THE DEFORMATION OF HARMONIC MAPS
GIVEN BY THE CLIFFORD TORI

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Introduction

The purpose of this paper is to provide some new results on deformations for harmonic maps. Let ϕ be a harmonic map of a compact Riemannian manifold M into a Riemannian manifold N. A one-parameter family $\phi(t)$ of harmonic maps such that $\phi(0)=\phi$ is called a *harmonic deformation of ϕ*. Then each $\phi(t)$ satisfies the harmonic map equations:

$$(0.1) \qquad \tau(\phi(t))\equiv 0,$$

where $\tau(\phi)$ denotes the tension field of ϕ. By taking a derivative of the equation (0.1) at $t=0$, we have the equation

$$(0.2) \qquad \frac{d}{dt}\tau(\phi(t))\Big|_{t=0}\equiv \mathcal{I}_\phi(\dot\phi)=0, \quad \dot\phi\in C^\infty(\phi^{-1}TN).$$

Here $\mathcal{I}_\phi$ denotes the Jacobi operator of the energy functional. If a section $v\in C^\infty(\phi^{-1}TN)$ of $\phi^{-1}TN$ satisfies the equation (0.2), then it is called an *infinitesimal harmonic deformation* (or a *harmonic i-deformation*) *of ϕ*. We denote by HID(ϕ) the vector space of all harmonic i-deformations of ϕ. The space HID(ϕ) just coincides with the vector space Ker$\mathcal{I}_\phi$ of all Jacobi fields of ϕ. If $v\in$HID(ϕ) generates harmonic deformations, then v is said to be *integrable*. Let Harm(M, N) denote the space of all harmonic maps of M into N. From the point of view of the deformation theory of harmonic maps, the following are fundamental problems;

    (1) to ask whether or not all harmonic i-deformations of ϕ are integrable,

    (2) to make its cause clear if an harmonic i-deformation which is not integrable appears,

    (3) to investigate the structure of a neighborhood in Harm(M, N) around ϕ,

    (4) to determine the connected component in Harm(M, N) containing ϕ and to examine its compactness, if it is noncompact, to construct its natural compactification.

Because of the finiteness of the dimension of Ker$\mathcal{I}_\phi$, we know that Harm(M,

N) is locally finite dimensional (cf. [S]). For instance, Sunada [S] showed that Harm(M, N) is a smooth submanifold of $L_k^2(M, N)(k > m/2, \ m = \dim M)$ with tangent space $T_\phi \operatorname{Harm}(M, N) = \operatorname{Ker}\mathcal{T}_\phi$ if N is a complete locally symmetric space with non-positive sectional curvature. Certain special harmonic deformations were studied in [T]. See also [EL], [K1] for other references of deformation theory for harmonic maps.

Now harmonic maps of Riemann surfaces Σ into a compact Lie group G or a compact symmetric space G/K are known to have several nice properties. We have come to understand the construction and the classification of such maps, moreover, their moduli space. Hitchin [H] showed that the equations for a harmonic map from a 2-torus T^2 into a 3-sphere S^3 reduce in the geometry of a spectral curve, and hence constructed the moduli space of such maps algebraic geometrically. However, it seems to be difficult to investigate the deformations for harmonic maps in general.

In this paper, to actualize the problems, we deal with harmonic maps defined by using the Clifford tori in S^3 and classify completely harmonic i-deformations of these maps. As we know, they are simplest and most fundamental examples of harmonic maps of T^2 into S^3. Explicitly, they are defined by $\varphi_s \colon T^2 \to S^3 \cong SU(2)(s \in \mathbf{R})$, where

$$(0.3) \qquad \varphi_s(e^{it_1}, e^{it_2}) = \begin{bmatrix} \cos s \cdot e^{it_1} & \sin s \cdot e^{it_2} \\ -\sin s \cdot e^{-it_2} & \cos s \cdot e^{-it_1} \end{bmatrix},$$

with $t_1, t_2 \in \mathbf{R}$. Here we endow T^2 with a conformal structure of a square. Let $\Phi_s := i \circ \varphi_s \colon T^2 \to \mathbf{R}^4$, where $i \colon S^3 \hookrightarrow \mathbf{R}^4$ denotes the standard inclusion. Then we see that each φ_s is harmonic, because of

$$(0.4) \qquad \Delta \Phi_s = -\Big(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2}\Big) \Phi_s = \Phi_s (= |d\Phi_s|^2 \Phi_s).$$

Up to isometry of S^3, it is sufficient for us to deal with harmonic maps φ_s with $0 \leq s \leq \pi/4$. Throughout this paper, we assume that $0 \leq s \leq \pi/4$ unless otherwise specified. The images of φ_s with $s \neq 0$ are the Clifford tori in S^3 with constant mean curvature. In particular, the image of $\varphi_{\pi/4}$ is the Clifford minimal torus in S^3. On the other hand, we note that when $s=0$, the image of φ_0 becomes a great circle in S^3. Since the group $SO(4)$ acts on S^3 as an identity component isometry group of S^3, it induces *trivial* harmonic deformations.

This paper is organized as follows. In Section 1, we consider the harmonic i-deformations in general case. The notion of infinitesimal harmonic deformations of higher order is also introduced. The notions are analogous to that of infinitesimal Einstien deformations in [K2], [K3]. Moreover, we treat the case of a harmonic map of Σ into G.

In Section 2, all harmonic i-deformations of φ_s are determined. Then an interesting jump phenomena of the dimensions of the vector spaces HID(φ_s) will

occur at $s=0$, $\pi/6$. In order to make the cause of this phenomena clear, we examine the behaviour of negative eigenvalues of Jacobi operator $\mathcal{I}_{\varphi_s}$. The index for each φ_s will also be determined.

In Section 3, we classify harmonic i-deformations of φ_s. However, in cases φ_0 and $\varphi_{\pi/6}$, there exist some harmonic i-deformations whose integrability can not be easily decided. Therefore, we shall examine whether or not they are integrable up to second order, further, up to third order. We prove the following.

PROPOSITION 3.4, THEOREMS 3.6, 3.10.

(1) *For $\varphi_s(s \neq 0, \pi/6)$, all harmonic i-deformations of φ_s are integrable.*

(2) *For $\varphi_{\pi/6}$, there exist harmonic i-deformations of $\varphi_{\pi/6}$ which are not integrable up to second order, and hence are not integrable.*

(3) *For φ_0, there exist harmonic i-deformations of φ_0 which are integrable up to second order but not up to third order, and hence are not integrable.*

The results in Section 4 are the implications from those obtained in previous sections. It follows that for $s \neq 0$ a neighborhood in $\mathrm{Harm}(T^2, S^3)$ around φ_s, even around $\varphi_{\pi/6}$ at which nonintegrable harmonic i-deformations occur, becomes a 7-dimensional smooth manifold. On the other hand, the space $\mathrm{Harm}(T^2, S^3)$ has a singular point at φ_0. Finally, we obtain the following.

