E. MUES AND M. REINDERS
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MEROMORPHIC FUNCTIONS SHARING ONE VALUE AND UNIQUE RANGE SETS

E. MUES AND M. REINDERS

Abstract

We show that there exists a set S with 13 elements such that the condition $E_f(S)=E_g(S)$ implies $f=g$ for any pair of non-constant meromorphic functions f and g. The main tool is a general estimate on two meromorphic functions sharing only one value CM.

1. Introduction and Results

In this paper a meromorphic function is always meromorphic in the complex plane $\boldsymbol{C}$. We use the standard notations of Nevanlinna theory such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $S(r, f)$ etc. (see [2], for example). For $s\in\boldsymbol{N}$ we denote by $N_{[s]}(r, f)$ the Nevanlinna counting function of the poles of f where a p-fold pole is counted with multiplicity $\min(s, p)$. ∂_f is the divisor of the meromorphic function f.

We say that two meromorphic functions f and g share the value $a\in\hat{\boldsymbol{C}}$ IM (ignoring multiplicities) if $f^{-1}(\{a\})=g^{-1}(\{a\})$. f and g share the value a CM (counting multiplicities) if a k-fold a-point z_0 of f is also a k-fold a-point of g and vice versa, $k=k(z_0)$.

Let S be a subset of $\hat{\boldsymbol{C}}$. For a meromorphic function f we define

$$E_f(S)=\bigcup_{a\in S}\{(z, p)\,|\,f(z)=a \text{ with multiplicity } p\geqq 1\}.$$

S is called a *unique range set for meromorphic functions* (URSM) if for any two non-constant meromorphic functions f and g the condition $E_f(S)=E_g(S)$ implies $f=g$.

Note that $E_f(S)=E_g(S)$ if and only if $f(z)=a\in S$ implies $g(z)=b$ for some $b\in S$ with the same multiplicity, and vice versa.

Li and Yang [7, 8] proved that there are URSM with finitely many elements. In particular, they gave examples of URSM with 15 elements. On the

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other hand, they showed that any URSM must have at least 5 elements.

In this paper, we show that there are URSM with 13 elements. This is a consequence of the following theorem (compare also Theorem 1 in [8]).

THEOREM. *Let $m \geqq 2$, $n \geqq 2m+9$ be relatively prime integers and $a, b \in \boldsymbol{C} \setminus \{0\}$ such that the polynomial $w^n + aw^{n-m} + b$ has only simple zeros. Then the set $S = \{w \in \boldsymbol{C} \mid w^n + aw^{n-m} + b = 0\}$ is a URSM.*

To prove this theorem we state a general lemma on meromorphic functions sharing one value only.

LEMMA. *Let F and G be non-constant meromorphic functions sharing the value 1 CM. If $F \neq G$ and $FG \neq 1$ then*

$$T(r, F) \leqq \tilde{N}\left(r, \frac{1}{F}, \frac{1}{G}\right) + \tilde{N}(r, F, G) + o(T(r, F) + T(r, G)) \tag{1}$$

for $r \to \infty$ outside a set of finite measure.

Here $\tilde{N}(r, F, G)$ is a Nevanlinna counting function of the points z_0 where $F(z_0) = \infty$ or $G(z_0) = \infty$. Each points z_0 is counted in the following way:

- If $\partial_F(z_0) = -p < 0$ and $\partial_G(z_0) \geqq 0$ then z_0 is counted in $\tilde{N}$ with multiplicity $\min(p, 2)$.
- If $\partial_F(z_0) \geqq 0$ and $\partial_G(z_0) = -q < 0$ then z_0 is counted in $\tilde{N}$ with multiplicity $\min(q, 2)$.
- If $\partial_F(z_0) = -p < 0$ and $\partial_G(z_0) = -q < 0$ then z_0 is counted in $\tilde{N}$ with multiplicity 3 if $p \neq q$ and with multiplicity 2 if $p = q$.

Note that

$$\tilde{N}(r, F, G) \leqq N_{[2]}(r, F) + N_{[2]}(r, G),$$

$$\tilde{N}(r, F, G) \leqq 3\bar{N}(r, F) \qquad \text{if } F \text{ and } G \text{ share } \infty \text{ IM},$$

$$\tilde{N}(r, F, G) \leqq 2\bar{N}(r, F) \qquad \text{if } F \text{ and } G \text{ share } \infty \text{ CM}.$$

It turns out that the lemma also allows a unified access to some unicity theorems which arise from shared value problems (see [3, 6, 7, 8, 9, 10, 11], for example). This will be discussed in section 4.

