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EFFECTIVE BASE POINT FREENESS

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Abstract

It is an interesting problem to know when the adjoint bundle K_X+mL is free or very ample. Recently Ein-Lazarsfeld states very explicit numerical conditions about L such that K_X+L becomes free on a smooth 3-fold. In this paper, the author wants to enlarge their results on a singular 3-fold.

1. Introduction

The purpose of this paper is to enlarge Ein-Lazarsfeld's result on a singular 3-fold.

It is an interesting problem when adjoint bundle K_X+mL or pluricanonical system is free or very ample. Reider shows in [11] the following theorem:

THEOREM 1. *Let S be a smooth surface and L be a nef Cartier divisor on S. Assume that*:
(1) $L^2 \geqq 5$,
(2) $L \cdot C \geqq 2$ *for all curves on* S.
Then the adjoint bundle K_S+L is free.

From the above theorem, we can deduce the adjoint bundle K_S+mL is free if $m \geqq 3$. Also Sakai shows in [12] that the similar statement holds on a normal surface. In higher dimension, Fujita conjectured in [4] the adjoint bundle K_X+mL is free if $m \geqq \dim X+1$ and very ample if $m \geqq \dim X+2$. Ein-Lazarsfeld got in [2] the following theorem:

THEOREM 2. *Let X be a smooth 3-fold and L be a nef Cartier divisor. Fix one point $x \in X$. If L satisfies the following criterion, the adjoint bundle K_X+L is free at x.*
(1) $L^3 \geqq 92$,
(2) $L^2 \cdot S \geqq 7$, *for the all surfaces S such that $x \in S$*,
(3) $L \cdot C \geqq 3$, *for the all curves C such that $x \in C$*.

By the above theorem, K_X+mL is free if $m \geqq 5$. We want to enlarge the

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above result on a singular 3-fold, because it is natural to consider a variety which has some mild singularities by minimal model program. Recently Ein-Lazarsfeld-Masek states the freeness of adjoint bundle on the terminal 3-fold in [3]. Our results are different from the above results at two points. First, we work on the variety which has only log-terminal singularities. Second, we consider the nef Cartier divisor L such that $L-(K_X+\Delta)$ satisfies some numerical conditions. It will not be suitable to consider the adjoint bundle on a singular 3-fold, since the canonical divisor is not a Cartier divisor in general. In such situation, Kollár shows in [10] the following theorem:

THEOREM 3. *Let X be a n-dimensional normal projective variety and Δ be an effective $\boldsymbol{Q}$-divisor on X. Assume that (X, Δ) has only log-terminal singularities. Let L be a nef Cartier divisor such that $aL-(K_X+\Delta)$ is nef and big for some integer a. Then $(2(a+n)(n+2)!)L$ is free.*

An analog of the conjecture of Fujita says that $(a+n+1)L$ is free. Kollár's result gave explicit estimates in all dimension, but it is far from above estimation. So we guess if we use the technique of Ein-Lazarsfeld's, we can do some improvement of Kollár's result in dimension 3, and we obtain the following theorems.

THEOREM 4. *Let X be a normal 3-fold and Δ be an effective $\boldsymbol{Q}$-divisor. Assume that (X, Δ) has only log-terminal singularities. Let L be a nef Cartier divisor on X such that $aL-(K_X+\Delta)$ is nef and big for some rational number $a>0$. Then nL is free if $n>a+11/2$, $(n\in\boldsymbol{N})$.*

THEOREM 5. *Let X and Δ are same objects in above Theorem. Assume that (X, Δ) has only weak log-terminal singularities and X is $\boldsymbol{Q}$-factorial. Let L be a nef Cartier divisor on X such that $aL-(K_X+\Delta)$ is ample for some rational number $a>0$. Then nL is free if $n>a+11/2$, $(n\in\boldsymbol{N})$.*

Similary Ein-Lazarsfeld-Masek [3], we can obtain the following result about a pluricanonical system.

THEOREM 6. *Let X be a minimal 3-fold of general type and r be an index of X. Then*

(1) *$6K_X$ is free if $r=1$,*

(2) *mrK_X is free if*

$$m>2+\frac{1}{r}+\frac{7\sqrt[3]{r}}{2r} \quad r\geqq 2\,.$$

By the above theorem, we can deduce that $5rK_X$ is free for $r\geqq 2$. If $r\geqq 4$ then $4rK_X$ is free and if $r\geqq 9$ then $3rK_X$ is free. Ein-Lazarsfeld-Masek proved in [3] that $10K_X$ is free if $r=1$, $7rK_X$ is free if $r\geqq 2$, $5rK_X$ is free if $r\geqq 3$ and $2rK_X$ is free if $r\geqq 27$.

The outline of the proof of theorems

Fix one point x on X. We construct a smooth variety Y and a birational morphism $f: Y \to X$ depending on the type of singularity at x. According to the argument of Kawamata-Reid-Shokurov, we show the restriction map

$$H^0(Y, f^*L-N) \longrightarrow H^0(E, (f^*L-N)|_E)$$

is surjective, where N is an effective divisor such that $f^{-1}(x) \cap N = \emptyset$ and E is a surface on Y such that $x \in f(E)$. Then we will show there is a section $s \in H^0(E, (f^*L-N)|_E)$ which is non-vanishing at some point of $f^{-1}(x)$.

We distinguish three cases, according to $\dim f(E)=0, 1, 2$. In the case $\dim f(E)=0$, $\mathcal{O}_E(f^*L-N) \cong \mathcal{O}_E$ because $x \notin f(N)$. Thus we are done. In the case $\dim f(E)=1$, we prove the restriction map $H^0(E, (f^*L-N)_E) \to H^0(Z, (f^*L-N)|_Z)$ is surjective, where Z is a fibre of $f|_E$ such that $Z \cap N = \emptyset$. Then $\mathcal{O}_Z(f^*L-N) \cong \mathcal{O}_Z$ and we are done. In the case $\dim f(E)=2$, we will produce the required section by the following theorem:

THEOREM 7. *Let S be a normal projective surface and Δ be an effective $\boldsymbol{Q}$-divisor. Assume that (S, Δ) has only log-terminal singularities. Fix one point x on S. Let Q be a Cartier divisor on S which satisfies the following conditions*:

(1) $M := Q(K_S+\Delta)$ *is nef and big,*

(2) $M^2 > 4$,

(3) $M \cdot C > 2$ *for the all curves* $x \in C$.

Then $x_0 \notin \mathrm{Bs}|Q|$.

This Theorem is an extension of Theorem 2.1 in [2]. In Theorem 2.1 in [2], it is assumed that S has only rational double points.

In this paper, §2 devoted to prove two elementary lemmas which used later. In §3, we prove Theorem 7. The statement of main theorem appears in §4, where we also introduce Kawamata-Reid-Shokurov argument. We will show the existence of desirable section according to $\dim f(E)$ in §5 and will construct a smooth variety Y and a birational morphism f depending on the type of singularity at x in §6.

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2. Preliminary

We work throughout over the complex numbers $\boldsymbol{C}$. Notation and terminology is same as in [7]. In this section, we prove two elementary lemmas which need the proof of Theorems. First we define a multiplicity of $\boldsymbol{Q}$-Cartier divisor D at a point x.

DEFINITION 1. For an effective Cartier divisor D on a projective variety X and a point $x \in X$, define a natural number $\nu_x(D)$ by

$$\nu_x(D) := \max\{n \in \boldsymbol{N} \mid \mathcal{O}_X(-D) \subset m_x^n\},$$

where m_x is a maximal ideal of $\mathcal{O}_{X,x}$. For an effective $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-Weil divisor D' on X, define a rational number $\nu_x(D')$ by

$$\nu_x(D') := \frac{1}{p}\nu_x(pD'),$$

where p is a natural number such that pD' is a Cartier divisor.

Then we show the following lemma.

LEMMA 1. *Let X be a projective variety of dimension d and M be a nef and big $\boldsymbol{Q}$-Cartier divisor on X such that $M^d > \alpha^d$. Fix a point x on X such that* $\operatorname{mult}_x X = a$. *Then there is an effective $\boldsymbol{Q}$-divisor B on X such that $B \sim_{\boldsymbol{Q}} M$ and*

$$\nu_x(B) > \frac{\alpha}{\sqrt[d]{a}(1-\sigma)} 0 < \sigma \ll 1 .$$

Proof. Consider the following exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{O}_X(mM \otimes m_x^l)) \longrightarrow H^0(X, \mathcal{O}_X(mM)) \longrightarrow H^0(X, \mathcal{O}_X/m_x^l).$$

By Kodaira's Lemma, we can choose an ample divisor A, an effective divisor D, and small numbers $0 < \delta$, $\sigma \ll 1$ such that $M \sim_{\boldsymbol{Q}} A + \delta D$, $A^d > (\alpha/(1-\sigma))^d$. For a sufficiently large number m such that mM and $m(M-\delta D)$ are Cartier,

$$h^0(mM) \geqq h^0(m(M-\delta D)) = \frac{A^d}{d!} m^d + (\text{lower terms of } m) > \frac{A^d - \varepsilon}{d!} m^d,$$

where $0 < \varepsilon \ll \sigma \ll 1$. On the other hand,

$$h^0(\mathcal{O}_X/m_x^l) = a\frac{a}{d!} l^d + (\text{lower terms of } l) < \frac{a+\varepsilon}{d!} l^d$$

for a sufficiently large l. If we choose a suitable large m, there exists an integer l which satisfy the following inequality,

$$m\sqrt[d]{A^d - \varepsilon} > (\sqrt[d]{a+\varepsilon})^l > \frac{m\alpha}{1-\sigma}.$$

For these m and l, $h^0(mM) > h^0(\mathcal{O}_X/m_x^l)$. So we can choose a section $t \in H^0(mM)$ such that $\nu_x(\operatorname{div}(t)) \geqq l$. If we put $B := (1/m)\operatorname{div}(t)$ then the assertion of lemma follows. □

LEMMA 2. *Let S be a normal surface and M be a nef and big $\boldsymbol{Q}$-Cartier* *divisor on S. Fix one point x_0 on S. Assume that (S, x_0) is a normal quotient surface singularity and $M^2>(\sigma_2)^2$. Let $\pi: S_1\to S$ be a minimal resolution of (S, x_0) and Z be a rational curve such that $Z\subset\pi^{-1}(x_0)$. Then there is an effective $\boldsymbol{Q}$-divisor B on S_1 which satisfies the following two conditions*:

(1) $B\sim_{\boldsymbol{Q}}\pi^*M$,

(2) $B-\tau Z\geqq 0,\quad \tau>\dfrac{\sigma_2}{\sqrt{-Z^2}},\quad \tau\in\boldsymbol{Q}$.