THEOREM 4.2. *The set of harmonic maps from T^2 into S^3 obtained by acting the group $SO(4)$ on the union of four families $\{\varphi_s^\omega\} \cup \{\tilde{\varphi}_s^\omega\} \cup \{\hat{\varphi}_s^\omega\} \cup \{\bar{\varphi}_s^\omega\}$ with $0 \leq s \leq 2\pi$ becomes a connected component in $\mathrm{Harm}(T^2, S^3)$ containing φ_s. Hence, the connected component is compact.*

§1. The infinitesimal harmonic deformation

We study the infinitesimal harmonic deformation (or the harmonic i-deformation). Let (M, g) be a compact Riemannian manifold of dimension m and (N, h) a Riemannian manifold. First we deal with a harmonic map $\phi : M \to N$. The tension field $\tau(\phi)$ of ϕ is defined by $\tau(\phi) = \sum_{i=1}^m (\nabla d\phi)(e_i, e_i)$, where $\{e_i\}$ is an orthonormal basis on M. Then the harmonic map equations becomes $\tau(\phi) \equiv 0$.

DEFINITION. A one-parameter family $\phi(t)$ of harmonic maps with $\phi(0) = \phi$ is called a *harmonic deformation of ϕ*.

DEFINITION. A harmonic deformation $\phi(t)$ of ϕ is said to be *trivial* if there exist isometries $\sigma_M(t)$, $\sigma_N(t)$ of M, N with $\sigma_M(0) = \mathrm{id}_M$, $\sigma_N(0) = \mathrm{id}_N$ such that $\phi(t) = \sigma_N(t)^{-1} \circ \phi \circ \sigma_M(t)$.

Assume that $\phi(t)$ is a deformation of ϕ. If we express the Taylor expansion of $\tau(\phi(t))$ as

(1.1) $$\tau(\phi(t))=c_0(\phi)+tc_1(\dot{\phi})+\frac{t^2}{2!}c_2(\dot{\phi},\ddot{\phi})+\frac{t^3}{3!}c_3(\dot{\phi},\ddot{\phi},\dddot{\phi})+\cdots,$$

where

$$\dot{\phi}=\left.\frac{\partial\phi(t)}{\partial t}\right|_{t=0},\quad \ddot{\phi}=\left.\nabla^N_{\partial/\partial t}\frac{\partial\phi(t)}{\partial t}\right|_{t=0},\quad \dddot{\phi}=\left.\nabla^N_{\partial/\partial t}\nabla^N_{\partial/\partial t}\frac{\partial\phi(t)}{\partial t}\right|_{t=0}\in C^\infty(\phi^{-1}TN),$$

then the equation $\tau(\phi(t))=0$ implies the following system of the equations:

$$c_0(\phi)(=\tau(\phi))\equiv0,\quad c_1(\dot{\phi})=0,\quad c_2(\dot{\phi},\ddot{\phi})=0,\quad c_3(\dot{\phi},\ddot{\phi},\dddot{\phi})=0,\ \ldots.$$

By the direct computation, we have the following.

LEMMA 1.1. *Under the above situation, for a harmonic map $\phi: M\to N$, we have*

(1.2) $$c_1(\dot{\phi})=-\mathcal{I}_\phi(\dot{\phi}),$$

(1.3) $$c_2(\dot{\phi},\ddot{\phi})=-\mathcal{I}_\phi(\ddot{\phi})+4\sum_{i=1}^m R^N(\dot{\phi},d\phi(e_i))(\nabla^\phi_{e_i}\dot{\phi})$$
$$+[\text{terms of }\nabla R^N]$$

(1.4) $$c_3(\dot{\phi},\ddot{\phi},\dddot{\phi})=-\mathcal{I}_\phi(\dddot{\phi})+6\sum_{i=1}^m R^N(\ddot{\phi},d\phi(e_i))(\nabla^\phi_{e_i}\dot{\phi})$$

$$+6\sum_{i=1}^m R^N(\dot{\phi},d\phi(e_i))(\nabla^\phi_{e_i}\ddot{\phi})+4\sum_{i=1}^m R^N(\dot{\phi},\nabla^\phi_{e_i}\dot{\phi})(\nabla^\phi_{e_i}\dot{\phi})$$

$$+4\sum_{i=1}^m R^N(\dot{\phi},d\phi(e_i))(R^N(\dot{\phi},d\phi(e_i))\dot{\phi})$$

$$+[\text{terms of }\nabla R^N,\nabla\nabla R^N].$$

Here

(1.5) $$\mathcal{I}_\phi v=(\nabla^\phi)^*(\nabla^\phi)v-\sum_{i=1}^m R^N(v,d\phi(e_i))d\phi(e_i),\quad v\in C^\infty(\phi^{-1}TN)$$

is the Jacobi operator of ϕ, where $(\nabla^\phi)^*(\nabla^\phi)$ denotes the rough Laplacian with respect to the connection ∇^ϕ.

Now let us review here the Jacobi operator $\mathcal{I}_\phi$ in the above lemma from the point of view of the variational problem. It is a self-adjoint linear elliptic differential operator, which appears in the second variational formula of the energy for a harmonic map ϕ. We have $C^\infty(\phi^{-1}TN)=\oplus_\lambda V_\lambda$, where V_λ is the eigenspace of $\mathcal{I}_\phi$ with eigenvalue λ. Then the *nullity* and *index* of ϕ are defined by

(1.6)   nullity(ϕ)=dim V_0=dim Ker $\mathcal{T}_\phi$,

(1.7)   index(ϕ)= $\sum\limits_{\lambda<0}$ dim V_λ,

respectively. We note that each number is finite. A harmonic map ϕ is said to be *stable* if index(ϕ)=0.

Then we return to the discussion of harmonic i-deformations of ϕ.

DEFINITION. Let $\phi: M \to N$ be a harmonic map. We call that

(1) a section v of $\phi^{-1}TN$ *is a harmonic i-deformation of* ϕ if $c_1(v)=0$;

(2) a pair (v, w) of sections of $\phi^{-1}TN$ *is a harmonic i-deformation of second order of* ϕ if v is a harmonic i-deformation of ϕ and $c_2(v, w)=0$;

(3) a triple (v, w, z) of sections of $\phi^{-1}TN$ is a *harmonic i-deformation of third order of* ϕ if (v, w) is a harmonic i-deformation of second order of ϕ and $c_3(v, w, z)=0$.

DEFINITION. A harmonic i-deformation v of ϕ is said to be *trivial* if v is of the form of a linear combination of $d\phi(X)$ and $Y \circ \phi$. Here X is a Killing vector field on M if dim $M \neq 2$, or a conformal vector field if dim $M=2$, and Y is a Killing vector filed on N.

We denote by HID(ϕ) the vector space of all harmonic i-deformations of ϕ. Then it follows that

(1.8)   dim HID(ϕ)=nullity(ϕ).

DEFINITION. Let v be a harmonic i-deformation of a harmonic map ϕ: $M \to N$.

(1) If there exists $w \in C^\infty(\phi^{-1}TN)$ such that (v, w) becomes a harmonic i-deformation of second order of ϕ, then v is said to be *integrable up to second order*.

