H. UEDA
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UNICITY THEOREMS FOR MEROMORPHIC FUNCTIONS SHARING FIVE OR SIX VALUES IN SOME SENSE

Dedicated to Professor Mitsuru Nakai
on the occasion of his 60th birthday

HIDEHARU UEDA

1. Notations

In this paper the term "meromorphic function" will mean a meromorphic function in C. We will use the standard notations of Nevanlinna theory: $T(r, f)$, $m(r, c, f)$, $N(r, c, f)$, $\bar{N}(r, c, f)$, $N_1(r, f)$, $\Theta(c, f)$ $(c\in C\cup\{\infty\})$, and we assume that the reader is familiar with the basic results in Nevanlinna theory as found in [2]. Further, we will use the notations defined in the following (i)-(iv):

(i) Let f and g be distinct nonconstant meromorphic functions. For $r>0$, put $T(r)=\max\{T(r, f), T(r, g)\}$. We write $\sigma(r)=S(r)$ for every function $\sigma:(0, \infty)\to(-\infty, \infty)$ satisfying $\sigma(r)/T(r)\to 0$ for $r\to\infty$ possibly outside a set of finite Lebesgue measure.

(ii) For a nonconstant meromorphic function f, $c\in C\cup\{\infty\}$ and a positive integer k, we denote by $\bar{n}(r, c, f; k)$ the number of distinct roots of the equation $f=c$ with multiplicity k in $|z|\leqq r$. We write

$$\bar{N}(r, c, f; k)=\int_0^r\{\bar{n}(t, c, f; k)-\bar{n}(0, c, f; k)\}/t\ dt+\bar{n}(0, c, f; k)\log r.$$

(iii) For a nonconstant meromorphic function f, $c\in C\cup\{\infty\}$ and a positive integer k, we denote by $\bar{n}(r, c, f; \leqq k)$ the number of distinct roots of the equation $f=c$ with multiplicities less than or equal to k in $|z|\leqq r$. We write

$$\bar{N}(r, c, f; \leqq k)=\int_0^r\{\bar{n}(t, c, f; \leqq k)-\bar{n}(0, c, f; \leqq k)\}/t\ dt+\bar{n}(0, c, f; \leqq k)\log r.$$

(iv) Let f be a nonconstant meromorphic function. If $c\in C\cup\{\infty\}$ and k is a positive integer or $+\infty$, then we write $E_k(c, f)=\{z\in C: z$ is a root of $f=c$ of order less than or equal to $k.\}$.

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2. Results

The starting point of our argument in this paper is the following facts:

THEOREM A. *Let f and g be nonconstant meromorphic functions. Assume that there exist distinct 6 elements $a_1, \cdots, a_6$ in $C\cup\{\infty\}$ such that $E_2(a_j, f)=E_2(a_j, g)$ for $j=1, \cdots, 6$. Then $f\equiv g$.*

THEOREM B. *Let f and g be nonconstant meromorphic functions. Assume that there exist distinct 7 elements $a_1, \cdots, a_7$ in $C\cup\{\infty\}$ such that $E_1(a_j, f)=E_1(a_j, g)$ for $j=1, \cdots, 7$. Then $f\equiv g$.*

These two results are due to Bhoosnurmath and Gopalakrishna [1]. As we have already pointed out in [4, p. 458], in the above two results, the assumption on the number of distinct elements $\{a_j\}$ satisfying $E_k(a_j, f)=E_k(a_j, g)$ cannot be improved. Without loss of generality, we may assume that $a_1=\infty$, $a_2=0$, $a_3=1$, $a_4=a$, $a_5=b$ (, $a_6=c$). Then our examples in [4] show that

(I) if $\{a, b\}=\{\omega, \omega^2\}$, there exists a pair of distinct nonstant meromorphic functions F and G satisfying $F^3\equiv G^3$ and $E_2(a_j, F)=E_2(a_j, G)=\emptyset$ for $j=3, 4, 5$, where ω ($\neq 1$) is a cubic root of 1 (Clearly, F and G share two values 0 and ∞ CM (=counting multiplicities).), and

(II) if $\{a, b, c\}=\{i, -1, -i\}$, there exists a pair of distinct nonconstant meromorphic functions ϕ and χ satisfying $\phi^4\equiv\chi^4$ and $E_1(a_j, \phi)=E_1(a_j, \chi)=\emptyset$ for $j=3, 4, 5, 6$. (Clearly, ϕ and χ share two values 0 and ∞ CM.).