Proof. Consider the following exact sequence:

$$0\longrightarrow H^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-(k+1)Z))$$
$$\longrightarrow H^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-kZ))\longrightarrow H^0(Z, \mathcal{O}_Z(m\pi^*M-kZ)),$$

where m is an integer such that $m\pi^*M$ become a Cartier divisor. We obtain the following inequality,

$$h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-(k+1)Z))\geqq h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-kZ))-h^0(S_1, \mathcal{O}_Z(-kZ)).$$

By Riemann-Roch formula,

$$h^0(Z, \mathcal{O}_Z(-kZ))-h^0(Z, \omega_Z\otimes\mathcal{O}_Z(kZ))=\deg\mathcal{O}_Z(-kZ)+1-g(Z).$$

Since $Z^2<0$ and $g(Z)=0$, $h^0(Z, \mathcal{O}_Z(-kZ))=-kZ^2+1$. Thus

$$h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-(k+1)Z))\geqq h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-kZ))-(kZ^2+1).$$

From this inequality, we obtain

$$h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M-lZ))\geqq h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M)-\sum_{k=0}^{l-1}(-kZ^2+1).$$

By Kodaira's Lemma, there are an ample divisor A and an effective divisor D such that $\pi^*M\sim_{\boldsymbol{Q}}A+\varepsilon D$, $A^2>(\sigma_2/(1-\sigma))^2$, where σ and ε are small numbers such that $0<\sigma$, $\varepsilon\ll 1$. If we take m and l large enough, we may assume $m\pi^*M$, $m(\pi^*M-\varepsilon D)$ are Cartier divisor and

$$\begin{aligned} h^0(S_1, \mathcal{O}_{S_1}(m\pi^*M))&\geqq h^0(S_1, \mathcal{O}_{S_1}(m(\pi^*M-\varepsilon D))) \\ &=\frac{m^2}{2}A^2+(\text{lower term of } m) \\ &>\frac{A^2-\varepsilon'}{2}m^2, \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{l-1}(-kZ^2+1)&=\frac{-Z^2}{2}l^2+(\text{lower term of } l) \\ &<\frac{-Z^2}{2}(l+\varepsilon')^2,\quad 0<\varepsilon'\ll\sigma\ll 1. \end{aligned}$$

Furthermore we may assume m and l satisfy the following inequality:

$$m\sqrt{A^2-\varepsilon'}>(l+\varepsilon')\sqrt{-Z^2}>\frac{m\sigma_2}{1-\sigma}.$$

For these m and l, there is a section $t\in H^0(m\pi^*M-lZ)$. Let $B:=(1/m)(\operatorname{div}(t)+lZ)$. Then the assertion of lemma follows.

3. Effective base point freeness for surface

THEOREM 7. *Let S be a normal projective surface and Δ be an effective $\boldsymbol{Q}$-divisor on S. Assume that (S, Δ) has only log-terminal singularities. Fix one point x_0 on S. Let Q be a Cartier divisor on S which satisfies the following condition*:

(1) *$M:=Q-(K_S+\Delta)$ is nef and big,*
(2) *$M^2>4$,*
(3) *$M\cdot C>2$, for all curves $C\subset S$ such that $x_0\in C$.*

Then $x_0\notin \operatorname{Bs}|Q|$.

Proof of Theorem. If (S, x_0) is a smooth point or rational double point, Theorem 7 follows by the following theorem.

THEOREM 8 ([2] Theorem 2.1). *Let S be a normal projective surface which has only rational double points. Fix one point x_0 on S. Let M be a nef and big $\boldsymbol{Q}$-divisor which has the following numerical criterion*:

(1) *$M^2>4$.*
(2) *$M\cdot C>2$, for all curves C such that $x_0\in C$.*

Then $x_0\notin \operatorname{Bs}|K_S+\ulcorner M\urcorner|$.

Thus we may assume (S, x_0) is quotient singularity. Let $\pi: S_1\to S$ be the minimal resolution. For an effective $\boldsymbol{Q}$-divisor B such that $B\sim_{\boldsymbol{Q}}\pi^*M$ and prime divisors $\{E_i\}$, $(0\leqq i\leqq m)$ on S_1 such that $\cup E_i=\operatorname{Supp} B\cup\pi^{-1}(x_0)$, we define rational numbers $\{b_i\}$ and $\{e_i\}$ by the following formulae:

$$B=\sum b_iE_i,$$

$$K_{S_1}\sim_{\boldsymbol{Q}}\pi^*(K_S+\Delta)+\sum e_iE_i.$$

Let

$$c:=\min\left\{\frac{e_i+1}{b_i}\,|\,x_0\in\pi(E_i),\ b_i>0\right\}.$$

CLAIM 1. (1) *$cb_i-e_i\geqq 0$, for any i.*
(2) *If $cb_i-e_i>1$, then $x_0\notin\pi(E_i)$.*

Proof. Since (S, Δ) has only log-terminal singularities, we obtain $e_i>-1$

and $c>0$. Because π is the minimal resolution, $e_i \leqq 0$. Thus $cb_i - e_i \geqq 0$.

(2) If $b_i>0$ and $x_0 \in \pi(E_i)$, then $c \leqq (e_i+1)/b_i$. Thus $cb_i - e_i \leqq 1$. If $b_i=0$ and $cb_i - e_i > 1$, then $e_i < -1$. But this contradicts the hypothesis that (S, Δ) has only log-terminal singularities.

We go back the proof of Theorem 7. Let $R := \sum (cb_i - e_i) E_i$. By changing indices, we may assume the following:

(1) $cb_i - e_i = 1$ and E_i is not π-exceptional for $0 \leqq i \leqq m'$.

(2) $cb_i - e_i = 1$ and E_i is π-exceptional for $m'+1 \leqq i \leqq m_1$

(3) $cb_i - e_i > 1$ for $m_1+1 \leqq i \leqq m_2$.

Let

$$E' := \sum_{i=0}^{m'} E_i, \quad E'' := \sum_{i=m'+1}^{m_1} E_i \quad \text{and} \quad N := \sum_{i=m_1+1}^{m_2} [cb_i - e_i] E_i .$$

We define

$$T := \pi^* Q - N - E' - E'' - K_{S_1} - \langle R \rangle .$$

Then

$$T \sim_{\boldsymbol{Q}} (1-c) M .$$

PROPOSITION 1. *For a suitable B, $c<1$ holds.*

T is nef and big by Proposition 1. First we consider the case $E'' \neq 0$. We need the following lemma.

LEMMA 3 ([2] Lemma 1.1, 2.4). *Let S be a smmoth surface and M be a nef and big $\boldsymbol{Q}$-divisor on S. Then*

(1) $H^1(S, K_S + \ulcorner M \urcorner) = 0$.

(2) *Let $\{E_i\}$ be curves on S such that* $\mathrm{ord}_{E_i}(\langle M \rangle) = 0$ *and* $M \cdot E_i > 0$. *Then*

$$H^1(S, K_S + \ulcorner M \urcorner + E_1 + \cdots + E_k) = 0 .$$

By Lemma 3, we obtain

$$H^1(S_1, \mathcal{O}_{S_1}(K_{S_1} + \ulcorner T \urcorner + E')) \cong H^1(S_1, \mathcal{O}_{S_1}(\pi^* Q - N - E'')) = 0.$$

Thus the restriction map

$$H^0(S_1, \mathcal{O}_{S_1}(\pi^* Q - N)) \longrightarrow H^0(E'', \mathcal{O}_{E''}(\pi^* Q - N))$$

is surjective. Since E'' is π-exceptional and $x_0 \notin \pi(N)$ by Claim 1, $\mathcal{O}_{E''}(\pi^* Q - N) \cong \mathcal{O}_{E''}$. Therefore we obtain a section $t \in H^0(S_1, \mathcal{O}_{S_1}(\pi^* Q - N))$ which doesn't vanish at some points of $\pi^{-1}(x_0)$. Because $x_0 \notin \pi(N)$, the existence of t shows $x_0 \notin \mathrm{Bs}|Q|$.

Next we consider the case $E''=0$. By Lemma 3,

$$H^1(S_1, \mathcal{O}_{S_1}(K_{S_1} + \ulcorner T \urcorner + E' - E_0)) \cong H^1(S_1, \mathcal{O}_{S_1}(\pi^* Q - N - E_0)) = 0 .$$

Thus the restriction map

$$H^0(S_1, \mathcal{O}_{S_1}(\pi^*Q-N)) \longrightarrow H^0(E_0, \mathcal{O}_{E_0}(\pi^*Q-N))$$

is surjective.

PROPOSITION 2. *For a suitable B, $\ulcorner T \urcorner \cdot E_0 > 1$.*

Since $\ulcorner T \urcorner \cdot E$ is integer, $\ulcorner T \urcorner \cdot E \geqq 2$ by Proposition 2. Then $\deg(\mathcal{O}_{E_0}(\pi^*Q-N)) \geqq 2p_a(E_0)$, because

$$\mathcal{O}_{E_0}(\pi^*Q-N) \cong \mathcal{O}_{E_0}(K_{E_0} + \ulcorner T \urcorner + E_1 + \cdots + E_{m'}) .$$

Since E_0 is Gorenstein, $\mathcal{O}_{E_0}(\pi^*Q-N)$ is globally generated (cf. [5]) and there is a section $t \in H^0(E_0, \mathcal{O}_{E_0}(\pi^*Q-N))$ such that $t(y) \neq 0$ for some point $y \in \pi^{-1}(x_0)$. Thus we can deduce $x_0 \notin \mathrm{Bs}|Q|$. For the rest of the proof of Theorem 7, it is enough to show Proposition 1, 2. Now we will give a proof of Proposition 1, 2.

Proof of Proposition 1. Let $M^2 = (\sigma_2)^2$ and $M \cdot C \geqq \sigma_1$ for all curves $C \subset S$ such that $x_0 \in C$. Note that $\sigma_2 > 2$ and $\sigma_1 > 2$ by the assumption of Theorem 7.

LEMMA 4. *For a suitable B,*

(1) $c < \dfrac{(n+1)\sqrt{3}}{(2n+1)\sigma_2}$, *when the dual graph of $\pi^{-1}(x_0)$ is of type A_n.*

(2) $c < \dfrac{\sqrt{3}}{2\sigma_2}$, *when the dual graph of $\pi^{-1}(x_0)$ is of type D_n.*

(3) $c < \dfrac{1}{\sigma_2}$, *when the dual graph of $\pi^{-1}(x_0)$ is of type E_n.*

We can obtain the assertion of Proposition 1 by the above lemma, because $\sigma_2 > 2$.