(2) If there exist $w, z \in C^\infty(\phi^{-1}TN)$ such that (v, w, z) becomes a harmonic i-deformation of third order of ϕ, then v is said to be *integrable up to third order*.

DEFINITION. Let $\phi: M \to N$ be a harmonic map. A harmonic i-deformation v of ϕ is said to be *integrable* if there exists a harmonic deformation $\phi(t)$ of ϕ such that

$$v = \frac{\partial \phi(t)}{\partial t}\bigg|_{t=0}.$$

Obviously, the following holds.

LEMMA 1.2. *If a harmonic i-deformation v is not integrable up to second or third order, then v is not integrable.*

Next we deal with a harmonic map ϕ from a Riemann surface Σ into a

Lie group G with Lie algebra $\mathfrak{g}$.   Indeed, we are more intereseted in harmonic i-deformations of such maps.   In this case, we can describe harmonic i-deformations Lie algebraically.   The harmonic map equation becomes

$$d*\theta = 0.$$

Here $\theta := \phi^{-1}d\phi$ denotes the pull-back of Maurer-Cartan form of G, which is a 1-form with values in $\phi^{-1}TG = \mathfrak{g}$.   Let $\phi(t)$ be a harmonic deformation of ϕ.   Put $\theta_t = \phi(t)^{-1}d\phi(t)$.   Then it is convenient to use the following expansion instead of (1.1):

$$(1.9) \qquad d*\theta_t = a_0 + ta_1(\xi) + \frac{t^2}{2!}a_2(\xi, \dot{\xi}) + \frac{t^3}{3!}a_3(\xi, \dot{\xi}, \ddot{\xi}) + \cdots,$$

in terms of ξ, $\dot{\xi} = (d\xi_t/dt)|_{t=0}$ and $\ddot{\xi} = (d^2\xi_t/dt^2)|_{t=0}$, where

$$\xi_t = \phi(t)^{-1}\frac{\partial\phi(t)}{\partial t} \in C^\infty(\underline{\mathfrak{g}})$$

with $\xi_0 = \xi$.   Note that the order of derivative of ξ in (1.9) is one order less than that of ϕ in (1.1).   We see that $a_0 = d*\theta \equiv 0$.   The Jacobi operator in the following lemma is of the form

$$(1.10) \qquad \mathcal{I}_\phi(\zeta) = -*d*d_\theta\zeta, \qquad \text{for} \quad \zeta \in C^\infty(\mathfrak{g}),$$

with $d_\theta = d + \mathrm{ad}\theta$.

LEMMA 1.3.   *Under the above situation, we have*

$$(1.11) \qquad a_1(\xi) = -*\mathcal{I}_\phi(\xi),$$

$$(1.12) \qquad a_2(\xi, \dot{\xi}) = -*\mathcal{I}_\phi(\dot{\xi}) - P(\xi),$$

$$(1.13) \qquad a_3(\xi, \dot{\xi}, \ddot{\xi}) = -*\mathcal{I}_\phi(\ddot{\xi}) - Q(\xi, \dot{\xi}),$$

putting

$$(1.14) \qquad P(\xi) := [[*\theta, \xi] \wedge d\xi],$$

$$(1.15) \quad Q(\xi, \dot{\xi}) := [*d\xi \wedge d\dot{\xi}] + 2[[*\theta, \xi] \wedge d\dot{\xi}] + [[*\theta, \dot{\xi}] \wedge d\xi] + [[*d_\theta\xi, \xi] \wedge d\xi].$$

Proof.   Since $(d\theta_t/dt) = d_{\theta_t}\xi_t$, we get

$$(1.16) \qquad \frac{d}{dt}(d*\theta_t) = d*d_{\theta_t}\xi_t,$$

and hence, at $t=0$, $a_1(\xi) = (d/dt)(d*\theta_t)|_{t=0} = -*\mathcal{I}_\phi(\xi)$.   Using

$$\frac{d}{dt}(d_{\theta_t}\xi_t) = d_{\theta_t}\frac{d\xi_t}{dt} + [d_{\theta_t}\xi_t, \xi_t],$$

we have

(1.17) $\dfrac{d}{dt}(d*d_{\theta_t}\xi_t)=d*d_{\theta_t}\dfrac{d\xi_t}{dt}+[d*d_{\theta_t}\xi_t,\ \xi_t]-[*d_{\theta_t}\xi_t\wedge d\xi_t\rfloor$

$$=d*d_{\theta_t}\frac{d\xi_t}{dt}-[*d_{\theta_t}\xi_t\wedge d\xi_t]\qquad\text{because }d*d_{\theta_t}\xi_t=0$$

$$=d*d_{\theta_t}\frac{d\xi_t}{dt}-[[*\theta_t,\ \xi_t]\wedge d\xi_t]\qquad\text{because }[*d\xi_t\wedge d\xi_t]=0.$$

By putting $t=0$, we have (1.12). Moreover, we compute

(1.18) $\dfrac{d}{dt}\Big(d*d_{\theta_t}\dfrac{d\xi_t}{dt}-[[*\theta_t,\ \xi_t]\wedge d\xi_t]\Big)$

$$=d*d_{\theta_t}\frac{d^2\xi_t}{dt^2}-\Big[*d_{\theta_t}\xi_t\wedge d\Big(\frac{d\xi_t}{dt}\Big)\Big]$$

$$-[[*d_{\theta_t}\xi_t,\ \xi_t]\wedge d\xi_t]-\Big[\Big[*\theta_t,\ \frac{d\xi_t}{dt}\Big]\wedge d\xi_t\Big]-\Big[[*\theta_t,\ \xi_t]\wedge d\Big(\frac{d\xi_t}{dt}\Big)\Big].$$

Hence at $t=0$,

$$\frac{d}{dt}\Big(d*d_{\theta_t}\frac{d\xi_t}{dt}-[[*\theta_t,\ \xi_t]\wedge d\xi_t]\Big)\Big|_{t=0}$$

$$=d*d_\theta\ddot{\xi}-[*d_\theta\xi\wedge d\dot{\xi}]$$

$$-[[*d_\theta\xi,\ \xi]\wedge d\xi]-[[*\theta,\ \dot{\xi}]\wedge d\xi]-[[*\theta,\ \xi]\wedge d\dot{\xi}]$$

$$=-*\mathcal{T}_\phi(\ddot{\xi})$$

$$-([*d\xi\wedge d\dot{\xi}]+2[[*\theta,\ \xi]\wedge d\dot{\xi}]+[[*\theta,\ \dot{\xi}]\wedge d\xi]+[[*d_\theta\xi,\ \xi]\wedge d\xi]).$$

$\square$

DEFINITION. For a harmonic map $\phi:\Sigma\to G$, we call that

(1) a section ξ of $\underline{g}$ is a *harmonic i-deformation of* ϕ if $a_1(\xi)=0$;

(2) a pair (ξ,η) of sections of $\underline{g}$ is a *harmonic i-deformation of second order of* ϕ if ξ is a harmonic i-deformation of ϕ and $a_2(\xi,\eta)=0$;

(3) a triple (ξ,η,γ) of sections of $\underline{g}$ is a *harmonic i-deformation of third order of* ϕ if (ξ,η) is a harmonic i-deformation of second order of ϕ and $a_3(\xi,\eta,\gamma)=0$.