The main results of this paper are the following:

THEOREM 1. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values 0 and ∞ CM, and further that they satisfy $E_2(a_j, f)=E_2(a_j, g)$ for $j=3, 4, 5$, where $a_3=1$, $a_4=a$, $a_5=b$. (i) If $\{a, b\}=\{\omega, \omega^2\}$, then $f^3\equiv g^3$. (ii) If $\{a, b\}\neq\{\omega, \omega^2\}$, then $f\equiv g$.*

THEOREM 2. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values 0 and ∞ CM, and further that they satisfy $E_1(a_j, f)=E_1(a_j, g)$ for $j=3, 4, 5, 6$, where $a_3=1$, $a_4=a$, $a_5=b$, $a_6=c$. (i) If $\{a, b, c\}=\{i, -1, -i\}$, then $f^4\equiv g^4$. (ii) If $\{a, b, c\}=\{\alpha, -1, -\alpha\}$ ($\alpha^2\neq -1$), then $f^2\equiv g^2$. (iii) If $\{a, b, c\}\neq\{\alpha, -1, -\alpha\}$, then $f\equiv g$.*

3. Elementary estimates on meromorphic functions satisfying $E_2(a_j, f)=E_2(a_j, g)$ for five distinct values a_j ($j=1, 2, 3, 4, 5$)

In this section, we assume that f and g are distinct nonconstant meromorphic functions satisfying $E_2(a_j, f)=E_2(a_j, g)$ for five distinct values a_j ($j=1, 2, 3, 4, 5$) in $C\cup\{\infty\}$. Under these assumptions we write $\bar{N}(r, a_j, f; \leqq 2)=\bar{N}(r, a_j, g; \leqq 2)=\bar{N}(r, a_j; \leqq 2)$.

THEOREM 3. *If $a_j \in \boldsymbol{C}$ $(j=1, 2, 3, 4, 5)$, then we have the following estimates*:

(3.1) $T(r, f)=T(r)+S(r), \quad T(r, g)=T(r)+S(r)$;

(3.2) $\sum_{j=1}^{5} \overline{N}(r, a_j; \leqq 2)=2T(r)+S(r)$;

(3.3) $N(r, 0, f-g)=\overline{N}(r, 0, f-g)+S(r)=\sum_{j=1}^{5} \overline{N}(r, a_j; \leqq 2)+S(r)$;

(3.4) *For any $c \neq a_j$ $(j=1, 2, 3, 4, 5)$ in $\boldsymbol{C} \cup \{\infty\}$*

$N(r, c, f)=\overline{N}(r, c, f)+S(r)=T(r)+S(r)$, *and*

$N(r, c, g)=\overline{N}(r, c, g)+S(r)=T(r)+S(r)$;

(3.5) $\overline{N}(r, a_j; \leqq 2)=\overline{N}(r, a_j, f; 1)+S(r)=\overline{N}(r, a_j, g; 1)+S(r)$ $(j=1, 2, 3, 4, 5)$;

(3.6) $N(r, a_j, f)=\overline{N}(r, a_j, f; 1)+3\overline{N}(r, a_j, f; 3)+S(r)=T(r)+S(r)$,

$N(r, a_j, g)=\overline{N}(r, a_j, g; 1)+3\overline{N}(r, a_j, g; 3)+S(r)=T(r)+S(r)$

$(j=1, 2, 3, 4, 5)$;

(3.7) $m(r, 0, f-g)=S(r)$;

(3.8) $T(r, f-g)=2T(r)+S(r)$;

(3.9) *If $N_1'(r, f)$ refers only to those multiple points of f such that $f \neq a_j$ $(j=1, 2, 3, 4, 5)$ and if $N_1'(r, g)$ is similary defined, then $N_1'(r, f)=S(r)$ and $N_1'(r, g)=S(r)$.*