Proof of Lemma 4. We show (1) and (2) in parallel. By changing indices, we may assume $\pi^{-1}(x_0) = \sum_{i=1}^{n} E_i$. We define rational numbers a_i by the formula $K_{S_1} \sim_{\mathbf{Q}} \pi^* K_S + \sum a_i E_i$.

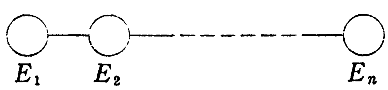

Figure 1. The dual graph of type A_n.

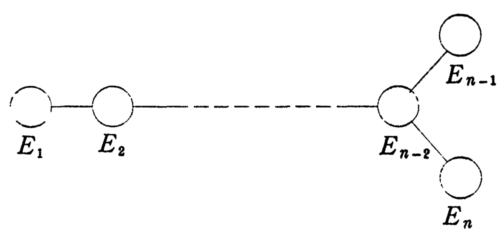

Figure 2. The dual graph of type D_n.

CLAIM 2. (1) *If the dual graph of* $\pi^{-1}(x_0)$ *is as in Figure* 1, *then*

$$a_i \leqq -1+\left(\frac{1}{i}+\frac{1}{n-i+1}\right)\left(\delta_i-\frac{i-1}{i}-\frac{n-i}{n-i+1}\right)^{-1}, \quad \textit{for any } i,$$

where $\delta_i := -(E_i)^2$.

(2) *If the dual graph of* $\pi^{-1}(x_0)$ *is as in Figure* 2, *then*

$$a_i \leqq -1+\frac{1}{i}\left(\delta_i-\frac{i-1}{i}-1\right)^{-1}, \quad \textit{for any } i,$$

where $\delta_i := -(E_i)^2$.

First we shall show the assertion of Lemma 4 (1), (2) by assuming Claim 2. Let E_k be the component which has the minimum self intersection number $(E_k)^2$ in $\{E_i\}$. By Lemma 2, we can take an effective $\boldsymbol{Q}$-divisor B such that $B \sim_{\boldsymbol{Q}} \pi^* M$ and $B-(\sigma_2/(\sqrt{\delta_k}(1-\sigma)))E_k \geqq 0$, $(0<\sigma \ll 1)$. Then

$$b_k \geqq \frac{\sigma_2}{\sqrt{\delta_k}(1-\sigma)}.$$

If the dual graph of $\pi^{-1}(x_0)$ is of type A_n,

$$e_k \leqq a_k \leqq -1+\left(\frac{1}{k}+\frac{1}{n-k+1}\right)\left(\delta_k-\frac{k-1}{k}-\frac{n-k}{n-k+1}\right)^{-1}, \quad \text{for any } k.$$

Thus

$$\begin{aligned} c &\leqq \frac{e_k+1}{b_k} \\ &= \left(\frac{1}{k}+\frac{1}{n-k+1}\right)\left(\delta_k-\frac{k-1}{k}-\frac{n-k}{n-k+1}\right)^{-1}\frac{(1-\sigma)\sqrt{\delta_k}}{\sigma_2} \\ &= \left(\frac{1}{k}+\frac{1}{n-k+1}\right)\left(\sqrt{\delta_k}-\frac{k-1}{k}-\frac{n-k}{n-k+1}\right)\frac{1}{\sqrt{\delta_k}}\Big)^{-1}\frac{1-\sigma}{\sigma_2}. \end{aligned}$$

The function

$$\delta_k \longmapsto \left(\sqrt{\delta_k}-\left(\frac{k-1}{k}+\frac{n-k}{n-k+1}\right)\frac{1}{\sqrt{\delta_k}}\right)^{-1}\frac{1-\sigma}{\sigma_2}$$

is decreasing in δ_k, and because $\delta_k \geqq 3$,

$$\begin{aligned} &\left(\frac{1}{k}+\frac{1}{n-k+1}\right)\left(\delta_k-\frac{k-1}{k}-\frac{n-k}{n-k+1}\right)^{-1}\frac{(1-\sigma)\sqrt{\delta_k}}{\sigma_2} \\ &\leqq \left(\frac{1}{k}+\frac{1}{n-k+1}\right)\left(3-\frac{k-1}{k}-\frac{n-k}{n-k+1}\right)^{-1}\frac{(1-\sigma)\sqrt{3}}{\sigma_2} \\ &< \frac{(n+1)\sqrt{3}}{(k(n-k+1)+n+1)\sigma_2}. \end{aligned}$$

Let $N(k):=k(n-k+1)$. Then $N(k)\geqq n$ because $1\leqq k\leqq n$. Therefore

$$c\leqq\frac{(n+1)\sqrt{3}}{(2n+1)\sigma_2}.$$

This completes the proof of Lemma 4 (1).

If the dual graph of $\pi^{-1}(x_0)$ is type D_n,

$$e_k\leqq a_k\leqq -1+\frac{1}{k}\left(\delta_k-\frac{k-1}{k}-1\right)^{-1}.$$

Thus

$$\begin{aligned}c&\leqq\frac{e_k+1}{b_k}\\&=\frac{1}{k}\left(\delta_k-\frac{k-1}{k}-1\right)^{-1}\frac{\sqrt{\delta_k}(1-\sigma)}{\sigma_2}\\&=\frac{1}{k}\left(\sqrt{\delta_k}-\left(\frac{k-1}{k}+1\right)\frac{1}{\sqrt{\delta_k}}\right)^{-1}\frac{1-\sigma}{\sigma_2}.\end{aligned}$$

The function

$$\delta_k\longmapsto\frac{1}{k}\left(\sqrt{\delta_k}-\left(\frac{k-1}{k}+1\right)\frac{1}{\sqrt{\delta_k}}\right)^{-1}$$

is decreasing in δ_k and because $\delta_k\geqq 3$,

$$\begin{aligned}&\frac{1}{k}\left(\delta_k-\frac{k-1}{k}-1\right)^{-1}\frac{\sqrt{\delta_k}(1-\sigma)}{\sigma_2}\\&\leqq\frac{1}{k}\left(3-\frac{k-1}{k}-1\right)^{-1}\frac{\sqrt{3}(1-\sigma)}{\sigma_2}=\frac{\sqrt{3}(1-\sigma)}{1+k}\\&\leqq\frac{\sqrt{3}(1-\sigma)}{2\sigma_2}<\frac{\sqrt{3}}{2\sigma_2}.\end{aligned}$$

This completes the proof of Lemma 4 (2).

Proof of Claim 2. (1) We consider $K_{S_1}E_j=\pi^*K_SE_j+\sum a_iE_iE_j$. Then we obtain the following linear equations:

$$\begin{aligned}&\delta_1-2=-\delta_1a_1+a_2,\\&\delta_k-2=-\delta_ka_k+a_{k-1}+a_{k+1},\quad(2\leqq k\leqq n-1)\\&\delta_n-2=-\delta_na_n+a_{n-1}.\end{aligned}$$

Thus

$$a_1=-1+P_n+\prod_{j=1}^{n}P_j,$$

and

$$a_k=-1+\Big(\prod_{j=1}^{k-1}Q_j+\prod_{j=1}^{n-k}P_j\Big)(\delta_k-P_{n-k}-Q_{k-1})^{-1},\quad k\neq 1\,,$$

where

$$P_j:=\frac{1}{\delta_{n-j+1}}-\frac{1}{\delta_{n-j+2}}-\cdots-\frac{1}{\delta_n},\quad (j\geqq 2),$$

$$P_1:=\frac{1}{\delta_n},$$

$$Q_j:=\frac{1}{\delta_j}-\frac{1}{\delta_{j-1}}-\cdots-\frac{1}{\delta_1},\quad (j\leqq 2),$$

$$Q_1:=\frac{1}{\delta_1}.$$

Because $\delta_i\geqq 2$,

$$P_j\leqq\frac{j}{j+1}\quad\text{and}\quad Q_j\leqq\frac{j}{j+1}.$$

Thus we obtain

$$\begin{aligned}a_1&=-1+\frac{1}{\delta_1-P_{n-1}}+\frac{1}{\delta_1-P_{n-1}}\prod_{j=1}^{n-1}P_j\\&\leqq-1+\frac{1}{\delta_1-(n-1)/n}+\frac{1}{\delta_1-(n-1)/n}\prod_{j=1}^{n-1}\frac{j}{j+1}\\&=-1+\frac{n+1}{n\delta_1-n+1}\end{aligned}$$

and

$$\begin{aligned}a_k&\leqq-1+\Big(\prod_{j=1}^{k-1}\frac{j}{j+1}+\prod_{j=1}^{n-k}\frac{j}{j+1}\Big)\Big(\delta_k-\frac{k-1}{k}-\frac{n-k}{n-k+1}\Big)\\&=-1+\Big(\frac{1}{k}+\frac{1}{n-k+1}\Big)\Big(\delta_k-\frac{k-1}{k}-\frac{n-k}{n-k+1}\Big),\quad (k\geqq 2).\end{aligned}$$