DEFINITION. Let ξ be a harmonic i-deformation of a harmonic map $\phi:\Sigma\to G$.

(1) If there exists $\eta\in C^\infty(\underline{g})$ such that (ξ,η) becomes a harmonic i-deformation of second order of ϕ, then ξ is said to be *integrable up to second order*.

(2) If there exist $\eta,\gamma\in C^\infty(\underline{g})$ such that (ξ,η,γ) becomes a harmonic i-deformation of third order of ϕ, then ξ is said to be *integrable up to third order*.

DEFINITION.   Let $\phi: \Sigma \to G$ be a harmonic map.   A harmonic i-deformation ξ of ϕ is said to be *integrable* if there exists a harmonic deformation $\phi(t)$ of ϕ such that

$$\xi = \phi^{-1} \frac{\partial \phi(t)}{\partial t}\bigg|_{t=0}.$$

By Lemma 1.3, the following is obvious.

LEMMA 1.4.   *Let* $\phi: \Sigma \to G$ *be a harmonic map and* ξ *a harmonic i-deformation of* ϕ.   *Then*

(1) ξ *is integrable up to second order if and only if* $*P(\xi)$ *is orthogonal to* $\mathrm{Ker}\,\mathcal{T}_\phi$.

(2) ξ *is integrable up to third order if and only if* $*Q(\xi, \eta)$ *is orthogonal to* $\mathrm{Ker}\,\mathcal{T}_\phi$ *for some* η *such that* $a_2(\xi, \eta)=0$.

Here we use the L^2-inner product on $C^\infty(\mathfrak{g})$.

§ 2.   The infinitesimal harmonic deformations of φ_s

Let us focus on harmonic maps $\varphi_s: T^2 \to SU(2)$ defined by using the Clifford tori in (0.3).   We begin by determining completely the harmonic i-deformations of φ_s.   In this section, it is sufficient to take $0 \leq s \leq \pi/4$.   From (1.10) and (1.11), the equation $a_1(\xi)=0$ of the harmonic i-deformations becomes

$$(2.1) \qquad \frac{\partial^2 \xi}{\partial t_1^2} + \frac{\partial^2 \xi}{\partial t_2^2} + \left[(\theta_s)_1, \frac{\partial \xi}{\partial t_1}\right] + \left[(\theta_s)_2, \frac{\partial \xi}{\partial t_2}\right] = 0, \qquad \text{for} \quad \xi \in C^\infty(\mathfrak{g}).$$

Here the pull-back $\theta_s = \varphi_s^{-1} d\varphi_s$ of Maurer-Cartan form of $G = SU(2)$ by φ_s is splitting as $\theta_s = (\theta_s)_1 dt_1 + (\theta_s)_2 dt_2$, where

$$(2.2) \qquad (\theta_s)_1 = i \begin{bmatrix} \cos^2 s & \cos s \sin s \cdot e^{-it_1} e^{it_2} \\ \cos s \sin s \cdot e^{it_1} e^{-it_2} & -\cos^2 s \end{bmatrix},$$

$$(2.3) \qquad (\theta_s)_2 = i \begin{bmatrix} -\sin^2 s & \cos s \sin s \cdot e^{-it_1} e^{-it_2} \\ \cos s \sin s \cdot e^{it_1} e^{-it_2} & \sin^2 s \end{bmatrix}.$$

PROPOSITION 2.1.   *The vector space* $\mathrm{HID}(\varphi_s)$ *of harmonic i-deformations of* φ_s *becomes*

$$(2.4) \qquad \mathrm{HID}(\varphi_s) = \begin{cases} \sum_{i=1}^7 \boldsymbol{R}\xi_i, & \text{if} \quad s \neq 0, \ \pi/6, \\ \sum_{i=1}^9 \boldsymbol{R}\xi_i, & \text{if} \quad s = \pi/6, \\ \sum_{i=1}^7 \boldsymbol{R}\xi_i + \sum_{i=8}^9 \boldsymbol{R}\tilde{\xi}_i, & \text{if} \quad s = 0. \end{cases}$$

Here $\xi_i\,(i=1, \ldots, 9)$ *and* $\tilde{\xi}_i\,(i=8, 9)$ *are defined as*

$$\xi_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\xi_4 = i \begin{bmatrix} -\tan s \cdot (e^{-it_1}e^{-it_2} + e^{it_1}e^{it_2}) & e^{-2it_1} - \tan^2 s \cdot e^{2it_2} \\ e^{2it_1} - \tan^2 s \cdot e^{-2it_2} & \tan s \cdot (e^{-it_1}e^{-it_2} + e^{it_1}e^{it_2}) \end{bmatrix},$$

$$\xi_5 = \begin{bmatrix} -\tan s \cdot (e^{-it_1}e^{-it_2} - e^{it_1}e^{it_2}) & e^{-2it_1} + \tan^2 s \cdot e^{2it_2} \\ -e^{2it_1} - \tan^2 s \cdot e^{-2it_2} & \tan s \cdot (e^{-it_1}e^{-it_2} - e^{it_1}e^{it_2}) \end{bmatrix},$$

$$\xi_6 = i \begin{bmatrix} 0 & e^{-it_1}e^{it_2} \\ e^{it_1}e^{-it_2} & 0 \end{bmatrix}, \quad \xi_7 = \begin{bmatrix} 0 & e^{-it_1}e^{it_2} \\ -e^{it_1}e^{-it_2} & 0 \end{bmatrix},$$

$$\xi_8 = i \begin{bmatrix} \tan s \cdot (e^{-it_1} + e^{it_1}) & e^{it_2} - \tan^2 s \cdot e^{-2it_1}e^{it_2} \\ e^{-it_2} - \tan^2 s \cdot e^{2it_1}e^{-it_2} & -\tan s \cdot (e^{-it_1} + e^{it_1}) \end{bmatrix},$$

$$\xi_9 = \begin{bmatrix} \tan s \cdot (e^{-it_1} + e^{it_1}) & e^{it_2} + \tan^2 s \cdot e^{-2it_1}e^{it_2} \\ -e^{-it_2} - ant^2 s \cdot e^{2it_1}e^{-it_2} & -\tan s \cdot (e^{-it_1} + e^{it_1}) \end{bmatrix},$$

$$\tilde{\xi}_8 = i \begin{bmatrix} 0 & e^{-it_1}e^{-it_2} \\ e^{it_1}e^{it_2} & 0 \end{bmatrix}, \quad \tilde{\xi}_9 = \begin{bmatrix} 0 & e^{-it_1}e^{-it_2} \\ -e^{it_1}e^{it_2} & 0 \end{bmatrix}.$$