Proof. By the second fundamental theorem

(3.10)
$$\begin{aligned} 3T(r, f) &\leqq \textstyle\sum_{j=1}^{5} \overline{N}(r, a_j, f)-N_1'(r, f)+S(r, f) \\ &\leqq \textstyle\sum_{j=1}^{5} \{2\overline{N}(r, a_j; \leqq 2)+N(r, a_j, f)\}/3-N_1'(r, f)+S(r, f) \\ &\leqq (2/3) \textstyle\sum_{j=1}^{5} \overline{N}(r, a_j; \leqq 2)+(5/3)T(r, f)-N_1'(r, f)+S(r, f) \\ &\leqq (2/3)\overline{N}(r, 0, f-g)+(5/3)T(r, f)-N_1'(r, f)+S(r, f) \\ &\leqq (2/3)N(r, 0, f-g)+(5/3)T(r, f)-N_1'(r, f)+S(r, f) \\ &\leqq (2/3)T(r, f-g)+(5/3)T(r, f)-N_1'(r, f)+S(r, f) \\ &\leqq (2/3)\{T(r, f)+T(r, g)\}+(5/3)T(r, f)+S(r, f) \\ &\leqq (7/3)T(r, f)+(2/3)T(r, g)+S(r, f), \quad \text{i.e.,} \end{aligned}$$

(3.11) $T(r, f) \leqq T(r, g)+S(r, f)$.

(3.10) is still valid when we exchange f and g, so that

(3.12) $$T(r, g) \leqq T(r, f)+S(r, g).$$

From (3.11) and (3.12), (3.1) follows, and further we see that equality (up to an $S(r)$ term) must hold everywhere in (3.10). Hence (3.2), (3.3), (3.5)-(3.9) are derived immediately. Using the second fundamental theorem once again, we have for any $c \neq a_j$ $(j=1, 2, 3, 4, 5)$

$$4T(r) \leqq \sum_{j=1}^{5} \bar{N}(r, a_j, f) + \bar{N}(r, c, f) + S(r)$$
$$= 3T(r) + \bar{N}(r, c, f) + S(r) \leqq 4T(r) + S(r).$$

This estimate is still valid when we replace f by g, so that (3.4) follows by the first fundamental theorem. ■

THEOREM 3'. *If $a_1 = \infty$, then we have the following estimates*:

(3.1)' $T(r, f) = T(r) + S(r)$, $T(r, g) = T(r) + S(r)$;

(3.2)' $\sum_{j=1}^{5} \bar{N}(r, a_j; \leqq 2) = 2T(r) + S(r)$;

(3.3)' $N(r, 0, f-g) = \bar{N}(r, 0, f-g) + S(r) = \sum_{j=2}^{5} \bar{N}(r, a_j; \leqq 2) + S(r)$;

(3.4)' *For any $c \neq a_j$ $(j=1, 2, 3, 4, 5)$ in $\boldsymbol{C}$*

$N(r, c, f) = \bar{N}(r, c, f) + S(r) = T(r) + S(r)$, *and*

$N(r, c, g) = \bar{N}(r, c, g) + S(r) = T(r) + S(r)$;

(3.5)' $\bar{N}(r, a_j; \leqq 2) = \bar{N}(r, a_j, f; 1) + S(r) = \bar{N}(r, a_j, g; 1) + S(r)$ $(j=1, 2, 3, 4, 5)$;

(3.6)' $N(r, a_j, f) = \bar{N}(r, a_j, f; 1) + 3\bar{N}(r, a_j, f; 3) + S(r) = T(r) + S(r)$,

$N(r, a_j, g) = \bar{N}(r, a_j, g; 1) + 3\bar{N}(r, a_j, g; 3) + S(r) = T(r) + S(r)$ $(j=1, 2, 3, 4, 5)$;

(3.7)' $m(r, 0, f-g) = S(r)$;

(3.8)' $T(r, f-g) + \bar{N}(r, \infty; \leqq 2) = 2T(r) + S(r)$;

(3.9)' *If $N_1'(r, f)$ refers only to those multiple points of f such that $f \neq a_j$ $(j=1, 2, 3, 4, 5)$ and if $N_1'(r, g)$ is similarly defined, then $N_1'(r, f) = S(r)$ and $N_1'(r, g) = S(r)$.*