(2) We consider $K_{S_1}E_j=\pi^*K_SE_j+\sum a_iE_iE_j$. Then we obtain the following linear equations:

$$\delta_1-2=-\delta_1a_1+a_2,$$

$$\delta_k-2=-\delta_ka_k+a_{k-1}+a_{k+1},\quad (2\leqq k\leqq n-3),$$

$$\delta_{n-2}-2=-\delta_{n-2}a_{n-2}+a_{n-3}+a_{n-1}+a_n,$$

$$\delta_j-2=-\delta_ja_j+a_{n-2},\quad (j=n-1,\ n).$$

Note that $\delta_j=2$ for $j=n-1$ or $j=n$. Thus

$$a_1=-1+S_{n-2}$$

$$a_k=-1+\Big(\prod_{j=1}^{k-1}T_j\Big)(\delta_k-T_{k-1}-S_{n-k-2})^{-1}, \quad (2\leqq k\leqq n-3)$$

$$a_{n-2}=-1+\Big(\prod_{j=1}^{n-3}T_j\Big)(\delta_{n-2}-T_{n-3}-1)^{-1},$$

where

$$S_j:=\frac{1}{\delta_{n-j-1}}-\frac{1}{\delta_{n-j}}-\cdots-\frac{1}{\delta_{n-2}-1}, \quad (j\geqq 2),$$

$$S_1:=\frac{1}{\delta_{n-2}-1},$$

$$T_j:=\frac{1}{\delta_j}-\frac{1}{\delta_{j-1}}-\cdots-\frac{1}{\delta_1}, \quad (j\geqq 2),$$

$$T_1:=\frac{1}{\delta_1}.$$

Because $\delta_i\geqq 2$,

$$S_j\leqq 1 \quad \text{and} \quad T_j\leqq\frac{j}{j+1}.$$

Thus we obtain

$$a_1=-1+\frac{1}{\delta_1-S_{n-3}}\leqq -1+\frac{1}{\delta_1-1}$$

and

$$a_k\leqq -1+\Big(\prod_{j=1}^{k-1}\frac{j}{j+1}\Big)\Big(\delta_k-\frac{k-1}{k}-1\Big)^{-1}$$

$$=-1+\frac{1}{k}\Big(\delta_k-\frac{k-1}{k}-1\Big)^{-1}, \quad (k\geqq 2).$$

We have completed the proof of Claim 2. □

We go back the proof of Lemma 4. We consider (3). Define rational numbers $\{a_i\}$ by $K_{S_1}\sim_{\boldsymbol{Q}}\pi^*K_S+\sum a_iE_i$. We denote by $(\mu; a, b, c; d, e)$ the dual graph of type E_n as in Figure 3. Then the classification of dual graphs of type E_n listed in Table 1. By changing indices, we may assume $(E_1)^2=-\mu$. We can take an effective $\boldsymbol{Q}$-divisor B such that $B\sim_{\boldsymbol{Q}}\pi^*M$ and $B-(\sigma_2/(1-\sigma)\sqrt{\mu})E_1$, $(0<\sigma\ll 1)$ by Lemma 2. Considering $K_{S_1}E_j=\pi^*K_SE_j+\sum a_iE_iE_j$, a_1 is computed as in Table 2. We can write $\sqrt{\mu}(a_1+1)$ as

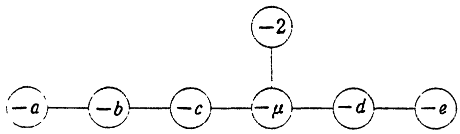

Figure 3. The dual graph of type E_n.

Table 1. List of dual graphs of type E_n.

Type	dual graph
1	μ; 2,2; 2,2
2	μ; 2,2; 3
3	μ; 3; 3
4	μ; 2,2; 2,2,2
5	μ; 3; 2,2,2
6	μ; 2,2; 4
7	μ; 3; 4
8	μ; 2,2; 2,2,2,2
9	μ; 3; 2,2,2,2
10	μ; 2,2; 3,2
11	μ; 3; 3,2
12	μ; 2,2; 2,3
13	μ; 3; 2,3
14	μ; 2,2; 5
15	μ; 3; 5

Table 2. List of a_1.

Type		
1	$a_1=-1+1/(6\mu-11)$	$\mu\geqq 3$
2	$a_1=-1+1/(6\mu-9)$	$\mu\geqq 2$
3	$a_1=-1+1/(6\mu-7)$	$\mu\geqq 2$
4	$a_1=-1+1/(12\mu-23)$	$\mu\geqq 3$
5	$a_1=-1+1/(12\mu-19)$	$\mu\geqq 2$
6	$a_1=-1+1/(12\mu-17)$	$\mu\geqq 2$
7	$a_1=-1+1/(12\mu-13)$	$\mu\geqq 2$
8	$a_1=-1+1/(30\mu-59)$	$\mu\geqq 3$
9	$a_1=-1+1/(30\mu-49)$	$\mu\geqq 2$
10	$a_1=-1+1/(30\mu-47)$	$\mu\geqq 2$
11	$a_1=-1+1/(30\mu-37)$	$\mu\geqq 2$
12	$a_1=-1+1/(30\mu-53)$	$\mu\geqq 2$
13	$a_1=-1+1/(30\mu-43)$	$\mu\geqq 2$
14	$a_1=-1+1/(30\mu-41)$	$\mu\geqq 2$
15	$a_1=-1+1/(30\mu-31)$	$\mu\geqq 2$

$$\sqrt{\mu}(a_1+1)=\frac{\sqrt{\mu}}{\alpha\mu-\beta}=\frac{1}{\alpha\sqrt{\mu}-(\beta/\sqrt{\mu})},$$

where α and β are positive integers. The function

$$\mu \longmapsto \frac{1}{\alpha\sqrt{\mu}-(\beta/\sqrt{\mu})}$$

is decreasing function in μ. Then we can deduce $\sqrt{\mu}(a_1+1)<1$ by Table 2. Thus

$$c\leqq\frac{e_1+1}{b_1}\leqq\frac{a_1+1}{b_1}\leqq\frac{(a_1+1)\sqrt{\mu}(1-\sigma)}{\sigma_2}<\frac{1}{\sigma_2}.$$

This completes the proof of Lemma 4 (3). We have now completed the proof of Lemma 4.

Proof of Proposition 2. We will prove Proposition 2 dividing into two cases.

CASE 1. The dual graph of $\pi^{-1}(x_0)$ is of type A_n, $(n\geqq3)$, D_n or E_n.

CASE 2. The dual graph of $\pi^{-1}(x_0)$ is of type A_n, $(n\leqq2)$.

In Case 1, Proposition 2 follows by the following Claim.

CLAIM 3. *If the dual graph of* $\pi^{-1}(x_0)$ *is of type* A_n, $(n\geqq3)$, D_n *or* E, *then* $c<1/2$.

Since $T\sim_{Q}(1-c)M$,

$$T\cdot E_0=(1-c)M\cdot E_0\geqq(1-c)\sigma_1 .$$

By Claim 3

$$T\cdot E_0>\left(1-\frac{1}{2}\right)\sigma_1=\frac{\sigma_1}{2}.$$

Thus we obtain

$$\ulcorner T\urcorner\cdot E_0\geqq T\cdot E_0>\frac{\sigma_1}{2}>1$$

because $\sigma_1>2$. This completes the proof of Proposition 2 in Case 1. □

Proof of Claim 3. If the dual graph of $\pi^{-1}(x_0)$ is of type A_n, $c<((n+1)\sqrt{3})/(2n+1)\sigma_2$ by Lemma 4. The function

$$n \longmapsto \frac{n+1}{2n+1}$$

is decreasing function in n. Hence

$$c<\frac{(n+1)\sqrt{3}}{(2n+1)\sigma_2}<\frac{4\sqrt{3}}{7\sigma_2}<\frac{4\sqrt{3}}{14}<\frac{1}{2}$$

by Lemma 4, $n\geqq 3$ and $\sigma_2>2$.

If the dual graph of $\pi^{-1}(x_0)$ is of type D_n. By Lemma 4, $c<\sqrt{3}/(2\sigma_2)$. Since $\sigma_2>2$, we obtain $c<\sqrt{3}/4<1/2$.

If the dual graph of $\pi^{-1}(x_0)$ is of type E_n, $c<1/\sigma_2$ by Lemma 4. Because $\sigma_2>2$, we can obtain $c<1/2$. Then we completed the proof of Claim 3. □

We go back the proof of Proposition 2. We consider Case 2. By changing indices, we may assume $\pi^{-1}(x_0)=\sum_{i=1}^{n} E_i$. Define rational numbers $\{a_i\}$ by the following formula $K_{S_1}\sim_Q \pi^*K_S+\sum a_iE_i$. Since $E''=0$, $cb_i-e_i<1$ for $1\leqq i\leqq n$. So $\ulcorner T\urcorner - T\geqq\sum_{i=1}^{n}(cb_i-e_i)E_i$. Hence

$$\ulcorner T\urcorner\cdot E_0\geqq T\cdot E_0+\Big(\sum_{i=1}^{n}(cb_i-e_i)E_i\Big)E_0\,.$$

Because $x_0\in\pi(E_0)$, $E_i\cdot E_0>0$ for some i $(1\leqq i\leqq n)$. We obtain

$$\Big(\sum_{i=1}^{n}(cb_i-e_i)E_i\Big)E_0\geqq\Big(\sum_{i=1}^{n}-e_iE_i\Big)E_0\geqq-\max\{e_i\,|\,1\leqq i\leqq n\}\,.$$

Therefore

$$\ulcorner T\urcorner\cdot E_0\geqq T.E_0-\max\{e_i\,|\,1\leqq i\leqq n\}\,.$$

CLAIM 4. (1) *If* $n=1$, *then* $\max\{e_i\,|\,1\leqq i\leqq n\}\leqq -1/3$.
(2) *If* $n=2$, *then* $\max\{e_i\,|\,1\leqq i\leqq n\}\leqq -1/5$.

Proof. (1) By considering $K_{S_1}E_1=\pi^*K_SE_1+a_1(E_1)^2$, we obtain the linear equation:

$$\delta_1-2=-a_1\delta_1\,,$$

where $\delta_1:=-(E_1)^2$. Thus $a_1=1+2/\delta_1$. Because $\delta_1\geqq 3$,

$$e_1\leqq a_1\leqq -1+\frac{2}{3}=-\frac{1}{3}\,.$$

(2) By considering $K_{S_1}E_i=\pi^*K_SE_i+\sum_{j=1}^{2}a_jE_jE_i$, we obtain the linear equations:

$$\delta_1-2=-a_1\delta_1+a_2$$

$$\delta_2-2=-a_2\delta_2+a_1,$$

where $\delta_i:=-(E_i)^2$. Then we obtain

$$a_1=-1+\frac{1}{\delta_1-(1/\delta_2)}+\frac{1}{\delta_1\delta_2-1}$$

$$a_2=-1+\frac{1}{\delta_2-(1/\delta_1)}+\frac{1}{\delta_1\delta_2-1}\,.$$

We may assume $\delta_1\geqq 3$ and $\delta_2\geqq 2$. Then

$$\frac{1}{\delta_1-(1/\delta_2)} \leqq \frac{3}{5}, \quad \frac{1}{\delta_2-(1/\delta_1)} \leqq \frac{2}{5} \quad \text{and} \quad \frac{1}{\delta_1\delta_2-1} \leqq \frac{1}{5}.$$

Thus

$$\max\{e_i \mid 1 \leqq i \leqq 2\} \leqq \max\{a_i \mid 1 \leqq i \leqq 2\} \leqq -\frac{1}{5}.$$

This completes the proof of Claim 3. □

We go back to the proof of Proposition 2 in Case 2. By Claim 4

$$\begin{aligned} \ulcorner T \urcorner \cdot E_0 &\geqq T \cdot E_0 - \max\{e_i \mid 1 \leqq i \leqq n\} \\ &= (1-c)M \cdot E_0 + \frac{1}{3} \quad (n=1), \\ &= (1-c)M \cdot E_0 + \frac{1}{5} \quad (n=2). \end{aligned}$$

Then by Lemma 4,

$$c < \frac{(n+1)\sqrt{3}}{(2n+1)\sigma_2}.$$

We obtain

$$\begin{aligned} (1-c)M \cdot E_0 &> \left(1 - \frac{2\sqrt{3}}{3\sigma_2}\right)\sigma_1 \quad (n=1), \\ &> \left(1 - \frac{3\sqrt{3}}{5\sigma_2}\right)\sigma_1 \quad (n=2). \end{aligned}$$

Because $\sigma_1 > 2$ and $\sigma_2 > 2$,

$$\begin{aligned} (1-c)M \cdot E_0 &> \left(2 - \frac{2\sqrt{3}}{3}\right) \quad (n=1), \\ &> \left(2 - \frac{3\sqrt{3}}{5}\right) \quad (n=2). \end{aligned}$$

Therefore

$$\begin{aligned} \ulcorner T \urcorner \cdot E_0 &> \left(2 - \frac{2\sqrt{3}}{3}\right) + \frac{1}{3} = \frac{7-2\sqrt{3}}{3} > 1 \quad (n=1), \\ &> \left(2 - \frac{3\sqrt{3}}{5}\right) + \frac{1}{5} = \frac{11-3\sqrt{3}}{5} > 1 \quad (n=2). \end{aligned}$$

We have completed the proof of Proposition 2. □

Now we have completed the proof of Theorem 7. Q.E.D.

