami equations (23) (this follows from the remark at the end of section 2 and by a result in [10], p. 62).

Let $d^k_{\bar{j}}$ be smooth functions defined on some neighborhood of $\bar{\Omega}_n$. The complex vector fields $\partial/\partial\bar{\zeta}^J - \sum_k d^k_{\bar{j}} \partial/\partial\zeta^k$ span a rank n complex vector subbundle $B \subset T(\Omega_n) \otimes \boldsymbol{C}$. For the Siegel domain Ω_n, the vector field ξ (determined by (6)) is given by $\xi = 2i\partial/\partial\zeta^1$. The CR isomorphism $\phi : \boldsymbol{H}_{n-1} \approx \partial\Omega_n$ maps the Lewy operators $L_{\bar{\alpha}}$ into $Z_{\bar{\alpha}} = \partial/\partial\bar{\zeta}^\alpha + \zeta^\alpha\bar{\xi}$, $2 \le \alpha \le n$. We establish the following

Proposition 2. *Let D be an open neighborhood of $\bar{\Omega}_n$ and $\mu : T^{1,0}(D) \to T^{1,0}(D)$ a fibrewise $\boldsymbol{C}$-antilinear bundle morphism which maps $T_{1,0}(\partial\Omega_n)$ into itself. Let $B_b \subset T(\partial\Omega_n) \otimes \boldsymbol{C}$ be the rank $n-1$ complex subbundle spanned* *by* $Z_{\bar{\alpha}} - \mu_{\bar{\alpha}}^{\beta} Z_{\beta}$, $2 \leq \alpha \leq n$, *where* $\mu_{\bar{\alpha}}^{\beta}$ *are given by* $\mu(Z_{\alpha}) = \mu_{\bar{\alpha}}^{\beta} Z_{\beta}$. *Let* $d_{\bar{j}}^{k}$ *be given by* $\mu(\partial/\partial\zeta^{j}) = d_{\bar{j}}^{k}\partial/\partial\zeta^{k}$ *and set* $h(\zeta) = 2i\sum_{\beta} d_{\bar{1}}^{\beta}\bar{\zeta}_{\beta} - d_{\bar{1}}^{1} - 1$. *Then*

$$B_b = [T(\partial\Omega_n) \otimes \boldsymbol{C}] \cap B$$

on $\partial\Omega_n \cap \{\zeta : h(\zeta) \neq 0\}$. *In particular, the trace on* $\partial\Omega_n$ *of any solution* $f \in C^{\infty}(\bar{\Omega}_n)$ *of the Beltrami equations* (22) *satisfies the tangential Beltrami equations* $Z_{\bar{\alpha}} f = \mu_{\bar{\alpha}}^{\beta} Z_{\beta} f$ *on the open set* $\{\zeta \in \partial\Omega_n : h(\zeta) \neq 0\}$.

Indeed, as $\mu(T_{1,0}(\partial\Omega_n)) \subseteq T_{1,0}(\partial\Omega_n)$,

$$\mu_{\bar{\alpha}}^{\beta} = d_{\bar{\alpha}}^{\beta} - 2i\zeta_{\alpha} d_{\bar{1}}^{\beta},$$

$$2i\mu_{\bar{\alpha}}^{\beta}\bar{\zeta}_{\beta} = d_{\bar{\alpha}}^{1} - 2i\zeta_{\alpha} d_{\bar{1}}^{1},$$

where $\zeta_{\alpha} = \zeta^{\alpha}$. Consequently $Z = a^{j}(\partial/\partial\bar{\zeta}^{j} - d_{\bar{j}}^{k}\partial/\partial\zeta^{k})$ is tangent to $\partial\Omega_n \cap \{h \neq 0\}$ if and only if $a^{1} = -2i\zeta_{\alpha} a^{\alpha}$, i.e. $Z \in \Gamma^{\infty}(B_b)$. q.e.d.

