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A NOTE ON THE POINCARÉ-BENDIXSON INDEX THEOREM

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Abstract

The local scheme for an equilibrium state of an analytic planar dynamical systems is investigated. Upper bounds of the numbers of elliptic and hyperbolic sectors are derived. Methods of singularity theory are applied to obtain appropriate estimations in terms of indices of maps explicitly constructed from a vector field.

I. Introduction

The study of geometric differential equations was founded by H. Poincaré in his classical "Mémoire" [PCR1] (see also [PCR2], [PCR3]).

At 15 years distance, Poincaré's ideas was followed by Bendixson's whose attention was mainly turned to the local phase-portrait around a critical point. In his major paper [BDX] Bendixson derived the index formula

$$\deg(F)=1+\frac{\mathcal{E}-\mathcal{H}}{2}$$

where $\deg(F)$ is the index of a stationary point of a planar vector field and $\mathcal{E}$, $\mathcal{H}$ are respectively the numbers of elliptic and hyperbolic sectors. This equality, known in bibliography as the Poincaré-Bendixson formula, gives an interesting application of topological methods to planar differential equations.

Under some additional assumptions one can give another Poincaré-Bendixson formula

$$\deg(F)=1+\frac{n_e-n_h}{2}$$

where n_e, n_h are respectively the numbers of internal and external tangent points of a vector field to a C_1 Jordan curve going around a stationary point.

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Consider an autonomous system

$$(1.1)\qquad \begin{cases} \dot{x}=P_m(x, y)+\phi(x, y)\equiv P(x, y) \\ \dot{y}=Q_m(x, y)+\psi(x, y)\equiv Q(x, y) \end{cases}$$

where P_m and Q_m are homogeneous polynomials of degree $m>1$, $P_m^2+Q_m^2\neq 0$, ϕ, $\psi=o(r^m)$, $r=\sqrt{x^2+y^2}$, for which the origin is an isolated singularity.

It is proved in [BDX] that

$$(1.2)\qquad \mathcal{E}<2\cdot m$$

and

$$(1.3)\qquad \mathcal{H}<2\cdot m+2.$$

In further developments, Berlinskii showed in [BER1], [BER2] that one has always

$$(1.4)\qquad \mathcal{E}<2\cdot m-1 \quad\text{and}\quad \mathcal{E}+\mathcal{H}<2\cdot m+2.$$

He also gave an example such that

$$\mathcal{E}=2\cdot m-1.$$

Moreover, it was proved that

$$(1.5)\qquad \text{if } \mathcal{E}\neq 0 \text{ then } \mathcal{E}+\mathcal{H}<2\cdot m.$$

Problems concerning estimations of the number of separatrices of an equilibrium state, the number of parabolic regions and the total number of regions in the neighbourhood of an isolated critical point have also been considered by Sagalovich in [SAG1], [SAG2] and Schecter and Singer in [S. S. 1], [S. S. 2].

The following simple example

$$(1.6)\qquad \begin{cases} \dot{x}=(x^2-y^2)\cdot(x^{2\cdot k}+y^{2\cdot k}) \\ \dot{y}=2\cdot x\cdot y\cdot(x^{2\cdot k}+y^{2\cdot k}) \end{cases}$$

shows that the difference between estimation of the number of elliptic regions given by Berlinskii ($2\cdot m-1=4\cdot k+3$) and the real number of elliptic sectors ($\mathcal{E}=2$) can be made arbitrarily high.

The aim of this paper is to compute the numbers n_e, n_h and consequently to give upper bounds for the numbers $\mathcal{E}$ and $\mathcal{H}$ in terms of indices of maps constructed explicitly from a vector field (P, Q). We would like to notice that using methods developed in this article we obtain for the problem (1.6) that $\mathcal{E}=2$.

After this introduction our paper is organized in the following way. In Section 2, we recall relevant material on the Poincaré-Bendixson theory. Next, in Section 3 we formulate and prove propositions which allows us to express numbers n_e and n_h in terms of indices of maps constructed directly from a vector field (Prop. 3.1).

Subsequently, we derive the upper bound for the number of pairs $(\mathcal{E}, \mathcal{H})$, which can appear in the Poincaré-Bendixson formula (Rem. 3.1). At the end of this section we apply our theorems to the Conley index theory for planar dynamical systems. Namely, our results allow us to verify if a sufficiently small disc centered at an equilibrium point is an isolating block. In such a situation we give a formula for the Conley index (Prop. 3.3) (cf. [A. F. S.] and [SFR1]).