4. Statement of main theorem

THEOREM 9. *Let X be a normal projective 3-fold and Δ be an effective $\boldsymbol{Q}$-divisor on X. Assume that (X, Δ) has only log-terminal singularities. Let X_0 be a normal projective variety and $g: X \to X_0$ be a projective morphism. Let Q be a Cartier divisor on X_0 and D be an ample $\boldsymbol{Q}$-Cartier divisor on X_0. Assume that $g^*Q-(K_X+\Delta+g^*D)$ is nef and big. Fix one point x_0 on X_0.*

In the case of $\dim X_0=3$, *suppose that g is a birational morphism and D has the following numerical criterion*:

(1) $D^3>(\sigma_3)^3$,

(2) $D^2 \cdot S>(\sigma_2)^2$, *for all surfaces $S \subset X_0$ such that $x_0 \in S$,*

(3) $D \cdot C>\sigma_1$, *for all curves $C \subset X_0$ such that $x_0 \in C$.*

Then if σ_1, σ_2 and σ_3 satisfy the following conditions:

(1) $\sigma_3 \geqq \dfrac{7}{2}$,

(2) $\left(1-\dfrac{7}{2\sigma_3}\right)\sigma_1 \geqq 2$, *and* $\left(1-\dfrac{7}{2\sigma_3}\right)\sigma_2 \geqq 2$,

$x_0 \notin \mathrm{Bs}\,|Q|$.

In the case $\dim X_0=2$, *suppose g has only connected fibres and D has the following numerical criterion*:

(1) $D^2>(\sigma_2)^2$,

(2) $D \cdot C>\sigma_1$, *for all curves $C \subset X_0$ such that $x_0 \in C$.*

Then if σ_1 and σ_2 satisfy the following conditions:

(1) $\sigma_2 \geqq 3$,

(2) $\left(1-\dfrac{3}{\sigma_2}\right)\sigma_1 \geqq 2$,

$x_0 \notin \mathrm{Bs}\,|Q|$.

In the case of $\dim X_0=1$, *suppose that g has only connected fibres and D has the following numerical criterion*:

(1) $\deg D>1$.

Then $x_0 \notin \mathrm{Bs}\,|Q|$.

Proof of theorems stated in introduction. First we consider Theorem 4. By Base point free theorem [7], we can obtain a normal variety X_0, a projective morphism $g: X \to X_0$ and an ample Cartier divisor H on X_0 such that $L \sim g^*H$ and $g_*\mathcal{O}_X \cong \mathcal{O}_{X_0}$. Let $Q:=nH$ and $D:=(n-a)H$. If $\dim X_0=3$ then the inequalities in Theorem 9 are satisfied with $\sigma_i=11/2$, $(i=1, 2, 3)$. If $\dim X_0=2$ then the inequalities in Theorem 9 are satisfied with $\sigma_i=11/2$, $(i=1, 2)$. If $\dim X_0=1$ then the inequalities in Theorem 9 are satisfied because $\deg D>11/2$. Next we consider Theorem 5. The pair $(X, (1-\varepsilon)\Delta)$ has only log-terminal singularities for $0<\varepsilon<1$, because X is $\boldsymbol{Q}$-factorial. On the other hand, $aL-(K_X+(1-\varepsilon)\Delta)$ is still ample for $0<\varepsilon \ll 1$. Thus we can reduce this theorem to

Theorem 4 by replacing Δ with $(1-\varepsilon)\Delta$. Finally we consider Theorem 6. Similarly we can obtain a normal 3-fold, a birational morphism $g: X \to X_0$ and an ample $\boldsymbol{Q}$-Cartier divisor K_{X_0} such that $K_X \sim_{\boldsymbol{Q}} g^*K_{X_0}$. If $r=1$ then we put $Q := 6K_X$ and $D := (5-\varepsilon)K_X$, $0<\varepsilon\ll 1$. Note that $K_{X_0}^3$ is even. Thus we can put

$$\sigma_1 = \frac{24\sqrt[3]{3}}{5} \quad \text{and} \quad \sigma_i = \frac{24}{5}, \quad \text{for } i=1, 2,$$

in the inequalities of Theorem 9. If $r \geqq 2$ then we put $Q := mrK_{X_0}$ and $D := (mr-1-\varepsilon)K_{X_0}$, $0<\varepsilon\ll 1$. Note that $K_{X_0}^3 \in (1/r)\boldsymbol{Z}$. We can put

$$\sigma_1 = \frac{mr-1-\varepsilon}{\sqrt[3]{r}} \quad \text{and} \quad \sigma_i = \frac{mr-1-\varepsilon}{r}, \quad \text{for } i=1, 2,$$

in the inequalities of Theorem 9.

Proof of Theorem. We may assume that X has only $\boldsymbol{Q}$-factorial terminal singularities, because by Kawamata [8], there is a normal projective 3-fold X', an effective $\boldsymbol{Q}$-divisor Δ', and a birational morphism $\pi: X' \to X$ which satisfy the following conditions:

(1) X' has only $\boldsymbol{Q}$-factorial terminal singularities.

(2) $K_{X'} + \Delta' \sim_{\boldsymbol{Q}} \pi^*(K_X + \Delta)$.

(3) (X', Δ') has only log-terminal singularities.

If we replace X, Δ, g by X', Δ', $g \circ \pi$ all assumptions of Theorem are satisfied.

Let B be an effective $\boldsymbol{Q}$-divisor on X such that $B \sim_{\boldsymbol{Q}} g^*D$, Y be a smooth projective 3-fold and f be a birational morphism $f: Y \to X$. We consider a pair (B, Y, f) which satisfy the following conditions:

(1) There is a simple normal crossing divisor $\sum E_i$ on Y.

(2) $K_Y \sim_{\boldsymbol{Q}} f^*(K_X + \Delta) + \sum e_i E_i$, $e_i > -1$.

(3) $f^*B = \sum b_i E_i$, $b_i \geqq 0$.

(4) There is an ample $\boldsymbol{Q}$-Cartier divisor A on Y such that $f^*(g^*Q - (K_X + \Delta + g^*D)) \sim_{\boldsymbol{Q}} A + \sum p_i E_i$, $0 < p_i \ll 1$ and $e_i + 1 - p_i > 0$.

We define

$$c := \min_i \left\{ \frac{e_i + 1 - p_i}{b_i} \mid x_0 \in g \circ f(E_i),\ b_i > 0 \right\}.$$

By changing indices, if necessary, we may assume that the minimum c attained only at a unique index $i=0$. We obtain the following lemma.

LEMMA 5. (1) *If $cb_i - e_i + p_i < 0$, then E_i is a f-exceptional divisor.*

(2) *If $cb_i - e_i + p_i > 1$, then $x_0 \notin g \circ f(E_i)$.*

Proof. (1) If $cb_i - e_i + p_i < 0$, then $e_i > cb_i + p_i$. By $b_i \geqq 0$, $p_i > 0$, and $c > 0$, we obtain $e_i > 0$.

(2) If $b_i > 0$ and $x_0 \in g \circ f(E_i)$, then $c \leqq (e_i + 1 - p_i)/b_i$. From this inequality, we obtain $cb_i - e_i + p_i \leqq 1$. If $b_i = 0$ and $cb_i - e_i + p_i > 1$, then $e_i < -1 + p_i$. But

this inequality contradicts $e_i+1-p_i>0$. □

We go back the proof of Theorem 9. Let $R:=\sum(cb_i-e_i+p_i)E_i$. Then we can write $[R]=E_0+N-P$, where E_0, N and P are effective Cartier divisors which satisfy the following conditions:

(1) E_0, N and P have no common components.

(2) P is a composite of the f-exceptional divisors.

(3) N is a composite of the divisors E_i such that $x_0 \notin g\circ f(E_i)$.

We define

$$T:=(g\circ f)^*Q+P-E_0-N-K_Y-\langle R\rangle\,.$$

Then

$$T\sim_{\boldsymbol{Q}}(1-c)(g\circ f)^*D+A\,,$$

by

(1) $$(g\circ f)^*Q+P-N-E_0\sim_{\boldsymbol{Q}}K_Y+(1-c)(g\circ f)^*D+A+\sum\langle cb_i-e_i+p_i\rangle E_i.$$

PROPOSITION 3. *For a suitable pair* (B, Y, f),

(1) $c<\dfrac{7}{2\sigma_3}$, *if* $\dim X_0=3$.

(2) $c<\dfrac{3}{\sigma_2}$, *if* $\dim X_0=2$.

(3) $c<\dfrac{1}{\deg D}$, *if* $\dim X_0=1$.

We will prove Proposition 3 in section 6. By the conditions of theorem:

(1) $\sigma_3\geqq\dfrac{7}{2}$, $(\dim X_0=3)$.

(2) $\sigma_2\geqq 3$, $(\dim X_0=2)$.

(3) $\deg D>1$, $(\dim X_0=1)$.

and Proposition 3, we get $1-c>0$. Thus T is nef and big. Supp $\langle T\rangle=$Supp $\langle R\rangle$ is a simple normal crossing divisor. Therefore by Kawamata-Viehweg Vanishing Theorem,

$$H^1(Y, K_Y+\ulcorner T\urcorner)\cong H^1(Y, (g\circ f)^*Q+P-N-E_0)=0\,.$$

Hence the restriction map

$$H^0(Y, (g\circ f)^*Q+P-N)\longrightarrow H^0(E_0, ((g\circ f)^*Q+P-N)|_{E_0})$$

is surjective.