2. The proof of the lemma

In order to prove the lemma we use a special case of Cartan's second main theorem on holomorphic curves. Cartan's theorem seems to be more flexible here than Nevanlinna's theorem on Borel's identities ([4]) which is used by the authors cited above in similar cases.

THEOREM A (Cartan). *Let g_1, g_2, g_3 be linearly independent entire functions* *without common zeros and* $g_4=g_1+g_2+g_3$. *Then for* $k, l\in\{1, 2, 3, 4\}$

$$T\left(r, \frac{g_k}{g_l}\right)\leqq \sum_{j=1}^{4} N\left(r, \frac{1}{g_j}\right)-N\left(r, \frac{1}{W}\right)+S(r). \tag{2}$$

Here $W=W(g_1, g_2, g_3)$ *is the Wronkian of* g_1, g_2, g_3 *and*

$$S(r)=o\left(T\left(r, \frac{g_2}{g_1}\right)+T\left(r, \frac{g_3}{g_1}\right)\right)$$

for $r\to\infty$ *outside a set of finite measure.*

For a proof see [1] or [5].

Let us make some remarks on how to estimate the term

$$N^*(r)=\sum_{j=1}^{4} N\left(r, \frac{1}{g_j}\right)-N\left(r, \frac{1}{W}\right) \tag{3}$$

in Cartan's theorem. First we note that

$$W(g_1, g_2, g_3)=W(g_1, g_2, g_4)=W(g_1, g_4, g_3)=W(g_4, g_2, g_3).$$

Let $z_0\in \boldsymbol{C}$ and suppose that

$$\partial_{g_j}(z_0)=p\geqq 1 \qquad \text{for some } j\in\{1, 2, 3, 4\}.$$

Since g_1, g_2, g_3 have no common zeros there are exactly two cases to consider:

(i) $\partial_{g_k}(z_0)=0$ for $k\neq j$,

(ii) $\partial_{g_k}(z_0)=q\geqq 1$ for some $k\neq j$ and $\partial_{g_l}(z_0)=0$ for $l\neq j, k$.

In case (i) we have $\partial_W(z_0)\geqq p-2$ if $p\geqq 2$, hence

(4) $\qquad z_0$ contributes at most $\min(p, 2)$ to $N^*(r)$.

In case (ii) if $p\neq q$ we have $p+q\geqq 3$ and $\partial_W(z_0)\geqq p+q-3$, so

(5) $\qquad z_0$ contributes at most 3 to $N^*(r)$ if $p\neq q$.

If $p=q$ we have $\partial_W(z_0)\geqq 2p-2$ and thus

(6) $\qquad z_0$ contributes at most 2 to $N^*(r)$ if $p=q$.

Now let F and G be non-constant meromorphic functions sharing the value 1 CM. Define the meromorphic function h by

$$h=\frac{F-1}{G-1}. \tag{7}$$

Then

$$F+h-hG=1. \tag{8}$$

Suppose first that the functions F, h and $-hG$ are linearly independent. Let P be a Weierstraßproduct with zeros exactly at the poles of F and with the corresponding multiplicities. Then

(9) $$PF+Ph-PhG=P$$

and the functions

(10) $$g_1=PF,\quad g_2=Ph,\quad g_3=-PhG \quad\text{and}\quad g_4=P$$

satisfy the hypotheses of Cartan's theorem. It follows that

(11) $$T(r, F)=T\left(r, \frac{g_1}{g_4}\right)\leqq N^*(r)+S(r)$$

where $N^*(r)$ is defined in (3) and the error term satisfies

(12) $$S(r)=o\left(T\left(r, \frac{h}{F}\right)+T\left(r, \frac{hG}{F}\right)\right)=o(T(r, F)+T(r, G))$$

for $r\to\infty$ outside a set of finite measure. Using (7) and (10) we see that

$$N\left(r, \frac{1}{g_1}\right)=N\left(r, \frac{1}{F}\right),\quad N\left(r, \frac{1}{g_2}\right)=N(r, G),$$

$$N\left(r, \frac{1}{g_3}\right)=N\left(r, \frac{1}{G}\right),\quad N\left(r, \frac{1}{g_4}\right)=N(r, F).$$

By the remarks (4), (5) and (6) made in estimating the term $N^*(r)$ we get

(13) $$N^*(r)\leqq \tilde{N}\left(r, \frac{1}{F}, \frac{1}{G}\right)+\tilde{N}(r, F, G).$$

If we combine (11), (12) and (13) we get the desired estimate (1).