In order to illustrate theory derived in this article we consider in Section 4 two examples. For planar polynomial dynamical systems we construct suitable maps and compute their indices. We use, as a tool, a computer program written by Andrzej Łęcki (Gdańsk University) which is able to compute the local indices of polynomial map-germs $f:(R^n, 0)\rightarrow(R^n, 0)$. This program is based on Eisenbud and Levine results [E. L.] and is briefly described in [SFR2]. The advantage of methods presented here is that our results can be described in comparatively simple way.

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2. Pleliminaries

Consider the following planar system

$$(2.1)\qquad \begin{cases} \dot{x}=P(x, y) \\ \dot{y}=Q(x, y) \end{cases}$$

where $F=(P, Q):(R^2, 0)\rightarrow(R^2, 0)$ is a continuous vector field and assume that at each point $(x, y)\in R^2$ sufficient conditions to the existence and uniqueness of the solutions of (2.1) are fulfilled.

Let C be a positively oriented Jordan curve of class C^1 in R^2 with the property that a vector field F does not vanish on C and that F is tangent to C at only a finite number of points $(x_i, y_i)\in C$ for $i=1, \cdots, k$.

The solution arc $(x(t), y(t))$ of (2.1) with $(x(0), y(0))=(x_0, y_0)\in C$ is said to be *internally* (or *externally*) tangent to C at (x_0, y_0) if there exists an $\varepsilon>0$ such that $(x(t), y(t))$ is interior (or exterior) to C for $0<|t|\leqq\varepsilon$.

Let us denote by N_e the set of points $(x_i, y_i)\in C$, the solution arc $(x(t), y(t))$ of the equation (2.1), $(x(0), y(0))=(x_i, y_i)$, is internally tangent to C at (x_i, y_i).

Similarly, the set of points $(x_i, y_i)\in C$, the solution arc $(x(t), y(t))$ of the equation (2.1), $(x(0), y(0))=(x_\iota, y_\iota)$, is externally tangent to C at (x_ι, y_ι) will be denoted by N_h. The number of elements of set N_e, N_h we will denote by n_e and n_h, respectively.

If the field $F: (\Omega, 0)\to(R^2, 0)$ has an isolated zero at $0\in R^2$ and there are no other zeros in Ω then the Brouwer topological degree of F will be shortly denoted by $\deg(F)$ instead of $\deg(F, \Omega, 0)$.

We recall the following famous Poincaré-Bendixson theorem which gives a relation between the index of an isolated singular point of a vector field F and numbers n_e and n_h.

THEOREM 2.1 (Poincaré-Bendixson). *Let $\Omega\subset R^2$ be a simply connected bounded set with boundary $\partial\Omega$ which is a positively oriented Jordan curve of class C^1 satisfying all conditions stated above. Then,*

$$2\cdot\deg(F)=2+n_e-n_h\,. \qquad \square$$

A solution $(x(t), y(t))\neq(0, 0)$ of (2.1) defined on an interval $[0, \omega)$ (or an interval $(-\omega, 0]$) for $0<\omega\leqq\infty$ is called a *positive* (or *negative*) *null solution* if $(x(t), y(t))\to(0, 0)$ as $t\to\omega$ (or $-\omega$). When the solution of (2.1) with $(x(0), y(0))=(0, 0)$ is unique, then necessarily $\omega=\infty$.

Let C be a positively oriented Jordan curve surrounding $(0, 0)\in R^2$. A solution $(x(t), y(t))$ of (2.1) is called a *positive* or *negative base solution* for C if $(x(t), y(t))$ is defined for either $t\geqq0$ or $t\leqq0$, $(x(0), y(0))\in C$, $(x(t), y(t))$ is interior to C for $t\neq0$, and $(x(t), y(t))$ is a null solution.

Let $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ be base solutions for C. The open set S of the interior of C with boundary consisting of $(0, 0)$, the arcs $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ and the (oriented closed) subarc C_{12} from $(x_1(0), y_1(0))$ to $(x_2(0), y_2(0))$ will be called the *sector* of C (determined by the ordered pair $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$. It is not excluded that $(x_1(0), y_1(0))=(x_2(0), y_2(0))$ so that C_{12} can be C or reduced to a point.