LEMMA 6. *In the following diagram,*

$$\begin{array}{ccc} H^0(Y, (g\circ f)^*Q+P-N) & \xrightarrow{l_1} & H^0(E_0, ((g\circ f)^*Q+P-N)|_{E_0}) \\ v_1 \uparrow & & \uparrow v_2 \\ H^0(Y, (g\circ f)^*Q-N) & \xrightarrow{l_2} & H^0(E_0, ((g\circ f)^*Q-N)|_{E_0}) \end{array}$$

the bottom horizontal map l_2 is surjective.

Proof. For the proof of this lemma, we need the following lemma.

LEMMA 7 ([7] Th. 1-5-2). *Let X be a smooth projective variety, Y be a normal projective variety, and $f: X \to Y$ be a birational morphism. Let Q be a f-exceptional divisor on X. Then $f_*\mathcal{O}_Q(Q)=0$.*

P and N have no common components. So by Lemma 7, $f_*\mathcal{O}_P(P-N)=0$. Hence v_1 is isomorphism. Moreover v_2 is injective and l_1 is surjective. By the commutatibity of this diagram, l_2 is surjective. □

PROPOSITION 4. *There is a section $t\in H^0(E_0, ((g\circ f)^*Q-N)|_{E_0})$ which dose not vanish at $(g\circ f)^{-1}(x_0)$.*

By Proposition 4 and Lemma 6, there is a section $s\in H^0(Y, (g\circ f)^*Q-N)$ which doesn't vanish at $(g\circ f)^{-1}(x_0)$. Because $x_0 \notin g\circ f(N)$, we can deduce $x_0 \notin \mathrm{Bs}|Q|$.

For completing the proof of theorem, it is enough to show Proposition 3 and Proposition 4. We will show Proposition 4 at the section 5 and Proposition 3 at the section 6.

5. The existence of desirable section

Proof of Proposition 4.

The Case $\dim g\circ f(E_0)=0$.

Since $x_0 \notin g\circ f(N)$, $\mathcal{O}_{E_0}((g\circ f)^*Q-N)\cong \mathcal{O}_{E_0}$. Thus the assertion of Proposition 4 follows immediately.

The Case $\dim g\circ f(E_0)=1$.

Let $C:=g\circ f(E_0)$. Take a stein factorization $E_0 \xrightarrow{\rho} C' \xrightarrow{\nu} C$ of $E_0 \xrightarrow{g\circ f} C$. Then C' is a smooth curve and ρ is a flat morphism. Fix one point $x_1\in \nu^{-1}(x_0)$. Let Z be a scheme theoretic inverse of x_1.

LEMMA 8. *The restriction map*

$$H^0(E_0, ((g\circ f)^*Q-N)|_{E_0}) \xrightarrow{\lambda} H^0(Z, ((g\circ f)^*Q-N)|_Z)$$

is nonzero map.

Since $H^0(C', \nu^*Q) \cong H^0(E_0, ((g\circ f)^*Q)|_{E_0})$, the section $s \in H^0(E_0, ((g\circ f)^*Q)|_{E_0})$ which vanish at some point of $(g\circ f)^{-1}(x_1)$ is identically zero on Z. Thus, if Lemma 8 is valid, there is a section $t \in H^0(E_0, ((g\circ f)^*Q-N)|_{E_0})$ which doesn't vanish some point of $(g\circ f)^{-1}(x_0)$.

Proof of Lemma 8. First we prove the following claim.

CLAIM 5. (1) $H^1(E_0, ((g\circ f)^*Q+P-N)|_{E_0}-Z)=0$.
(2) $H^0(Z, ((g\circ f)^*Q+P-N)|_Z) \neq 0$.

Proof. (1) Let $\Delta_{E_0} := \sum \langle cb_i - e_i + p_i \rangle (E_i|_{E_0})$. We define

$$U := ((g\circ f)^*Q+P-N)|_{E_0} - Z - K_{E_0} - \Delta_{E_0} .$$

Then

$$U \sim_{\mathbf{Q}} (1-c)(\rho\circ\nu)^*(D|_C) + A|_{E_0} - Z ,$$

$$\sim_{\mathbf{Q}} \rho^*\mathcal{O}_{C'}((1-c)\nu^*(D|_C) - x_1) + A|_{E_0} ,$$

because

$$((g\circ f)^*Q+P-N)|_{E_0} \sim_{\mathbf{Q}} K_{E_0} + \Delta_{E_0} + (1-c)((g\circ f)^*D)|_{E_0} + A|_{E_0} ,$$

by the equation (1) and $((g\circ f)^*D)|_{E_0} = (\rho\circ\nu)^*(D|_C)$. We obtain

$$\deg \rho^*\mathcal{O}_{C'}((1-c)\nu^*(D|_C) - x_1) > (1-c)\sigma_1 - 1 .$$

By Proposition 3,

$$c < \frac{7}{2\sigma_3}, \quad (\dim X_0 = 3), \quad \text{and} \quad c < \frac{3}{\sigma_2}, \quad (\dim X_0 = 2).$$

Thus

$$(1-c)\sigma_1 > \left(1 - \frac{7}{2\sigma_3}\right)\sigma_1 \geqq 2, \quad (\dim X_0 = 3),$$

and

$$(1-c)\sigma_1 > \left(1 - \frac{3}{\sigma_2}\right)\sigma_1 \geqq 2, \quad (\dim X_0 = 2),$$

by the assumption of theorem. Hence U is ample. On the other hand $\mathrm{Supp}\langle U \rangle = \mathrm{Supp}\langle \Delta_{E_0} \rangle$ is simple normal crossing divisor. Therefore by Kawamata-Viehweg Vanishing Theorem

$$H^1(E_0, K_{E_0} + \lceil U \rceil) \cong H^1(E_0, ((g\circ f)^*Q+P-N)|_{E_0} - Z) = 0 .$$

(2) Since $x \notin g\circ f(N)$, there is a point x' on C' such that $x' \notin g\circ f(N)$. For $Z' := \rho^{-1}(x')$, $H^0(Z', ((g\circ f)^*Q+P-N)|_{Z'}) = H^0(Z', P|_{Z'}) \neq 0$, Because ρ is flat, $H^0(Z, ((g\circ f)^*Q+P-N)|_Z) \neq 0$ by semicontinuity. □

We go back the proof of Lemma 8. Consider the following diagram:

$$\begin{array}{ccc} H^0(E_0, ((g\circ f)^*Q+P-N)|_{E_0}) & \xrightarrow{l_1} & H^0(Z, ((g\circ f)^*Q+P-N)|_Z) \\ v_1 \uparrow & & \uparrow v_2 \\ H^0(E_0, ((g\circ f)^*Q-N)|_{E_0}) & \xrightarrow{l_2} & H^0(Z, ((g\circ f)^*Q-N)|_Z). \end{array}$$

From Claim 5, l_1 is surjection and v_2 is nonzero map and injection. On the other hand v_1 is isomorphism. Thus l_2 is surjection. Moreover $H^0(Z, ((g\circ f)^*Q-N)|_Z)\neq 0$, because $\mathcal{O}_Z((g\circ f)^*Q-N)\cong\mathcal{O}_Z$. Therefore the assertion of Lemma 8 follows.

The Case $\dim g\circ f(E_0)=2$

Let $S_1:=g\circ f(E_0)$. We define that P_1 is composition of the components E_ι of P such that $x_0\in g\circ f(E_\iota)$. Let $P_2:=P-P_1$. We denote by f' (resp. g') the restriction morphism $f|_{E_0}$ (resp. $g|_{f(E_0)}$).

LEMMA 9. (1) *S_1 is normal in a neighborhood of x_0.*
(2) *$P_1|_{E_0}$ is f'-exceptional.*

Proof. Consider the following diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & (g\circ f)_*\mathcal{O}_Y(-E_0+P_1) & & \\ & & \downarrow & & \\ 0 \longrightarrow \mathcal{O}_{X_0} & \longrightarrow & (g\circ f)_*\mathcal{O}_Y(P_1) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 \longrightarrow \mathcal{O}_{S_1} & \longrightarrow & (g'\circ f')_*\mathcal{O}_{E_0}(P_1|_{E_0}) & & \\ & & \downarrow & & \\ & & R_1(g\circ f)_*\mathcal{O}_Y(-E_0+P_1). & & \end{array}$$

We define

$$T':=-E_0+P_1-K_Y-\sum\langle cb_i-b_i+p_i\rangle E_\iota\,.$$

Then

$$T'\sim_{Q}-(g\circ f)^*Q-P_2+N+A+(1-c)(g\circ f)^*D\,,$$

by the equation (1). For a curve such that $C\subset(g\circ f)^{-1}(x_0)$, $P_2\cdot C=N\cdot C=0$, because $\mathrm{Supp}(P_2+N)\cap(g\circ f)^{-1}(x_0)=\emptyset$. Thus T' is $g\circ f$-ample in a some neighborhood V of x_0. $\mathrm{Supp}\langle T'\rangle$ is a simple normal crossing divisor. Therefore, by Kawamata-Viehweg Vanishing Theorem,

$$R^1(g\circ f)_*\mathcal{O}_Y(K_Y+\ulcorner T'\urcorner)|_V\cong R^1(g\circ f)_*\mathcal{O}_Y(-E_0+P_1)|_V=0\,.$$

Hence $(g'\circ f')_*\mathcal{O}_{E_0}(P_1|_{E_0})|_V\cong\mathcal{O}_{S_1}|_V$. The assertions of Lemma follows by this isomorphism. □

We go back the proof of Proposition 4. Since S_1 is normal in a some neighborhood V of x_0, by Lemma 9, there is an open neighborhood V' of x_0 such that $g'\circ f'$ is an isomorphism over $V'\setminus\{x_0\}$. Let $Z:=(g'\circ f')^{-1}(x_0)$. For Z, we define $p: E_0\to S_0$ be a contraction morphism of Z, and $q: S_0\to S_1$ be an induced morphism. Then q is isomorphism over V'. Let $\Delta_{E_0}:=\sum\langle cb_i-e_i+p_i\rangle(E_i|_{E_0})$. Take a relative log minimal model (S, Δ_S) of (E_0, Δ_{E_0}) over S_1.

LEMMA 10. $S\cong S_0$.