Now we assume that the functions F, h and $-hG$ are linearly dependent. Then

(14) $$c_1F+c_2h-c_3hG=0$$

where c_1, c_2, c_3 are constants not all equal to zero. If $c_1=0$ it follows that F or G is constant. So $c_1\neq 0$ and we may assume that $c_1=1$. From (14) we get

(15) $$G=\frac{(c_2-1)F-c_2}{(c_3-1)F-c_3}$$

where

(16) $$c_2\neq c_3$$

since G is not constant. We consider three cases:

Case 1: $c_2\neq 0, 1$. From (15) we see that $G(z_0)=0$ if and only if $F(z_0)-c_2/(c_2-1)=0$. Using the second main theorem we get

$$T(r,\ F)+S(r,\ F)\leqq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-c_2/(c_2-1)}\right)+\overline{N}(r,\ F)$$

$$=\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,\ F)$$

$$\leqq\tilde{N}\left(r,\frac{1}{F},\frac{1}{G}\right)+\tilde{N}(r,\ F,\ G).$$

Thus (1) holds in this case.

Case 2: $c_3\neq 0,\ 1$. In this case we get the inequality (1) in a similar way.

Case 3: $c_2\in\{0,\ 1\}$ and $c_3\in\{0,\ 1\}$. If $c_2=0$ then $c_3=1$ because of (16). Substituting these values in (15) gives $F=G$. If $c_2=1$ then $c_3=0$ and (15) gives $FG=1$.

3. The proof of the theorem

Let f and g be non-constant meromorphic functions satisfying $E_f(S)=E_g(S)$. We have to show that $f=g$. Without loss of generality we may assume that

$$T(r,\ g)\leqq T(r,\ f),\qquad r\in I \tag{17}$$

for some set $I\subset(0,\ \infty)$ of infinite Lebesgue measure. The functions F and G defined by

$$F=-\frac{1}{b}(f^n+af^{n-m}),\qquad G=-\frac{1}{b}(g^n+ag^{n-m}) \tag{18}$$

share the value 1 CM. We denote the zeros of w^m+a by $u_1,\ \cdots,\ u_m$. According to the lemma, we distinguish three cases.

Case 1: $F\neq G$ and $FG\neq 1$. Then

$$T(r,\ F)\leqq N_{[2]}\left(r,\frac{1}{F}\right)+N_{[2]}\left(r,\frac{1}{G}\right)+N_{[2]}(r,\ F)+N_{[2]}(r,\ G)$$

$$+o(T(r,\ F)+T(r,\ G)),\qquad r\notin E.$$

Using (18) and (17) this gives

$$nT(r,\ f)\leqq 2\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{m}N_{[2]}\left(r,\frac{1}{f-u_j}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+\sum_{j=1}^{m}N_{[2]}\left(r,\frac{1}{g-u_j}\right)$$

$$+2\overline{N}(r,\ f)+2\overline{N}(r,\ g)+o(T(r,\ f)+T(r,\ g)),\qquad r\notin E$$

$$\leqq(2m+8)T(r,\ f)+o(T(r,\ f)),\qquad r\in I\backslash E.$$

It follows that $n\leqq 2m+8$. Since we assumed $n\geqq 2m+9$ this case can not occur.