Consider the case that there exists a solution $(x_0(t), y_0(t))$, $-\infty<t<\infty$, of (2.1) which is interior or on C for all t and $(x_0(t+t_1), y_0(t+t_1))\equiv(x_1(t), y_1(t))$ for $t\geqq0$, $(x_0(t+t_2), y_0(t+t_2))\equiv(x_2(t), y_2(t))$ for $t\leqq0$, for some t_1, t_2 $(\leqq t_1)$.

The point $(0, 0)$ and the arc $(x_0(t), y_0(t))$, $-\infty<t<\infty$, form a Jordan curve J with interior I. If S contains I, then it is called an *elliptic sector*. When $t_1=t_2$ (so that $(x_1(0), y_1(0))=(x_2(0), y_2(0))=(x_0(t_1), y_0(t_1))$, and C_{12} reduces to the point $(x_0(t_1), y_0(t_1))$, then S is elliptic and coincides with I. When $t_1\neq t_2$, S can contain points not in I.

A sector S with the properties that it is not an elliptic sector and that $S\cup C_{12}$ contains no base solution is called a *hyperbolic sector*.

Let us denote by $\mathcal{E}$ (resp. $\mathcal{H}$) the number of elliptic (resp. hyperbolic) sectors (cf. [HRT]). Using this notation one can express another Poincaré-Bendixson theorem in the following way.

THEOREM 2.2 (Poincaré-Bendixson formula). *If the origin is the only zero of F in Ω then*

$$2\cdot\deg(F)=2+\mathcal{E}-\mathcal{H}. \qquad \square$$

3. Main results

Throughout the paper we shall assume that F is an analytic vector field, i.e. functions P, Q are real analytic and that $C=\partial\Omega$ is a small circle centered at $0\in R^2$.

Define an analytic function $S:(R^2, 0)\rightarrow(R, 0)$ by the formula

$$S(x, y)=x\cdot P(x, y)+y\cdot Q(x, y).$$

Clearly, S has a critical point at the origin. Moreover, $F(x, y)$ is tangent to C at $(x, y)\in C$ if and only if $S(x, y)=0$. Thus, if $S\equiv 0$ then each solution arc is a circle.

If $S\neq 0$ then the set $S^{-1}(0)\cap\Omega-\{0\}$ consists of a finite (possibly zero) number of analytic branches, i.e. connected components, emanating from $0\in R^2$, where Ω is a sufficiently small disc.

It is proper to add that each branch of $S^{-1}(0)\cap\Omega-\{0\}$ is diffeomorphic to an open interval and is transversal to every small circle centered at $0\in R^2$. Hence our initial assumption that F is tangent to C at most at a finite number of points is satisfied.

Define a map $\Delta:(R^2, 0)\rightarrow(R, 0)$, $\Delta(x, y)=\partial(\omega, S)/\partial(x, y)$, where $\omega(x, y)=x^2+y^2$. As a consequence of the more general result proved by the third author in [SFR1] we claim that the number of branches of zeros of S emanating from the origin is equal to the local index of the map $H=(\Delta, S):(R^2, 0)\rightarrow(R^2, 0)$ multiplied by two.

PROPOSITION 3.1. *Assume that S has an isolated critical point at $0\in R^2$ then*

$$n_e+n_h=2\cdot\deg(H).$$

Proof. Since S has an isolated critical point at the origin we may assume that the gradient $\nabla S\neq 0$ everywhere in Ω, except at the origin. Hence, if $(x, y)\in S^{-1}(0)\cap\Omega-\{0\}$ then S changes a sign at (x, y). Because $S^{-1}(0)$ is transversal to every small circle then a restriction of S to C also changes a sign at each $(x, y)\in C\cap S^{-1}(0)$. Therefore $n_e+n_h=\text{card } C\cap S^{-1}(0)=2\cdot\deg(H)$. $\square$

The above formula has been originally proved by Kenji Aoki, Takuo Fukuda and Wei-Zhi Sun in [A. F. S.] (for a generalization to higher dimensions see also [A. F. N.]).

The authors are awared of the fact that there are other possible ways to find a formula for the sum n_e+n_h. Using methods developed in [ARD] and [WLL] one can prove the following.

PROPOSITION 3.2. *If S has an isolated critical point at $0\in R^2$ then*

$$n_e+n_h=2-2\cdot\deg(\nabla S). \qquad \square$$

From the Poincaré-Bendixson formula and Propositions 3.1, 3.2 we get.