Proof. Let

$$E_0 \xrightarrow{\rho} S \xrightarrow{\nu} S_0\,.$$

First we show $\rho_*(P_1|_{E_0})=0$. Assume the contrary. Because $P_1|_{E_0}$ is $g'\circ f'$-exceptional and (S_0, x_0) is a normal point, there is a curve $C\subset\operatorname{Supp}(\rho_*(P_1|_{E_0}))$ such that $\rho_*(P_1|_{E_0})\cdot C<0$. Since $\rho_*(K_{E_0}+\Delta_{E_0})=K_S+\Delta_S$ and

$$K_{E_0}+(1-c)((g\circ f)^*D)|_{E_0}+A|_{E_0}+\Delta_{E_0}\sim_{\mathbf{Q}}((g\circ f)^*Q+P-N)|_{E_0}$$

by the equation (1), we obtain

$$K_S+\Delta_S\sim_{\mathbf{Q}}(q\circ\nu)^*Q+\rho_*((P_1+P_2-N)|_{E_0})-\rho_*(A|_{E_0})-(1-c)(q\circ\nu)^*D\,.$$

Since $\operatorname{Supp}(P_2+N)\cap(g\circ f)^{-1}(x_0)=\emptyset$,

$$(K_S+\Delta_S)\cdot C=-\rho_*(A|_{E_0})\cdot C+\rho_*(P_1|_{E_0})\cdot C\,.$$

Because ρ is a birational morphism of surfaces, ample divisor's push-forward is also ample. Thus $(K_S+\Delta_S)\cdot C<0$, which contradicts with the assumption that (S, Δ_S) is log minimal model. Hence $\rho_*(P_1|_{E_0})=0$. Next we show $\nu^{-1}(x_0)$ is one point. Assume the contrary. Take a curve $C\subset\nu^{-1}(x_0)$. Then

$$(K_S+\Delta_S)\cdot C=-\rho_*(A|_{E_0})\cdot C<0\,,$$

which is contradiction. Therefore ν is isomorphism. □

We go back the proof of Proposition 4. For completing the proof of Proposition 4, we need the following lemma:

LEMMA 11. *(S, Δ_S), $p_*((g\circ f)^*Q+P-N)|_{E_0})$ are satisfy the assumptions of Theorem* 7.

By this lemma, we can choose a section $s\in H^0(S, p_*((g\circ f)^*Q+P-N)|_{E_0})$ such that $s(x_0)\neq 0$. Because P_1 is p-exceptional,

$$p^*(p_*((g\circ f)^*Q+P-N)|_{E_0})\sim((g\circ f)^*Q+P_2-N)|_{E_0}.$$

So there is a setion $s'\in H^0(E_0, (g\circ f)^*Q+P_2-N)|_{E_0})$ which does not vanish at some point of $E_0\cap(g\circ f)^{-1}(x_0)$. Then consider the following diagram.

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y((g\circ f)^*Q+P-N)) & \xrightarrow{l_1} & H^0(E_0, \mathcal{O}_{E_0}(((g\circ f)^*Q+P-N)|_{E_0})) \\ & & \uparrow v_{21} \\ v_1 \uparrow & & H^0(E_0, \mathcal{O}_{E_0}(((g\circ f)^*Q+P_2-N)|_{E_0})) \\ & & \uparrow v_{22} \\ H^0(Y, \mathcal{O}_Y((g\circ f)^*Q-N)) & \xrightarrow{l_2} & H^0(E_0, \mathcal{O}_{E_0}(((g\circ f)^*Q-N)_{E_0})) \end{array}$$

Already we show that v_1 is isomorphism and l_1, l_2 are surjective. Moreover v_{21}, v_{22} are injective, we can deduce v_{21}, v_{22} are isomorphism. Then the section $v_{22}^{-1}(s')$ has the desirable property.

Proof of Lemma 11. First we show the following claim.

CLAIM 6. *Let* $M:=p_*(((g\circ f)^*Q+P-N)|_{E_0})-(K_S+\Delta_S)$. *Then M satisfy the following conditions*:

(1) *M is nef and big,*

(2) $M^2>4$,

(3) $M\cdot C>2$ *for all curves C such that* $q^{-1}(x_0)\in C$.

Proof. By the equation (1),

$$((g\circ f)^*Q+P-N)|_{E_0})\sim_{\mathbf{Q}} K_{E_0}+(1-c)((g\circ f)^*D)|_{E_0}+A|_{E_0}+\Delta_{E_0}.$$

Since (S, Δ_S) is a log minimal model of (E_0, Δ_{E_0}), $p_*(K_{E_0}+\Delta_{E_0})=K_S+\Delta_S$. Because $((g\circ f)^*D)|_{E_0}=(g'\circ f')^*(D|_{S_1})$ and $g'\circ f'=q\circ p$, we obtain $p_*((g\circ f)^*D)|_{E_0}=q^*(D|_{S_1})$. Therefore

$$M\sim_{\mathbf{Q}}(1-c)q^*D|_{S_1}+p_*A|_{E_0}.$$

Then M is nef and big because $D|_{S_1}$ is ample, p is birational and $A|_{E_0}$ is ample. Furthermore

$$M^2=((1-c)q^*D|_{S_1}+p_*A|_{E_0})^2>(1-c)^2(q^*(D|_{S_1}))^2,$$

and

$$M\cdot C=((1-c)q^*D|_{S_1}+p_*A|_{E_0})\cdot C>((1-c)q^*(D|_{S_1}))\cdot C,$$

where C is a curve such that $q^{-1}(x_0)\in C$. By the assumption of theorem,

$$(1-c)^2(q^*(D|_{S_1}))^2=(1-c)^2(D|_{S_1})^2=(1-c)^2D^2\cdot S_1>((1-c)\sigma_2)^2,$$

and

$$(1-c)q^*(D|_{S_1})\cdot C=(1-c)D|_{S_1}\cdot q(C)=(1-c)D\cdot q(C)>(1-c)\sigma_1\,,$$

because q is an isomorphism in a neighborhood of $\{x_0\}$. By Proposition 3, $c<7/(2\sigma_3)$. Thus $1-c>1-7/(2\sigma_3)$. Then, again by the assumption of theorem

$$\left(1-\frac{7}{2\sigma_3}\right)\sigma_2\geqq 2,\quad \text{and}\quad \left(1-\frac{7}{2\sigma_3}\right)\sigma_1\geqq 2\,.$$

Hence we can deduce the assertion of claim. □

We go back the proof of Lemma 11. By Claim 6, we only have to show that $p_*(((g\circ f)^*Q+P-N)|_{E_0})$ is Cartier divisor.

$$p_*(((g\circ f)^*Q+P-N)|_{E_0})=p_*((g'\circ f')^*(Q|_{S_1})+P_1|_{E_0}+P_2|_{E_0}-N|_{E_0})\,.$$

Since $P_1|_{E_0}$ is p-excetional and $g'\circ f'=q\circ p$,

$$\begin{aligned}&p_*((g'\circ f')^*(Q|_{S_1})+P_1|_{E_0}+P_2|_{E_0}-N|_{E_0})\\&\quad=p_*(q\circ p)^*(Q|_{S_1})+p_*(P_2|_{E_0})-p_*(N|_{E_0})\\&\quad=q_*Q|_{S_1}+p_*(P_2|_{E_0})-p_*(N|_{E_0})\,.\end{aligned}$$

Then $p_*(P_2|_{E_0})-p_*(N|_{E_0})$ is Cartier divisor because p is isomorphism over $S\setminus\{q^{-1}(x_0)\}$. Therefore $p_*(((g\circ f)^*Q+P-N)|_{E_0})$ is Cartier divisor. This completes the proof of Lemma 11. □

Now we have completed the proof of Proposition 4. □

6. Estimation of c

Proof of Proposition 3.

(1) Let B be an effective $\boldsymbol{Q}$-divisor on X, Y be a smooth 3-fold and f be a birational morphism $f:Y\to X$. We will construct the pair (B, Y, f) which satisfy the following conditions:

(1) There is a simple normal crossing divisor $\sum E_i$ on Y.

(2) $K_Y\sim_{\boldsymbol{Q}} f^*(K_X+\Delta)+\sum e_iE_i$, $e_i>-1$.

(3) $f^*B=\sum b_iE_i$.

(4) There is an ample $\boldsymbol{Q}$-divisor A on Y such that $f^*(g^*Q-(K_X+\Delta+g^*D))\sim_{\boldsymbol{Q}} A+\sum p_iE_i$, $0<p_i\ll 1$ and $e_i+1-p_i>0$.

We construct these objects dividing into three cases:

CASE 1.1. There is a smooth point in $g^{-1}(x_0)$.

CASE 1.2. $g^{-1}(x_0)$ is one point, and (X, x_1) is terminal singularity of index one, where $x_1:=g^{-1}(x_0)$.

CASE 1.3. $g^{-1}(x_0)$ is one point, and (X, x_1) is terminal singularity of index r, $(r \geqq 2)$, where $x_1 := g^{-1}(x_0)$.

If there is no smooth points in $g^{-1}(x_0)$, $g^{-1}(x_0)$ is one point, because three dimensional terminal singularity is isolated singularity.

CASE 1.1. Let $x_1 \in g^{-1}(x_0)$ be a smooth point. By Lemma 1, we can choose an effective $\boldsymbol{Q}$-divisor B on X such that $\nu_{x_1}(B) > \sigma_3/(1-\sigma)$ and $B \sim_{\boldsymbol{Q}} g^*D$. Let $h: X' \to X$ be the blowing up with center x_1. We take a resolution of singularities $f_0: Y' \to X'$ such that the union of the exceptional locus and the proper transform of $\Delta + B$ by $h \circ f_0$ is a divisor with only simple normal crossings. By Kodaira's Lemma, we can take an effective divisor D' such that $(h \circ f_0)^*(g^*Q - (K_X + \Delta + g^*D)) - \delta D'$ is ample for $0 < \delta < 1$. Taking a succession of blowing-ups with nonsingular center, we can find a smooth variety Y and a birational morphism $f_1: Y \to Y'$ which satisfies the following properties:

(1) the union of the $h \circ f_0 \circ f_1$-exceptional locus, the proper transform of $\Delta + B$ by $h \circ f_0 \circ f_1$ and the proper transform of D' by f_1 is a divisor with only simple normal crossings.

(2) $(h \circ f_0 \circ f_1)^*(g^*Q - (K_X + \Delta + g^*D)) - \delta f_1^* D' - \sum d'_i E_i$ is ample for $0 < d_i \ll 1$, where $\sum d'_i E_i$ is an exceptional divisor.