Case 2: $FG=1$. In this case

$$f^{n-m}(f^m+a)g^{n-m}(g^m+a)=b^2.$$

If $f(z_0)=0$ or $f^m(z_0)+a=0$ then $g(z_0)=\infty$ and hence $g^{n-m}(g^m+a)$ has a pole of order at least n at z_0. It follows that every zero of f has multiplicity at least two and every zero of f^m+a has multiplicity at least n. The second main theorem gives

$$\begin{aligned}(m-1)T(r, f)+S(r, f)&\leqq \bar{N}\Big(r, \frac{1}{f}\Big)+\sum_{j=1}^{m}\bar{N}\Big(r, \frac{1}{f-u_j}\Big)\\ &\leqq \frac{1}{2}N\Big(r, \frac{1}{f}\Big)+\frac{1}{n}\sum_{j=1}^{m}N\Big(r, \frac{1}{f-u_j}\Big)\\ &\leqq \Big(\frac{1}{2}+\frac{m}{n}\Big)T(r, f)+O(1).\end{aligned}$$

Hence $m-1\leqq(1/2+m/n)$. Because of $m\geqq 2$ we conclude that

$$n\leqq\frac{m}{m-3/2}\leqq 4.$$

This is a contradiction to our assumptions.

Case 3: $F=G$. Then

$$f^n+af^{n-m}=g^n+ag^{n-m}.$$

As in [8] we set $h=f/g$ and get

$$g^m(h^n-1)=-a(h^{n-m}-1). \tag{19}$$

Let $z_0\in \boldsymbol{C}$ be a point with $h^n(z_0)=1$ but $h(z_0)\neq 1$. Then $h^{n-m}\neq 1$ since n and $n-m$ are relatively prime. Thus $h^n(z_0)=1$ with multiplicity at least m. It follows that h has $n-1$ completely ramified values. If h is not constant, the second main theorem implies $n-1\leqq 4$ in contrast to our assumptions. Hence h is constant. Since g is not constant, (19) gives $h=1$ which means that $f=g$.

This proves the theorem.

4. Concluding remarks

As we already mentioned in the introduction, there is a series of shared value problems which can be treated in a unified way with the help of the lemma. As an example, we quote the following result of Hua [3].

THEOREM B. *Let f and g be non-constant meromorphic functions. Suppose that f and g share the value* 1 *CM and that*

$$\Delta=\delta(0, f)+\delta(0, g)+\delta(\infty, f)+\delta(\infty, g)>3. \tag{20}$$

Then $f=g$ or $fg=1$.

Proof. Without loss of generality we may assume that there exists a set

$I \subset (0, \infty)$ of infinite measure such that $T(r, g) \leqq T(r, f)$ for $r \in I$. If $f \neq g$ and $fg \neq 1$, the lemma gives

$$T(r, f) \leqq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + N(r, f) + N(r, g) + S(r)$$

$$\leqq (4 + 4\varepsilon - \Delta) T(r, f) + S(r, f) \qquad \text{if } r \in I,\ \varepsilon > 0 .$$

It follows that $\Delta \leqq 3$. □

The example

$$f(z) = e^{2z} - e^{z}, \qquad g(z) = \frac{e^{2z}}{e^{z}+1} \tag{21}$$

shows that the bound 3 in (20) is best possible. It also shows that we may have equality in (1).

In a similar way one can use the lemma in all situations where $f^{(n)}$ and $g^{(n)}$ share the value 1 CM by setting $F = f^{(n)}$ and $G = g^{(n)}$.

Finally let us note the following corollary of the lemma.

COROLLARY. *Let f and g be non-constant meromorphic functions sharing the values 0 and ∞ IM and the value 1 CM. If*

$$\limsup_{r \to \infty} \frac{\bar{N}(r, 1/f) + \bar{N}(r, f)}{T(r, f)} < \frac{1}{3},$$

then $f = g$ or $fg = 1$.

Proof. If $f \neq g$ and $fg \neq 1$, the lemma gives

$$T(r, f) \leqq \tilde{N}\left(r, \frac{1}{f}, \frac{1}{g}\right) + \tilde{N}(r, f, g) + S(r)$$

$$\leqq 3\bar{N}\left(r, \frac{1}{f}\right) + 3\bar{N}(r, f) + S(r) . \quad □$$

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INSTITUT FÜR MATHEMATIK
UNIVERSITÄT HANNOVER
POSTFACH 6009
D-30060 HANNOVER
GERMANY
e-mail: mues@math.uni-hannover.de

INSTITUT FÜR MATHEMATIK
UNIVERSITÄT HANNOVER
POSTFACH 6009
D-30060 HANNOVER
GERMANY
e-mail: reinders@math.uni-hannover.de