Proof. Let $d \in \boldsymbol{C}$ be different from a_j $(j=2, 3, 4, 5)$, and let $b_j = (a_j - d)^{-1}$ $(j=1, 2, 3, 4, 5)$. Then $b_1, \cdots, b_5$ are all distinct and finite. If we put $F = (f-d)^{-1}$ and $G = (g-d)^{-1}$, then $E_2(b_j, F) = E_2(b_j, G)$ $(j=1, 2, 3, 4, 5)$. Hence (3.1)-(3.9) of Theorem 3 hold when f, g, a_j $(j=1, 2, 3, 4, 5)$ are replaced by F, G, b_j, respectively. Taking $T(r, F) = T(r, f) + O(1)$ and $T(r, G) = T(r, g) + O(1)$ into consideration, we immediately deduce (3.1)', (3.2)', (3.4)', (3.5)', (3.6)' and (3.9)'.

Next, from (3.4) it follows that $m(r, \infty, F) = S(r)$ and $m(r, \infty, G) = S(r)$. Combining these and (3.7), we have

$$m(r, 0, f-g) = m(r, \infty, (f-g)^{-1}) = m(r, \infty, FG/(G-F))$$
$$\leqq m(r, \infty, F) + m(r, \infty, G) + m(r, \infty, (G-F)^{-1}) = S(r),$$

which gives (3.7)'.

Finally we prove (3.3)' and (3.8)'. Since $F-G = (g-f)/(f-d)(g-d)$, we easily see that $\{z : z \text{ is a zero of } F-G\} = Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5$, where Z_1, Z_2, Z_3, Z_4 and Z_5 are defined as follows:

(i) Let $z_1 \in Z_1$. Then $f(z_1) \neq d, \infty$; $g(z_1) \neq d, \infty$ and $f(z_1)=g(z_1)$. In this case the multiplicity of the zero z_1 of $F-G$ is equal to the multiplicity of the zero z_1 of $f-g$.

(ii) Let $z_2 \in Z_2$. Then z_2 is a common d-point of f and g with the same multiplicity, say p, and further z_2 is a zero of $f-g$ with multiplicity $s \geqq 2p+1$. In this case the multiplicity of the zero z_2 of $F-G$ is equal to $s-2p$.

(iii) Let $z_3 \in Z_3$. Then z_3 is a common pole of f and g with the same multiplicity, say p, and further z_3 is a zero of $f-g$ with multiplicity s. In this case the multiplicity of the zero z_3 of $F-G$ is equal to $s+2p$.

(iv) Let $z_4 \in Z_4$. Then z_4 is a common pole of f and g with the same multiplicity, say p, and further $(f-g)(z_4) \neq 0, \infty$. In this case the multiplicity of the zero z_4 of $F-G$ is equal to $2p$.

(v) Let $z_5 \in Z_5$. Then z_5 is a common pole of f and g with the multiplicity, say p and q respectively, and further $(f-g)(z_4)=\infty$ with multiplicity s ($\leqq \max(p, q)$). In this case the multiplicity of the zero z_5 of $F-G$ is equal to $p+q-s \geqq \min(p, q)$.

Hence by (3.3)

$$\sum_{j=1}^{5} \bar{N}(r, a_j; \leqq 2)+S(r)=\sum_{j=1}^{5} \bar{N}(r, b_j, F; \leqq 2)+S(r)$$
$$=N(r, 0, F-G)=N(r, Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5) \geqq N(r, Z_1 \cup Z_3 \cup Z_4 \cup Z_5)$$
$$\geqq N(r, 0, f-g; f \neq d, g \neq d)+2\sum_{p=1}^{\infty} p\bar{N}(r, 0, f-g; f=g=\infty \text{ with multiplicity } p)$$
$$+2\sum_{p=1}^{\infty} p\bar{N}(r, \infty, f=g=\infty \text{ with multiplicity } p \text{ and } f-g \neq 0, \infty)$$
$$+\sum_{(p,q)} \{\min(p, q)\} \bar{N}(r, \infty, f-g; f=\infty \text{ with multiplicity } p \text{ and } g=\infty \text{ with multiplicity } q)$$
$$\geqq \sum_{j=2}^{5} \bar{N}(r, 0, f-g; f=g=a_j)+\bar{N}(r, \infty; \leqq 2)$$
$$+N_1(r, 0, f-g; f=a_j \ (j=1, 2, 3, 4, 5))$$
$$+N(r, 0, f-g; f \neq d, a_j \ (j=1, 2, 3, 4, 5))$$
$$+\sum_{p=1}^{\infty}(2p)\bar{N}(r, 0, f-g; f=g=\infty \text{ with multiplicity } p)$$
$$+\sum_{p=1}^{\infty}(2p-1)\bar{N}(r, \infty, f=g=\infty \text{ with multiplicity } p \text{ and } f-g \neq 0, \infty)$$
$$+\sum_{(p,q)} \{\min(p, q)-1\} \bar{N}(r, \infty, f-g; f=\infty \text{ with multiplicity } p \text{ and } g=\infty \text{ with multiplicity } q),$$