If we write $K_Y \sim_{\boldsymbol{Q}} (h \circ f_0 \circ f_1)^*(K_X + \Delta) + \sum e_i E_i$, then $e_i > -1$ because (X, Δ) has only log-terminal singularities. If we take δ and d'_i small enough, we can obtain $e_i + 1 - p_i > 0$ by the Logarithmic Ramification Formula. Thus the pair $(B, Y, f := h \circ f_0 \circ f_1)$ satisfies the above four conditions. Suppose F be the exceptional divisor of h. Let E_1 be the proper transform of F by $f_0 \circ f_1$. Then $b_1 > \sigma_3/(1-\sigma)$, $e_1 \leqq 2$. Thus

$$c \leqq \frac{e_1 + 1 - p_1}{b_1} < \frac{e_1 + 1}{b_1} < \frac{3(1-\sigma)}{\sigma_3}.$$

CASE 1.2. Let $x_1 := g^{-1}(x_0)$. Since x_1 is a cDV singularity, $\operatorname{mult}_{x_1} X = 2$. By Lemma 1, we can choose an effective $\boldsymbol{Q}$-divisor B on X such that $\nu_{x_1}(B) > \sigma_3/\sqrt[3]{2}(1-\sigma)$ and $B \sim_{\boldsymbol{Q}} g^*D$. Take the blowing up $h: X' \to X$ with center x_1. Then X' is a normal 3-fold which has only Gorenstein singularity and there is a reduced divisor F on X' such that $K_{X'} \sim h^* K_X + F$. Choose a smooth variety Y and a birational morphism $f': Y \to X'$ as in Case 1.1 such that the pair $(B, Y, f := h \circ f')$ satisfies the above four conditions. Let E_1 be the proper transform of F by f. Then $b_1 > \sigma_3/\sqrt[3]{2}(1-\sigma)$, $e_1 \leqq 1$. Thus

$$c \leqq \frac{e_1 + 1 - p_1}{b_1} < \frac{e_1 + 1}{b_1} < \frac{2\sqrt[3]{2}(1-\sigma)}{\sigma_3}.$$

CASE 1.3. By Kawamata [8], we can choose a normal variety X_2 and a birational morphism $h: X' \to X$ which satisfy the following conditions:

(1) h is an isomorphism over $X\backslash\{x_1\}$.

(2) $K_{X'}\sim_{\boldsymbol{Q}} h^*K_X+(1/r)G+$(other components), where G is a reduced component of the exceptional divisor of h and r is an index of (X, x_1).

Take a smooth point $x_2\in\operatorname{Supp} G$ of X'. By Lemma 1, we can choose an effective $\boldsymbol{Q}$-divisor B' on X' such that $\nu_{x_2}(B')>\sigma_3/(1-\sigma)$ and $B'\sim_{\boldsymbol{Q}}(g\circ h)^*D$. We put $B:=h_*B'$. After take the blowing-up $h':X''\to X'$ with center x_2, we select a smooth variety Y and a birational morphism f' as in Case 1.1 such that the pair $(B, Y, f:=h\circ h'\circ f')$ satisfies the above four conditions. Suppose F the exceptional divisor of h'. Let E_1 be the proper transform of F by f'. Then $b_1>\sigma_3/(1-\sigma)$, $e_1\leqq 2+1/r\leqq 5/2$. Thus

$$c\leqq\frac{e_1+1-p_1}{b_1}<\frac{e_1+1}{b_1}<\frac{7(1-\sigma)}{2\sigma_3}.$$

Now we complete the proof of Proposition 3 of (1).

(2) We will construct normal varieties X', X_1, and a reduced divisor F on X' which satisfy the following conditions:

(1) There is birational morphisms $h:X'\to X$, and $\bar{h}:X_1\to X_0$.

(2) There is a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & X_0 \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \bar{h}} \\ X' & \xrightarrow{\ \bar{g}\ } & X_1 \end{array}$$

(3) $\bar{h}(F)$ is one point x_1, where (X_1, x_1) is a smooth point and $x_1\in\bar{h}^{-1}(x_0)$.

(4) $K_{X'}\sim_{\boldsymbol{Q}} h^*K_X+aF+$(other components), $(a\leqq 2)$.

First we show if we construct these objects, the assertion of proposition follows. By Lemma 1, we can choose an effective $\boldsymbol{Q}$-divisor B' such that $B'\sim_{\boldsymbol{Q}}\bar{h}^*D$ and $\nu_{x_1}(B')>\sigma_2/(1-\sigma)$, $(0<\sigma\ll 1)$. Let $B:=h_*\bar{g}^*B'$. Take a smooth variety Y and a birational morphism $f':Y\to X'$ as in Case 1.1 such that the pair $(B, Y, f:=g\circ h\circ f')$ satisfies the four conditions in (1). Let E_1 be the proper transform of F. By the definition of c,

$$c\leqq\frac{e_1+1-p_1}{b_1}.$$

By construction of F and B, $e_1\leqq 2$ and $b_1>\sigma_2/(1-\sigma)$. Thus

$$\begin{aligned} c&\leqq\frac{(2+1-p_1)(1-\sigma)}{\sigma_2} \\ &<\frac{3}{\sigma_2}. \end{aligned}$$

We construct these objects dividing into six cases.

CASE 2.1. (X_0, x_0) is smooth point, and $\dim g^{-1}(x_0)=2$.

CASE 2.2. (X_0, x_0) is smooth point, $\dim g^{-1}(x_0)=1$, and there is a singular point of X in $g^{-1}(x_0)$.

CASE 2.3. (X_0, x_0) is smooth point, $\dim g^{-1}(x_0)=1$, and there is no smooth points of X in $g^{-1}(x_0)$.

CASE 2.4. (X_0, x_0) is singular point, and $\dim g^{-1}(x_0)=2$.

CASE 2.5. (X_0, x_0) is singular point, $\dim g^{-1}(x_0)=1$, and there is a singular point of X in $g^{-1}(x_0)$.

CASE 2.6. (X_0, x_0) is singular point, $\dim g^{-1}(x_0)=1$, and there is no singular points of X in $g^{-1}(x_0)$.

CONSTRUCTION OF (X', X_1, F).

CASE 2.1. This Case is verv easy. Let $X':=X$, $X_1:=X_0$. We take a divisor $F\subset g^{-1}(x_0)$. Then X', X_1 and F satisfy the conditions.

CASE 2.2. Let $X_1:=X_0$. Similarly to the Case 1.3, we take a normal variety X', birational morphism $h: X'\to X$, and the h-exceptional divisor F. Then X', X_1 and F satisfy the conditions.

CASE 2.3. Let $X_1:=X_0$. We choose a curve C in $g^{-1}(x_0)$ and take an embedded resolution of C, $\tilde{h}_1: \tilde{X}\to X$. Let C' be the proper transform of C. We take the blowing up along C', $\tilde{h}_2: X'\to\tilde{X}$. We define F is the exceptional divisor of $\tilde{h}_2$. Then X', X_1 and F satisfy the conditions.

CASE 2.4. Let $\bar{h}: X_1\to X_0$ be the minimal resolution of (X_0, x_0). Take a divisor $F'\subset g^{-1}(x_0)$. By Hironaka [6], we can choose a smooth variety $\tilde{X}$ and a birational morphism $\tilde{h}: \tilde{X}\to X$ which satisfy the following conditions:

(1) There is a morphism $\bar{g}: \tilde{X}\to X_1$ such that $g\circ\tilde{h}=\bar{h}\circ\bar{g}$.

(2) There is a simple normal crossing divisor $\sum$ on $\tilde{X}$ such that

$$\operatorname{Supp}\textstyle\sum=(\tilde{h}\text{-exceptional divisor})\cup(\text{The proper transform of } F').$$

Let F'' be the proper transform of F'. If $\bar{g}(F'')$ is one point, then we set $F:=F''$ and $X':=\tilde{X}$. If $\bar{g}(F'')$ is a curve I, we choose a point $x_1\in I$ such that $\bar{g}^{-1}(x_1)\cap F''$ is not contained any $\tilde{h}$-exceptional divisor. Then take a curve C such that $\bar{g}(C)=x_1$ and C is not contained any $\bar{h}$-exceptional divisors. We construct X' and F similarly to Case 2.3. Since C is not contained any $\bar{h}$-exceptional divisors,

$$K_{X'}\sim_{Q}\bar{h}^{*}K_X+F+(\text{other components}).$$

Other conditions are easily checked.

CASE 2.5. First we construct a normal variety $\tilde{X}$, a birational morphism $\tilde{g}: \tilde{X} \to X$ and $\tilde{g}$-exceptional divisor $\tilde{F}$ as Case 1.3. Then we can construct X', X_1 and F similarly to the Case 2.4. We only have to check condition 4. Since $K_{\tilde{X}} \sim_{\boldsymbol{Q}} \tilde{g}^* K_X + aF +$(other components), $(a \leqq 1)$,

$$K_{X'} \sim_{\boldsymbol{Q}} (\tilde{g} \circ \bar{h})^* K_X + (a+1)F + \text{(other components)}.$$

Other conditions are easily checked.

CASE 2.6. First we construct a normal variety $\tilde{X}$, a birational morphism $\tilde{g}: \tilde{X} \to X$ and $\tilde{g}$-exceptional divisor $\tilde{F}$ as Case 2.3. Then we can construct X', X_1 and F similarly to the Case 2.4. We only have to check condition 4. Since $K_{\tilde{X}} \sim_{\boldsymbol{Q}} \tilde{g}^* K_X + F +$(other components),

$$K_{X'} \sim_{\boldsymbol{Q}} (\tilde{g} \circ \bar{h})^* K_X + 2F + \text{(other components)}.$$

Other conditions are easily checked. Then we complete the proof of Proposition of (2).

(3) We take a rational number σ_1 such that $\deg D > \sigma_1 > 1$. Since x_0 is a smooth point of X_0, by Lemma 1 we can choose an effective $\boldsymbol{Q}$-divisor B' on X_0 such that $B' \sim_{\boldsymbol{Q}} D$ and $\nu_{x_0}(B') > \sigma_1/(1-\sigma)$. We define $B := g^* B'$. Select a smooth variety Y and a birational morphism f such that the pair (B, Y, f) satisfies the four conditions in (1). Suppose F be an irreducible component of $g^{-1}(x_0)$. Let E_1 be the proper transform of F by f. Then $b_1 > \sigma_1/(1-\sigma)$, $e_1 \leqq 0$. Thus

$$c \leqq \frac{e_1 + 1 - p_1}{b_1} < \frac{e_1 + 1}{b_1} < \frac{(1-\sigma)}{\sigma_1} < \frac{1}{\deg D}.$$

We complete of the proof of Proposition of (3). □

Now we have completed the proof of Theorem 9. Q.E.D.

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