Proof. By regarding $\xi = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ as an $\mathfrak{su}(2)^C (=\mathfrak{sl}(2, C))$-valued function on T^2, we express a Fourier expansion of ξ as $\xi = \sum_{k, l \in Z} \xi_{kl} e^{ikt_1} e^{ilt_2}$, $\xi_{kl} = \begin{bmatrix} a_{kl} & b_{kl} \\ c_{kl} & -a_{kl} \end{bmatrix}$. Then the equation (2.1) is equivalent to the system of linear algebraic equations with unknowns a_{kl}, b_{kl}, c_{kl} and parameter s:

$$(2.5) \quad \begin{cases} (k^2 + l^2)a_{kl} - (k+l)\cos s \sin s \cdot (b_{k-1 l+1} - c_{k+1 l-1}) = 0, \\ 2(k+l)\cos s \sin s \cdot a_{k+1 l-1} - \{(k^2 + l^2) + 2(k\cos^2 s - l\sin^2 s)\} b_{kl} = 0, \\ 2(k+l)\cos s \sin s \cdot a_{k-1 l+1} + \{(k^2 + l^2) - 2(k\cos^2 s - l\sin^2 s)\} c_{kl} = 0. \end{cases}$$

We need a long calculation to solve the equations (2.5). Then setting

$$a_0 := a_{00}, \quad b_0 := b_{00}, \quad c_0 := c_{00}, \quad b_1 := b_{-11}, \quad c_1 := c_{1-1},$$

$$b_2 := b_{-20}, \quad c_2 := c_{20}, \quad b_3 := b_{-1-1}, \quad c_3 := c_{11},$$

$$b_4 := b_{02} \quad \text{and} \quad c_4 := c_{0-2}$$

for simplicity, we obtain that each a, b, c is of the following form;

if $s \neq 0, \pi/6,$

(2.6)
$$\begin{cases} a = a_0 - b_2 \tan s \cdot e^{-it_1} e^{-it_2} - c_2 \tan s \cdot e^{it_1} e^{it_2} \\ b = b_0 + b_2 e^{-2it_1} + b_1 e^{-it_1} e^{it_2} - c_2 \tan^2 s \cdot e^{2it_2} \\ c = c_0 + c_2 e^{2it_1} + c_1 e^{it_1} e^{-it_2} - b_2 \tan^2 s \cdot e^{-2it_2}, \end{cases}$$

if $s = \pi/6$,

(2.7)
$$\begin{cases} a = a_0 - b_2 \tan s \cdot e^{-it_1} e^{-it_2} - c_2 \tan s \cdot e^{it_1} e^{it_2} + c_4 \tan s \cdot e^{-it_1} + b_4 \tan s \cdot e^{it_1} \\ b = b_0 + b_2 e^{-2it_1} + b_1 e^{-it_1} e^{it_2} - c_2 \tan^2 s \cdot e^{2it_2} - c_4 \tan^2 s \cdot e^{-2it_1} e^{it_2} + b_4 e^{it_2} \\ c = c_0 + c_2 e^{2it_1} + c_1 e^{it_1} e^{-it_2} - b_2 \tan^2 s \cdot e^{-2it_2} - b_4 \tan^2 s \cdot e^{2it_1} e^{-it_2} + c_4 e^{-it_2}, \end{cases}$$

if $s = 0$,

(2.8)
$$\begin{cases} a = a_0 \\ b = b_0 + b_2 e^{-2it_1} + b_1 e^{-it_1} e^{it_2} + b_3 e^{-it_1} e^{-it_2} \\ c = c_0 + c_2 e^{2it_1} + c_1 e^{it_1} e^{-it_2} + c_3 e^{it_1} e^{it_2}. \end{cases}$$

Using the conditions $a \in \sqrt{-1}\mathbf{R}$ and $b = -\bar{c}$ of $\xi \in \mathfrak{su}(2)$, we can show that $\xi \in$ HID(φ_s) can be written as a linear combination of ξ_i or $\tilde{\xi}_i$. Hence we complete the proof of Proposition 2.1.   □

Consequently, we have determined the dimension of each vector space HID(φ_s), namely, the nullity of φ_s as a harmonic map from (1.8). Then we encounter an interesting jump phenomena of the numbers.

COROLLARY 2.2. *We have*

$$\dim \mathrm{HID}(\varphi_s) = \begin{cases} 9, & \text{if } s = 0, \ \pi/6, \\ 7, & \text{otherwise}, \end{cases}$$

(*see Figuer* 2.1(i)).

To make the cause of this jump phenomena at $s = 0$ and $\pi/6$ clear, we next examine the behaviour of the negative eigenvalues of $\mathcal{J}_{\varphi_s}$.

PROPOSITION 2.3. *Take s with $0 \leq s \leq \pi/2$. Each Jacobi operator $\mathcal{J}_{\varphi_s}$ has negative eigenvalues*

$$\begin{cases} \lambda_1 := 1 - \sqrt{2(1 + \cos 2s)}, & \text{if } 0 \leq s \leq \pi/3, \\ \lambda_2 := 1 - \sqrt{2(1 - \cos 2s)}, & \text{if } \pi/6 \leq s \leq \pi/2, \end{cases}$$

with multiplicity 2(*see Figuer* 2.1(ii)). *Moreover, the eigenspace corresponding to λ_2 is spanned by ξ_8, ξ_9.*

Proof. Suppose $\lambda < 0$. Similar to the proof of Proposition 2.1, we find that

the equations $\mathcal{T}_{\varphi_s}(\xi)=\lambda\xi$ are equivalent to the system of linear algebraic equations with unknowns a_{kl}, b_{kl}, c_{kl}, and parameters s and λ:

$$(2.9) \quad \begin{cases} (k^2+l^2-\lambda)a_{kl}-(k+l)\cos s\sin s\cdot(b_{k-1l+1}-c_{k+1l-1})=0, \\ 2(k+l)\cos s\sin s\cdot a_{k+1l-1}-\{(k^2+l^2)+2(k\cos^2 s-l\sin^2 s)-\lambda\}b_{kl}=0, \\ 2(k+l)\cos s\sin s\cdot a_{k-1l+1}+\{(k^2+l^2)-2(k\cos^2 s-l\sin^2 s)-\lambda\}c_{kl}=0. \end{cases}$$