which implies that

(3.13) $N_1(r, 0, f-g; f=a_j \ (j=1, 2, 3, 4, 5))=S(r)$,

(3.14) $N(r, 0, f-g; f \neq d, a_j \ (j=1, 2, 3, 4, 5))=S(r)$,

(3.15) $\sum_{p=1}^{\infty}(2p)\bar{N}(r, 0, f-g; f=g=\infty \text{ with multiplicity } p)=S(r)$,

(3.16) $\sum_{p=1}^{\infty}(2p-1)\bar{N}(r, \infty, f=g=\infty \text{ with multiplicity } p \text{ and } f-g \neq 0, \infty)=S(r)$,

(3.17) $\sum_{(p,q)} \{\min(p, q)-1\} \bar{N}(r, \infty, f-g; f=\infty \text{ with multiplicity } p$ and $g=\infty$ with multiplicity $q)=S(r)$ and

(3.18) $\overline{N}(r, 0, f-g; f=g=a_j)=\overline{N}(r, a_j; \leqq 2)+S(r)$ $(j=2, 3, 4, 5)$.

Combining (3.13) and (3.15), we have

(3.19) $$N(r, 0, f-g; f=g=\infty)=S(r) \text{ and}$$
$$N_1(r, 0, f-g; f=a_j\ (j=2, 3, 4, 5))=S(r).$$

(3.14) and the arbitrariness of the selection of d give

(3.20) $N(r, 0, f-g; f\neq a_j\ (j=1, 2, 3, 4, 5))=S(r)$.

From (3.18)-(3.20) it follows that

$$N(r, 0, f-g)=\overline{N}(r, 0, f-g)+S(r)=\sum_{j=2}^{5}\overline{N}(r, a_j; \leqq 2)+S(r).$$

This proves (3.3)'. Further from (3.3)', (3.7)' and (3.2)' we easily obtain (3.8)'. This completes the proof of Theorem 3'. ■

4. Preparations for the proof of Theorem 1

Let $a_1=\infty$, $a_2=0$, $a_3=1$, $a_4=a$ and $a_5=b$. In this section, for these five distinct values $\{a_j\}$ we assume that two distinct nonconstant meromorphic functions f and g satisfy $E_2(a_j, f)=E_2(a_j, g)$. The following function Φ corresponds to the function ψ in [3, p. 171] and plays an important role in the proof of Theorem 1.

LEMMA 1. *The function*

$$\Phi=\frac{(f')^3(g')^3(f-g)^6}{f^3g^3\{(f-1)(g-1)(f-a)(g-a)(f-b)(g-b)\}^2}$$

satisfies

(4.1) $$m(r, \infty, \Phi)=S(r) \quad and \quad N(r, \infty, \Phi)$$
$$=3\{\overline{N}(r, 0, f; 3)+\overline{N}(r, 0, g; 3)+\overline{N}(r, \infty, f; 3)+\overline{N}(r, \infty, g; 3)\}+S(r).$$

Proof. From (3.6)' of Theorem 3' we have $m(r, a_j, f)=S(r)$ and $m(r, a_j, g)=S(r)$. From the fundamental estimate of the logarithmic derivative it follows that $m(r, \infty, f'/f)=S(r)$ and $m(r, \infty, g'/g)=S(r)$. Combining these, we have $m(r, \infty, \Phi)=S(r)$. The second estimate of (4.1) is an immediate consequence of (3.3)', (3.6)', (3.16) and (3.17). ■