THEOREM 3.1. *Let $F=(P, Q):(R^2, 0)\to(R^2, 0)$ be an analytic vector field having an isolated zero at $0\in R^2$. Then*

$$n_e=\deg(H)+\deg(F)-1,$$

$$n_h=\deg(H)-\deg(F)+1,$$

or equivalently

$$n_e=\deg(F)-\deg(\nabla S),$$

$$n_h=2-\deg(F)-\deg(\nabla S). \qquad \square$$

Notice that having determined numbers n_e and n_h we are able to give upper bounds for numbers $\mathcal{E}$ and $\mathcal{H}$ which appear in Poincaré-Bendixson formula. In particular we obtain the following inequalities

$$\max(0, 2\cdot\deg(F)-2)\leqq\mathcal{E}\leqq\deg(H)+\deg(F)-1, \tag{3.1}$$

$$\max(0, 2-2\cdot\deg(F))\leqq\mathcal{H}\leqq\deg(H)-\deg(F)+1, \tag{3.2}$$

or equivalently

$$\max(0, 2\cdot\deg(F)-2)\leqq\mathcal{E}\leqq-\deg(\nabla S)+\deg(F), \tag{3.3}$$

$$\max(0, 2-2\cdot\deg(F))\leqq\mathcal{H}\leqq-\deg(\nabla S)-\deg(F)+2. \tag{3.4}$$

Remark 3.1. Taking into account the above inequalities we get at most

$2-\deg(F)+\deg(H)$ pairs $(\mathcal{E}, \mathcal{H})$ in a case $\deg(F)\geqq 1$,

$\deg(F)+\deg(H)$ pairs $(\mathcal{E}, \mathcal{H})$ in a case $\deg(F)\leqq 1$,

or equivalently

$3-\deg(F)-\deg(\nabla S)$ pairs $(\mathcal{E}, \mathcal{H})$ in a case $\deg(F)\geqq 1$,

$1+\deg(F)-\deg(\nabla S)$ pairs $(\mathcal{E}, \mathcal{H})$ in a case $\deg(F)\leqq 1$,

which can appear in Poincaré-Bendixson formula. □

Now let us look more precisely at numbers n_e and n_h.
Put

1. $N_e^{+,(-)} = \{(x_i, y_i) \in N_e$ such that vectors (x_i, y_i) and $F(x_i, y_i)$ give a positive (negative) orientation$\}$,
2. $n_e^{+,(-)}$=the number of elements of $N_e^{+,(-)}$.
3. $N_h^{+,(-)} = \{(x_i, y_i) \in N_h$ such that vectors (x_i, y_i) and $F(x_i, y_i)$ give a positive (negative) orientation$\}$,
4. $n_h^{+,(-)}$=the number of elements of $N_h^{+,(-)}$.

Obviously $n_e^+ + n_e^- = n_e$ and $n_h^+ + n_h^- = n_h$.

Our purpose is to express numbers n_e^+, n_e^-, n_h^+, n_h^- in terms of indices at 0 of some explicitly given mappings.

Let us define

$$K(x, y) = \det\begin{vmatrix} x & y \\ P & Q \end{vmatrix}$$

$\Delta = \partial(K, S)/\partial(x, y)$ and $G = (\Delta, S)$. Since $\nabla S(0) = 0$ then $\Delta(0) = 0$, and then $G: (R^2, 0) \to (R^2, 0)$.

LEMMA 3.1. *If S has an isolated critical point at the origin then $0 \in R^2$ is isolated in $G^{-1}(0)$ and*

$$n_e^+ - n_e^- + n_h^+ - n_h^- = 2\cdot\deg(G),$$

$$n_e^+ - n_e^- - n_h^+ + n_h^- = 0.$$

Proof. If $(x, y) \in C \cap S^{-1}(0)$ then $F(x, y)$ is tangent to C at (x, y), and then $K(x, y) \neq 0$. Clearly, the number of branches of $S^{-1}(0) \cap \Omega - \{0\}$ on which K is positive (resp. negative) is equal to $n_e^+ + n_h^+$ (resp. $n_e^- + n_h^-$).

According to [SFR1], $0 \in R^2$ is isolated in $G^{-1}(0)$ and $n_e^+ + n_h^+ - n_e^- - n_h^- = 2\cdot\deg(G)$. The function S changes a sign in some neighbourhood of every point $(x, y) \in S^{-1}(0) \cap C$ which implies the second formula. □

We have obtained the following system of linear equations

$$n_e^+ + n_e^- = \deg(F) + \deg(H) - 1,$$

$$n_h^+ + n_h^- = 1 - \deg(F) + \deg(H),$$

$$n_e^+ - n_e^- + n_h^+ - n_h^- = 2\cdot\deg(G),$$

$$n_e^+ - n_e^- - n_h^+ + n_h^- = 0.$$

Equivalently we can write

$$n_e^+ + n_e^- = \deg(F) - \deg(\nabla S),$$

$$n_h^+ + n_h^- = 2 - \deg(F) - \deg(\nabla S),$$

$$n_e^+ - n_e^- + n_h^+ - n_h^- = 2\cdot\deg(G),$$

$$n_e^+ - n_e^- - n_h^+ + n_h^- = 0.$$

Obviously the above systems are not singular and therefore we have.