By a harder calculation than that for (2.5), we find that $a_{kl}=0$ for except (k, l) satisfying either (2.10) or (2.11) of the following:

$$(2.10) \qquad l=0 \quad \text{and} \quad \lambda^2-2k^2\lambda+k^4-2k^2(1-\cos 2s)=0$$

$$(2.11) \qquad l^2=1 \quad \text{and} \quad \lambda^2-2(k^2+1)\lambda+k^4+2k^2\cos 2s-1-2\cos 2s=0,$$

which are equivalent to

$$(2.12) \qquad l=0 \quad \text{and} \quad \lambda=k^2-\sqrt{2k^2(1-\cos 2s)}$$

$$(2.13) \qquad l^2=1 \quad \text{and} \quad \lambda=(k^2+1)-\sqrt{2(k^2-k^2\cos 2s+1+\cos 2s)},$$

respectively. Then it follows from $\lambda<0$ and $k\in\mathbf{Z}$ that

$$(2.14) \qquad (k, l, \lambda)=(\pm 1, 0, 1-\sqrt{2(1-\cos 2s)})$$

$$(2.15) \qquad (k, l, \lambda)=(0, \pm 1, 1-\sqrt{2(1+\cos 2s)}),$$

respectively. In particular, put $\lambda_2:=1-\sqrt{2(1-\cos 2s)}$. Using (2.9) again, we obtain that each a, b, c in $\xi=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ satisfying $\mathcal{T}_{\varphi_s}(\xi)=\lambda_2(\xi)$ is of the form:

$$\begin{cases} a=c_4\tan s\cdot e^{-it_1}+b_4\tan s\cdot e^{it_1} \\ b=-c_4\tan^2 s\cdot e^{-2it_1}e^{it_2}+b_4 e^{it_2} \\ c=-b_4\tan^2 s\cdot e^{2it_1}e^{-it_2}+c_4 e^{-it_2}. \end{cases} \qquad \square$$

By virtue of Proposition 2.3, it follows that the subspace of $\mathrm{HID}(\varphi_{\pi/6})$ which cause the jump phenomena of dimension 2 just coincides with the eigenspace corresponding to λ_2. The index of each φ_s can be also determined.

COROLLARY 2.4. *Take s with $0\leqq s\leqq\pi/2$. We have*

$$\mathrm{index}(\varphi_s)=\begin{cases} 2, & \text{if} \quad 0\leqq s\leqq\pi/6,\ \pi/3\leqq s\leqq\pi/2, \\ 4, & \text{if} \quad \pi/6<s<\pi/3, \end{cases}$$

(*see Figure* 2.1(iii)).

Remark. Proposition 2.3 implies that the harmonic maps from T^2 into 3-dimensional real projective space $\mathbf{R}P^3$ obtained by compositions of $\{\varphi_s\}$ and the

natural projection $S^3 \to RP^3$ are stable in the sense of index$=0$. Very recently Ejiri proves in [Ej] that the maps become non-zero homotopically energy minimizing. Thus our result supports Ejiri's one.
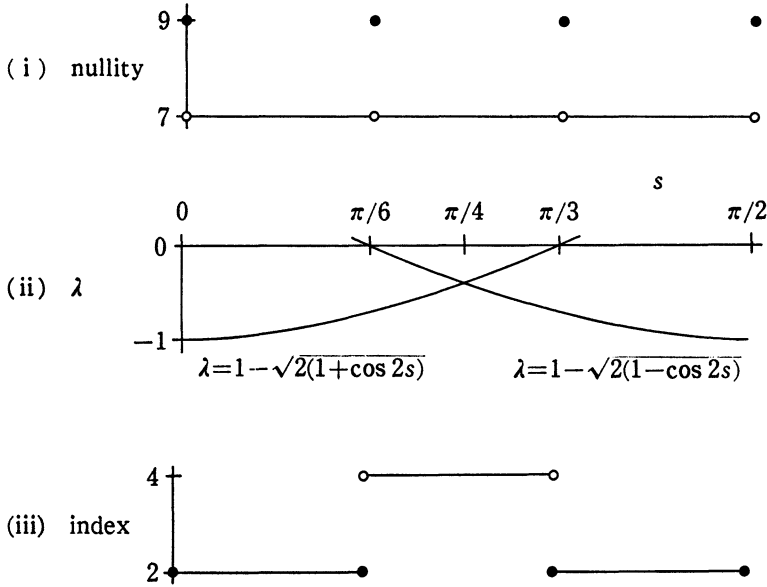

Figure 2.1.

§3. Classification of infinitesimal harmonic deformations

In this section, assume that $0 \leq s \leq \pi/4$. It is easy to check the following.

LEMMA 3.1. Let $f_{(a,b)} : T^2 \ni (e^{it_1}, e^{it_2}) \mapsto (e^{i(t_1+a)}, e^{i(t_2+b)}) \in T^2$ be a conformal transformation of T^2. Let R and L denote right and left translations of $SU(2)$, respectively. Then

$$\varphi_s \circ f_{(a,b)}(e^{it_1}, e^{it_2}) = (R_A \circ L_B \circ \varphi_s)(e^{it_1}, e^{it_2}),$$

where $A = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$, $B = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix} \in SU(2)$, putting $\alpha = (a+b)/2$, $\beta = (a-b)/2$.

LEMMA 3.2. Trivial harmonic i-deformations of φ_s span

$$\begin{cases} W_0 := \sum_{i=1}^6 R\xi_i, & \text{if } s \neq 0, \\ V_0 := \sum_{i=1}^5 R\xi_i, & \text{if } s = 0. \end{cases}$$

Proof. By Lemma 3.1, a trivial harmonic i-deformation of φ_s is of the form $Y \circ \varphi_s$, where Y is a Killing vector field on S^3. By a direct computation,

we obtain the statement.   □

Now let us examine harmonic i-deformations of φ_s which are not trivial.

3.1. The case of $s \neq 0, \pi/6$
We decompose as $\mathrm{HID}(\varphi_s) = W_0 \oplus W_1$, where $W_1 = \boldsymbol{R}\xi_7$.

LEMMA 3.3. *Let $s \neq 0, \pi/6$. Then a harmonic i-deformation of φ_s obtained by moving the parameter s spans W_1.*

Proof. We compute

$$\varphi_s^{-1} \frac{\partial \varphi_s}{\partial s} = \begin{bmatrix} \cos s \cdot e^{-it_1} & -\sin s \cdot e^{it_2} \\ \sin s \cdot e^{-it_2} & \cos s \cdot e^{it_1} \end{bmatrix} \begin{bmatrix} -\sin s \cdot e^{it_1} & \cos s \cdot e^{it_2} \\ -\cos s \cdot e^{-it_2} & -\sin s \cdot e^{-it_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{-it_1}e^{it_2} \\ -e^{it_1}e^{-it_2} & 0 \end{bmatrix} = \xi_7 .$$ □

Then we obtain the following.

PROPOSITION 3.4. *Let $s \neq 0, \pi/6$. Any element of $\mathrm{HID}(\varphi_s) = W_0 \oplus W_1$ is integrable.*

Proof. We see that harmonic i-deformations of φ_s, which are induced by compositions of harmonic maps obtained by moving the parameter s and isometries of $SU(2)$, span a direct sum $W_0 \oplus W_1$.   □

3.2. The case of $s = \pi/6$
We decompose as $\mathrm{HID}(\varphi_{\pi/6}) = W_0 \oplus W_1 \oplus W_2$, where $W_1 = \boldsymbol{R}\xi_7$, $W_2 = \sum_{i=8}^{9} \boldsymbol{R}\xi_i$. When $s = \pi/6$, the statement of Lemma 3.3 also holds. Hence we obtain the following similar to Proposition 3.4.

PROPOSITION 3.5. *Any element of $W_0 \oplus W_1 \subset \mathrm{HID}(\varphi_{\pi/6})$ is integrable.*

However, the integrability for an element of $\mathrm{HID}(\varphi_{\pi/6}) \backslash (W_0 \oplus W_1)$ can not be decided consequently. Then our next effort is to examine whether or not this element is integrable up to second order.