In what follows, for the sake of simplicity we write

$$[f]_1=3\frac{f''}{f'}-6\frac{f'}{f}-2\left\{\frac{f'}{f-1}+\frac{f'}{f-a}+\frac{f'}{f-b}\right\},$$

$$[f]_2=3\frac{f''}{f'}+6\frac{f'}{f}-2\left\{\frac{f'}{f-1}+\frac{f'}{f-a}+\frac{f'}{f-b}\right\},$$

$$\Psi_1=\{[f]_1-[g]_1\}^6-64a^4b^4(1+a^{-1}+b^{-1})^6\Phi \text{ and}$$
$$\Psi_2=\{[f]_2-[g]_2\}^6-64(1+a+b)^6\Phi\,.$$

LEMMA 2. (i) *For* $[f]_j-[g]_j$ $(j=1, 2)$ *we have*

$$N(r, \infty, [f]_j-[g]_j)\leqq\bar{N}(r, 0, f; 3)+\bar{N}(r, 0, g; 3)+\bar{N}(r, \infty, f; 3)+\bar{N}(r, \infty, g; 3)+S(r). \tag{4.2}$$

(ii) *If* z_0 *denotes a simple zero of* f *which is also a simple zero of* g, *then* $\Psi_1(z_0)=0$. *Similarly, if* z_∞ *is a common simple pole of* f *and* g, *then* $\Psi_2(z_\infty)=0$.

Proof. (i) Using (3.3)′, (3.6)′, (3.16) and (3.17), we obtain

$$N(r, \infty, [f]_j-[g]_j)=\bar{N}(r, 0, f; 3)+\bar{N}(r, 0, g; 3)+\bar{N}(r, \infty, f; 3)+\bar{N}(r, \infty, g; 3)-\bar{N}(r, f=0, g=\infty; 3)-\bar{N}(r, f=\infty, g=0; 3)+S(r),$$

where $\bar{N}(r, f=0, g=\infty; 3)$ refers to common roots of $f=0$ and $g=\infty$ with the same multiplicity 3, and $\bar{N}(r, f=\infty, g=0; 3)$ is also defined similarly. Hence (4.2) follows.

(ii) Simple calculations give

$$([f]_1-[g]_1)(z_0)=2(1+a^{-1}+b^{-1})\{f'(z)-g'(z_0)\},$$
$$\Phi(z_0)=a^{-4}b^{-4}\{f'(z_0)-g'(z_0)\}^6,$$

and so $\Psi_1(z_0)=0$. Next, if f and g have the following expansions at z_∞: $f(z)=A/(z-z_\infty)+O(1)$, $g(z)=B/(z-z_\infty)+O(1)$, then we have

$$([f]_2-[g]_2)(z_\infty)=2(1+a+b)\{A^{-1}-B^{-1}\}, \quad \Phi(z_\infty)=\{A^{-1}-B^{-1}\}^6.$$

Hence $\Psi_2(z_\infty)=0$. ■

LEMMA 3. *If there is a constant* $\tau\in[0, 1/15)$ *such that*

$$\bar{N}(r, 0, f; 3)+\bar{N}(r, \infty, f; 3)\leqq\tau T(r)+S(r),$$

then both $\Psi_1(z)\equiv 0$ *and* $\Psi_2(z)\equiv 0$ *hold.*

Proof. Assume that $\Psi_1(z)\not\equiv 0$. Using (3.1)′, (3.5)′, (3.6)′, (4.1), (4.2) and the fundamental estimate of the logarithmic derivative, we have

$$\begin{aligned} T(r, \Psi_1)&=m(r, \infty, \Psi_1)+N(r, \infty, \Psi_1)\\ &\leqq 6\{\bar{N}(r, 0, f; 3)+\bar{N}(r, 0, g; 3)+\bar{N}(r, \infty, f; 3)\\ &\quad+\bar{N}(r, \infty, g; 3)\}+S(r)\\ &=12\{\bar{N}(r, 0, f; 3)+\bar{N}(r, \infty, f; 3)\}+S(r). \end{aligned} \tag{4.3}$$

From (3.5)′ and Lemma 2 (ii) it follows that

(4.4) $$\bar{N}(r, 0; \leqq 2) \leqq N(r, 0, \Psi_1) + S(r) \leqq T(r, \Psi_1) + S(r).$$