THEOREM 3.2. *Under the assumptions of Lemma* 3.1 *we derive*

$$n_e^+=\frac{1}{2}\cdot(\deg(H)+\deg(G)+\deg(F)-1),$$

$$n_e^-=\frac{1}{2}\cdot(\deg(H)-\deg(G)+\deg(F)-1),$$

$$n_h^+=\frac{1}{2}\cdot(\deg(H)+\deg(G)-\deg(F)+1),$$

$$n_h^-=\frac{1}{2}\cdot(\deg(H)-\deg(G)-\deg(F)+1),$$

or equivalently

$$n_e^+=\frac{1}{2}\cdot(-\deg(\nabla S)+\deg(G)+\deg(F)),$$

$$n_e^-=\frac{1}{2}\cdot(-\deg(\nabla S)-\deg(G)+\deg(F)),$$

$$n_h^+=\frac{1}{2}\cdot(-\deg(\nabla S)+\deg(G)-\deg(F)+2),$$

$$n_h^-=\frac{1}{2}\cdot(-\deg(\nabla S)-\deg(G)-\deg(F)+2). \qquad \square$$

The number of elliptic sectors determined by arcs of solutions of (2.1) which give a positive (resp. negative) orientation at their tangent points to C we will denote by $\mathcal{E}^+$ (resp. $\mathcal{E}^-$).

The number of hyperbolic sectors determined by arcs of solutions of (2.1) which give a positive (resp. negative) orientation at their tangent points to C we will denote by $\mathcal{H}^+$ (resp. $\mathcal{H}^-$).

Remark 3.2. Let W be a simply connected bounded set with boundary $\partial\Omega=C$ and assume that $F^{-1}(0)\cap\Omega=\{0\}$. If the origin is an isolated critical point of S then

$$\mathcal{E}^+-\mathcal{H}^+=\mathcal{E}^--\mathcal{H}^-=\deg(F)-1. \qquad \square$$

Results presented in this paragraph have natural applications to the Conley index theory. Loosely speaking, Conley's generalized Morse index assigned to a compact isolated invariant set S is the homotopy type of a pointed topological spaces, i.e. a space with distinguished point in it. Very roughly, it is defined in the following way. Let N be a neighbourhood containing S compactly in its interior. An index pair $\langle N_1, N_2\rangle$ is a pair N_1, N_2 of compact subsets of N satisfying several conditions:

1) relative to N they are invariant under forward flow i.e. for $i=1, 2$ $\forall x_0 \in N_i$ and $\forall t>0$, the point $x(t, x_0)$ (i.e. the point on the orbit starting at x_0, after time t has elapsed) is either in N_i or is not in N,
2) $S \subset \text{int}[N_1 \backslash (N_1 \cap N_2)]$,
3) N_2 is roughly the set of points through which orbits eventually leave N under the flow, i.e. if some point x in N_1 eventually (its orbit) flows out of N then it first passes through N_2.

The generalized Morse index of S is the homotopy type

$$h(S)=[N_1/(N_1 \cap N_2)]$$

(cf. [CON], [SMR]). (Recall that two spaces A and B are homotopy equivalent if there exist two continuous mappings $\phi: A \to B$ and $\psi: B \to A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the identity in B and A, respectively.)

Proposition 3.1 can be treated as a criterium which allows us to verify if a sufficiently small disc centered at $0 \in R^2$ is an isolating block for the origin considered as a stationary solution of (2.1).

Using this proposition together with Poincaré-Bendixson theorem (Th. 2.1) one can compute numbers n_e and n_h. As an immediate consequence of definition of isolating block (cf. [CON]) we obtain the following.

PROPOSITION 3.3. *If S has an isolated critical point at the origin then a sufficiently small disc centered at the origin is an isolating block for* $0 \in R^2$ *iff* $n_e=0$.