THEOREM 3.6. *Any element of $\mathrm{HID}(\varphi_{\pi/6}) \backslash (W_0 \oplus W_1)$ is not integrable up to second order. Hence, it can not be integrable.*

Proof. Let $\xi = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathrm{HID}(\varphi_{\pi/6}) \backslash (W_0 \oplus W_1)$ satisfying (2.7). Note that $b_4 (= -\bar{c}_4) \neq 0$. By Lemma 1.4(1), we check whether or not

$$*P(\xi) = -\Big[[(\theta_{\pi/6})_1, \xi], \frac{\partial \xi}{\partial t_1} \Big] - \Big[[(\theta_{\pi/6})_2, \xi], \frac{\partial \xi}{\partial t_2} \Big]$$

is orthogonal to $\mathrm{Ker}\, \mathcal{T}_{\varphi_{\pi/6}}$. Choose

$$\xi_0 := \begin{bmatrix} 0 & -e^{-it_1}e^{it_2} \\ e^{it_1}e^{-it_2} & 0 \end{bmatrix} \in \mathrm{Ker}\, \mathcal{T}_{\varphi_{\pi/6}}.$$

Then by long calculations, we obtain

$$\int_{T^2} \langle *P(\xi), \xi_0 \rangle_{\mathfrak{su}(2)} = -2 \Big(3 - \tan^2 \frac{\pi}{6} \Big) \tan \frac{\pi}{6} \times (2b_4 c_4) = \frac{32\sqrt{3}}{9} |b_4|^2 \neq 0.$$

Hence ξ is not integrable up to second order. The second statement follows from Lemma 1.2. $\square$

3.3. The case of $s=0$

We decompose as $\mathrm{HID}(\varphi_0) = V_0 \oplus V_1 \oplus V_2$, where $V_1 = \sum_{i=6}^{7} \boldsymbol{R}\xi_i$, $V_2 = \sum_{i=8}^{9} \boldsymbol{R}\tilde{\xi}_i$. Let us define two maps φ_s^ω, $\tilde{\varphi}_s^\omega \colon T^2 \to SU(2)$ $(\omega \in \boldsymbol{R})$ by

$$(3.1) \qquad \varphi_s^\omega(e^{it_1}, e^{it_2}) := \begin{bmatrix} \cos s \cdot e^{it_1} & \sin s \cdot e^{i(t_2+\omega)} \\ -\sin s \cdot e^{-i(t_2+\omega)} & \cos s \cdot e^{-it_1} \end{bmatrix},$$

$$(3.2) \qquad \tilde{\varphi}_s^\omega(e^{it_1}, e^{it_2}) := \begin{bmatrix} \cos s \cdot e^{it_1} & \sin s \cdot e^{-i(t_2+\omega)} \\ -\sin s \cdot e^{i(t_2+\omega)} & \cos s \cdot e^{-it_1} \end{bmatrix}.$$

Then we see that $\varphi_0^\omega = \tilde{\varphi}_0^\omega = \varphi_0$ and that φ_s^ω, $\tilde{\varphi}_s^\omega$ are harmonic maps in the same way to (0.4).

LEMMA 3.7. *Infinitesimal harmonic deformations of* φ_s^ω (*respectively,* $\tilde{\varphi}_s^\omega$), *which are obtained by fixing the parameter* ω *arbitrarily and moving the parameter* s, *span* V_1 (*respectively,* V_2).

Proof. We see that

$$(\varphi_s^\omega)^{-1} \frac{\partial \varphi_s^\omega}{\partial s} = \begin{bmatrix} \cos s \cdot e^{-it_1} & -\sin s \cdot e^{i(t_2+\omega)} \\ \sin s \cdot e^{-i(t_2+\omega)} & \cos s \cdot e^{it_1} \end{bmatrix} \begin{bmatrix} -\sin s \cdot e^{it_1} & \cos s \cdot e^{i(t_2+\omega)} \\ -\cos s \cdot e^{-i(t_2+\omega)} & -\sin s \cdot e^{-it_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{i\omega}e^{-it_1}e^{it_2} \\ -e^{-i\omega}e^{it_1}e^{-it_2} & 0 \end{bmatrix}.$$

By putting $\omega = 0$, $\pi/2$, it becomes ξ_7, ξ_6, respectively. Similarly, since

$$(\tilde{\varphi}_s^\omega)^{-1} \frac{\partial \tilde{\varphi}_s^\omega}{\partial s} = \begin{bmatrix} \cos s \cdot e^{-it_1} & -\sin s \cdot e^{-i(t_2+\omega)} \\ \sin s \cdot e^{i(t_2+\omega)} & \cos s \cdot e^{it_1} \end{bmatrix} \begin{bmatrix} -\sin s \cdot e^{it_1} & \cos s \cdot e^{-i(t_2+\omega)} \\ -\cos s \cdot e^{i(t_2+\omega)} & -\sin s \cdot e^{-it_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{-i\omega}e^{-it_1}e^{-it_2} \\ -e^{i\omega}e^{it_1}e^{it_2} & 0 \end{bmatrix},$$

putting $\omega = 0$, $3\pi/2$, it becomes $\tilde{\xi}_9$, $\tilde{\xi}_8$, respectively.   □

We discuss the integrability of harmonic i-deformations of φ_0. By Lemmas 3.2 and 3.7, we have the following.

PROPOSITION 3.8.   *Any element of* $V_0 \oplus V_1$, $V_0 \oplus V_2 \subset \mathrm{HID}(\varphi_0)$ *is integrable.*

Therefore our problem is to investigate whether or not an element of $\mathrm{HID}(\varphi_0) \setminus \{(V_0 \oplus V_1) \cup (V_0 \oplus V_2)\}$ is integrable.

THEOREM 3.9.   *Any element of* $\mathrm{HID}(\varphi_0)$ *is integrable up to second order.*

Proof.   Fix arbitrary $\xi = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathrm{HID}(\varphi_0)$ satisfying (2.8). Then we obtain that $\eta = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \in C^\infty(\mathfrak{g})$ with $A \in \sqrt{-1}\boldsymbol{R}$, $B = -\bar{C}$ is a solution to $*\mathscr{T}_{\varphi_0}(\eta) + P(\xi) = 0$ if and only if each A, B, C is of the form

$$(3.3) \quad \begin{cases} A = \bar{a}_0 - b_0 c_2 e^{2it_1} + b_2 c_0 e^{-2it_1} \\ \qquad - (b_3 c_2 + b_0 c_1)e^{it_1}e^{-it_2} + (b_3 c_0 + b_2 c_1)e^{-it_1}e^{-it_2} \\ \qquad - (b_1 c_2 + b_0 c_3)e^{it_1}e^{it_2} + (b_2 c_3 + b_1 c_0)e^{-it_1}e^{it_2} \\ B = \bar{b}_0 + \bar{b}_2 e^{-2it_1} + \bar{b}_3 e^{-it_1}e^{-it_2} + \bar{b}_1 e^{-it_1}e^{it_2} \\ C = \tilde{c}_0 + \tilde{c}_2 e^{2it_1} + \tilde{c}_3 e^{it_1}e^{it_2} + \tilde{c}_1 e^{it_1}e^{-it_2}, \end{cases}$$

with $\tilde{a}_0 \in \sqrt{-1}\boldsymbol{R}$, $\bar{b}_i = -\bar{\tilde{c}}_i \in \boldsymbol{C}$ $(i = 0, 1, 2, 3)$ chosen arbitrarily.   □

Unfortunately, Theorem 3.9 has not given any answer to our problem yet. Then let us proceed a more step.