PROPOSITION 3. *Let* $F : \Omega_n \to \Omega_n$ *be a* C^{∞} *diffeomorphism, smooth up to the boundary, each of whose components* F^{j} *satisfies the PDE*

$$Z_{\bar{\alpha}} F^{j} = \mu_{\bar{\alpha}}^{\beta} Z_{\beta} F^{j}$$

in Ω_n, *for some* C^{∞} *functions* $\mu_{\bar{\alpha}}^{\beta} : \Omega_n \to \boldsymbol{C}$. *If F is a foliated map, i.e. it preserves the foliation* $\mathscr{F}_n$, *then for any* $\alpha > -1$ *there is* $f_{\alpha} \in C^{2}(\Omega_n)$, $f_{\alpha} \neq 0$ *everywhere, so that*

$$F^{*}\omega_{\alpha} \equiv f_{\alpha}\omega_{\alpha}, \ \mathrm{mod}\,\theta_{\alpha},\ d\rho_{\alpha}.$$

Proof. Set $V_{\alpha} = Z_{\alpha}(F^{j})\partial/\partial\zeta^{j}$ and $W_{\alpha} = Z_{\bar{\alpha}}(F^{j})\partial/\partial\zeta^{j}$. As

$$(dF)T(\mathscr{F}_n) = T(\mathscr{F}_n),$$

$$Z_{\alpha} - \mu_{\alpha}^{\bar{\beta}} Z_{\bar{\beta}} \in T(\mathscr{F}_n) \otimes \boldsymbol{C},$$

$$W_{\alpha} = \mu_{\bar{\alpha}}^{\beta} V_{\beta},$$

(where $\mu_{\alpha}^{\bar{\beta}} = \overline{\mu_{\bar{\alpha}}^{\beta}}$) one has

$$(dF)(Z_{\alpha} - \mu_{\alpha}^{\bar{\beta}} Z_{\bar{\beta}}) = V_{\alpha} - \mu_{\alpha}^{\bar{\beta}} W_{\beta} \in T_{1,0}(\mathscr{F}_n).$$

Note that

$$H(\mathscr{F}_n) \otimes \boldsymbol{C} = \mathrm{Re}\{B_b \oplus \overline{B_b}\}$$

and

$$(F^{*}\theta_{\alpha})\overline{B}_b \subseteq \theta_{\alpha}(T_{1,0}(\mathscr{F}_n)) = 0$$

hence

$$F^{*}\theta_{\alpha} = a\theta_{\alpha} + b\,d\rho_{\alpha},$$

for some C^∞ functions $a, b : \Omega_n \to \boldsymbol{R}$. Here $\theta_\alpha = (i/2)(\bar{\partial} - \partial)\rho_\alpha$. Also $\rho_\alpha \circ F = \lambda \rho_\alpha$ for some $\lambda \in C^2(\bar{\Omega}_n)$, $\lambda > 0$ everywhere. Next, one may use

$$\omega_\alpha = (n + 1 + \alpha)\left\{\frac{d\theta_\alpha}{\rho_\alpha} - \frac{d\rho_\alpha \wedge \theta_\alpha}{\rho_\alpha^2}\right\}$$

to conclude that

$$F^*\omega_\alpha = \frac{a}{\lambda}\omega_\alpha + \frac{n+1+\alpha}{\lambda \rho_\alpha}((da - a d \log \lambda) \wedge \theta_\alpha + (db - b d \log \lambda) \wedge d\rho_\alpha).$$

Finally $a \neq 0$ everywhere (for if $a(z_0) = 0$ at some $z_0 \in \Omega_n$ then

$$\theta_{\alpha, F(z_0)}(d_{z_0}F) = b(z_0)\, d_{z_0}\rho_\alpha,$$

i.e. $(d_{z_0}F)T(\mathscr{F}_n)_{z_0} \subseteq H(\mathscr{F}_n)_{F(z_0)}$, a contradiction).

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