Moreover, if $\deg(\nabla S)<1$ *then, the Conley index of the origin is a homotopy type of pointed space which is a wedge of one-dimensional spheres and the number of spheres is equal to* $-\deg(\nabla S)$.

In the case $\deg(\nabla S)=1$ *the origin is an attractor (or repeller) and therefore the Conley index is a homotopy type of a pointed zero-dimensional sphere (or two-dimensional sphere).* □

It is understood in the above proposition that the Conley index is equal to the trivial pointed space $[*, *]$, if $\deg(\nabla S)=0$.

4. Examples

Theory developed in this article can be applied as it is shown in the examples below.

In the first example we consider a dynamical system for which our results allow us to determine the numbers of elliptic and hyperbolic sectors whereas methods known till now give only some estimations of the numbers $\mathcal{E}$ and $\mathcal{H}$. On the other hand, it is not true that we can always determine $\mathcal{E}$ and $\mathcal{H}$.

In the second example we apply our results to the Conley index theory. We compute the Conley index of the origin which is an isolated invariant set of a given dynamical system.

Example 4.1. Let us consider the following dynamical system

$$\begin{cases} \dot{x}=P(x, y)=x^9+x^5\cdot y^4-3\cdot x^7\cdot y^2-3\cdot x^3\cdot y^6+x^3\cdot y^8-3\cdot x\cdot y^{10} \\ \dot{y}=Q(x, y)=3\cdot x^6\cdot y-x^4\cdot y^3+3\cdot x^2\cdot y^7-y^9+3\cdot x^6\cdot y^5-x^4\cdot y^7. \end{cases}$$

We then have

$$F(x, y)=(P(x, y), Q(x, y))$$

and

$$S(x, y)=x^{10}+x^6\cdot y^4-3\cdot x^8\cdot y^2-3\cdot x^4\cdot y^6-3\cdot x^2\cdot y^{10}+3\cdot x^6\cdot y^2$$
$$-x^4\cdot y^4+3\cdot x^2\cdot y^8-y^{10}+3\cdot x^6\cdot y^6.$$

By Bendixson's inequalities (1.2), (1.3) we obtain the following estimations

$$\mathcal{E}<14 \quad \text{and} \quad \mathcal{H}<16$$

which can be slightly improved by Berlinskii's results. Namely, inequalities (1.4) give us

$$\mathcal{E}<13 \quad \text{and} \quad \mathcal{E}+\mathcal{H}<16$$

Using the computer program mentioned in introduction we derive

$$\deg(F)=3$$

and therefore by Theorem 2.2 we have $\mathcal{E}\geqq 4$. In this situation we can apply (1.5) to obtain the following five pairs $(\mathcal{E}, \mathcal{H})$ which can appear in Poincaré-Bendixson formula:

$$(4, 0), (5, 1), (6, 2), (7, 3), (8, 4).$$

Now we apply once more the computer program in order to derive the index of the polynomial H defined in the beginning of the previous section

$$\deg(H)=2.$$

Thus by Theorem 3.1 we have

$$n_e=4 \quad \text{and} \quad n_h=0$$

and consequently by (3.1), (3.2) we obtain

$$\mathcal{E}=4 \quad \text{and} \quad \mathcal{H}=0.$$

Example 4.2. For the dynamical system

$$\begin{cases} \dot{x}=P(x, y)=x^5-3\cdot x^3\cdot y^2+x^3\cdot y^4-3\cdot x\cdot y^6 \\ y=Q(x, y)=-3\cdot x^6\cdot y^3-3\cdot x^4\cdot y^1+x^4\cdot y^5+x^2\cdot y^3-3\cdot x^2\cdot y^5+y^7 \end{cases}$$

we have

$$F(x, y)=(P(x, y), Q(x, y))$$

and

$$S(x, y)=x^6-3\cdot x^4\cdot y^2+x^4\cdot y^4-3\cdot x^2\cdot y^6-3\cdot x^6\cdot y^4-3\cdot x^4\cdot y^2 +x^4\cdot y^6+x^2y^4-3\cdot x^2\cdot y^5+y^7.$$

Applying again the computer program we obtain

$$\deg(F)=-3 \quad \text{and} \quad \deg(\nabla S)=-3$$

and, by Theorem 3.1,

$$n_e=0 \quad \text{and} \quad n_h=8.$$

From Proposition 3.3 we conclude that a sufficiently small disc centered at the origin is an isolating block for $\{0\}$. Moreover, the Conley index $h(\{0\})$ is a homotopy type of the wedge $S^1\vee S^1\vee S^1$.

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