THEOREM 3.10.   *Any element of* $\mathrm{HID}(\varphi_0) \setminus \{(V_0 \oplus V_1) \cup (V_0 \oplus V_2)\}$ *is not integrable up to third order.   Hence, it can not be integrable.*

Proof.   Let $\xi = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathrm{Ker}\mathscr{T}_{\varphi_0} \setminus \{(V_0 \oplus V_1) \cup (V_0 \oplus V_2)\}$ satisfying (2.8). Note that $b_1(= -\tilde{c}_1) \neq 0$ and $b_3(= -\tilde{c}_3) \neq 0$. Take also arbitrarily η such that (ξ, η) becomes a harmonic i-deformation of second order of φ_0, and then it satisfies (3.3). Our claim is to show $*Q(\xi, \eta)$ is not orthogonal to $\mathrm{Ker}\mathscr{T}_{\varphi_0}$. Choose

$$\tilde{\xi}_0 := \begin{bmatrix} 0 & b_3 e^{-it_1}e^{-it_2} + b_1 e^{-it_1}e^{it_2} \\ -\bar{b}_3 e^{it_1}e^{it_2} - \bar{b}_1 e^{it_1}e^{-it_2} & 0 \end{bmatrix} \in \mathrm{Ker}\mathscr{T}_{\varphi_0}.$$

Then by very long calculations, we can show

$$\int_{T^2} \langle *Q(\xi,\ \eta),\ \tilde{\xi}_0 \rangle_{\mathfrak{su}(2)} = -12b_1b_3(\bar{b}_3c_1+\bar{b}_1c_3)-12c_1c_3(-b_3b_1-b_1b_3)$$

$$= 48|b_1|^2|b_3|^2 \neq 0. \qquad \qquad \square$$

§ 4.   The structure of a neighborhood and the component in Harm(T^2, S^3)

Using the results in previous sections, we discuss the local structure of a neighborhood in Harm(T^2, S^3) around φ_s.

PROPOSITION 4.1.   *Let* $0 \leq s \leq \pi/4$. *For* $s \neq 0$, *a neighborhood in* Harm(T^2, S^3) *around* φ_s *becomes a 7-dimensional smooth manifold. On the other hand, the space* Harm(T^2, S^3) *has a singular point at* φ_0.

Remark.   Although nonintegrable harmonic i-deformations occur at $s=\pi/6$, a neighborhood in Harm(T^2, S^3) around $\varphi_{\pi/6}$ becomes a smooth manifold.

Let $0 \leq s \leq 2\pi$. We need to give attention to the case of $s=\pi/2$. It follows from Lemma 3.7 that the families φ_s^ω and $\tilde{\varphi}_s^\omega$ intersect orthogonally each other at φ_0. Similarly, we observe that up to deformations obtained by moving the parameter ω, φ_s^ω and $\hat{\varphi}_s^\omega$ intersect orthogonally each other at $\varphi_{\pi/2}$. Here $\hat{\varphi}_s^\omega$: $T^2 \rightarrow SU(2)$ $(\omega \in \mathbf{R})$ is a family of harmonic maps defined by

$$(4.1) \qquad \hat{\varphi}_s^\omega(e^{it_1},\ e^{it_2}) := \begin{bmatrix} \cos s \cdot e^{-it_1} & \sin s \cdot e^{i(t_2+\omega)} \\ -\sin s \cdot e^{-i(t_2+\omega)} & \cos s \cdot e^{it_1} \end{bmatrix}.$$

Finally let us determine the connected component in Harm(T^2, S^3) containing φ_s and discuss its compactness. We define a new family $\bar{\varphi}_s^\omega$: $T^2 \rightarrow SU(2)$ $(\omega \in \mathbf{R})$ of harmonic maps defined by

$$(4.2) \qquad \bar{\varphi}_s^\omega(e^{it_1},\ e^{it_2}) := \begin{bmatrix} \cos s \cdot e^{-it_1} & \sin s \cdot e^{-i(t_2+\omega)} \\ -\sin s \cdot e^{i(t_2+\omega)} & \cos s \cdot e^{it_1} \end{bmatrix}.$$

THEOREM 4.2.   *The set of harmonic maps from* T^2 *into* S^3 *obtained by acting the group* $SO(4)$ *on the union of four families* $\{\varphi_s^\omega\} \cup \{\bar{\varphi}_s^\omega\} \cup \{\hat{\varphi}_s^\omega\} \cup \{\bar{\varphi}_s^\omega\}$ *with* $0 \leq s \leq 2\pi$ *becomes a connected component in* Harm(T^2, S^3) *containing* φ_s. *Hence, the connected component is compact.*

Figure 4.1 indicates how four family $\{\varphi_s^\omega\}$, $\{\bar{\varphi}_s^\omega\}$, $\{\hat{\varphi}_s^\omega\}$ and $\{\bar{\varphi}_s^\omega\}$ with $0 \leq s \leq 2\pi$ of harmonic maps intersect orthogonally each other, up to deformations with respect to the parameter ω.

Remark.   The action of $SO(4)$ moves $\{\varphi_s^\omega\}$ to $\{\bar{\varphi}_s^\omega\}$ and $\{\tilde{\varphi}_s^\omega\}$ to $\{\hat{\varphi}_s^\omega\}$. Then the union of two circles $[\varphi_s^\omega]$ and $[\tilde{\varphi}_s^\omega]$, which intersect orthogonally each other at $[\varphi_0]$, becomes the connected component in the quotient space Harm(T^2, S^3)/$SO(4)$ containing $[\varphi_s]$.
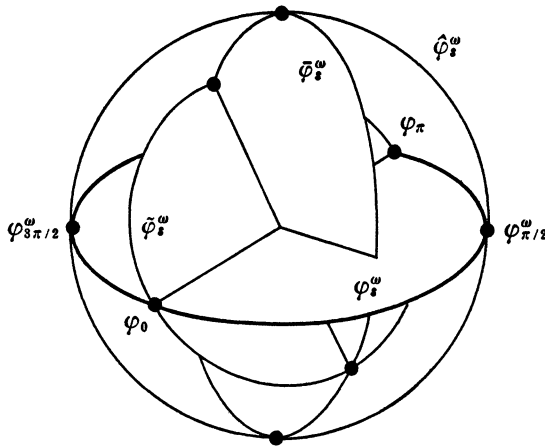
Figure 4.1.

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