Combining (4.3), (4.4), (3.5)′ and (3.6)′, we obtain

$$T(r) + S(r) \leqq 15\bar{N}(r, 0, f; 3) + 12\bar{N}(r, \infty, f; 3) + S(r) \leqq 15\tau T(r) + S(r),$$

which is impossible. This proves $\Psi_1(z) \equiv 0$. The proof of $\Psi_2(z) \equiv 0$ is much the same. ■

LEMMA 4. *If both $\Psi_1(z) \equiv 0$ and $\Psi_2(z) \equiv 0$ hold, then g/f is a constant.*

Proof. Consider first the case that $1 + a^{-1} + b^{-1} = 1 + a + b = 0$, i.e., $\{a, b\} = \{\omega, \omega^2\}$. In this case $[f]_1 - [g]_1 \equiv [f]_2 - [g]_2$ $(\equiv 0)$, and so $f'/f \equiv g'/g$. This leads to $g/f \equiv a$ constant.

Next, we consider the case that at least one of $1 + a^{-1} + b^{-1}$ or $1 + a + b$ is not zero. Without loss of generality, we assume that $1 + a + b \neq 0$. In this case

(4.5) $$[f]_1 - [g]_1 \equiv \lambda \{[f]_2 - [g]_2\},$$

where λ is a constant satisfying $\lambda^6 = a^4 b^4 (1 + a^{-1} + b^{-1})^6 / (1 + a + b)^6$. If $\lambda = 1$, then $f'/f \equiv g'/g$, which gives $g/f \equiv a$ constant.

Assume that $\lambda \neq 1$. We investigate the common zeros and poles of f and g. By the assumption $\Psi_2(z) \equiv 0$

(4.6) $$\{[f]_2 - [g]_2\}^6 \equiv 64(1 + a + b)^6 \Phi.$$

Let z_0 be a common zero of f and g whose multiplicities are p and q $(p \neq q)$, respectively. Then since the residue at z_0 of $[f]_2 - [g]_2$ is $9(p - q) \neq 0$, the left hand side of (4.6) has a pole of order 6 at z_0. On the other hand, z_0 is a regular point of Φ since $-3 - 3 + 6\min(p, q) \geqq 0$. This shows that if f and g have common zeros, then their multiplicities are identical. In the same way, we see that if f and g have common poles, then their multiplicities are identical.

Assume now that g/f is not a constant. Taking $E_2(0, f) = E_2(0, g)$ and $E_2(\infty, f) = E_2(\infty, g)$ into consideration, the above conclusions imply that the multiplicities of zeros and poles of g/f are all $\geqq 3$ if any. Thus $\Theta(0, g/f) \geqq 2/3$ and $\Theta(\infty, g/f) \geqq 2/3$.

From (4.5) we have

(4.7) $$(1-\lambda)\left[3\left(\frac{f''}{f'} - \frac{g''}{g'}\right) - 2\left(\frac{f'}{f-1} + \frac{f'}{f-a} + \frac{f'}{f-b} - \frac{g'}{g-1} - \frac{g'}{g-a} - \frac{g'}{g-b}\right)\right]$$
$$\equiv 6(1+\lambda)\left(\frac{f'}{f} - \frac{g'}{g}\right).$$

From integration of (4.7) we obtain

$$\frac{(f')^3\{(g-1)(g-a)(g-b)\}^2}{(g')^3\{(f-1)(f-a)(f-b)\}^2}\equiv A\left(\frac{f}{g}\right)^{\mu}, \tag{4.8}$$

where A is a nonzero constant and $\mu=6(1+\lambda)/(1-\lambda)$. Substituting (4.7) and (4.8) into (4.6), we have

$$64(1+a+b)^6\frac{(f')^6(f-g)^6}{Af^6(f/g)^{\mu-3}\{(f-1)(f-a)(f-b)\}^4}\equiv\{12/(1-\lambda)\}^6\left(\frac{f'}{f}-\frac{g'}{g}\right)^6,$$

and hence

$$\frac{(f')^3}{\{(f-1)(f-a)(f-b)\}^2}\equiv B\frac{f^3(f'/f-g'/g)^3}{(f-g)^3(g/f)^{(\mu-3)/2}}\equiv B\left\{\frac{(1-g/f)'}{(1-g/f)}\right\}^3\times (g/f)^{-(\mu+3)/2}, \tag{4.9}$$

where B is a nonzero constant. We easily see that the left hand side of (4.9) has poles of order at most 2. If g/f has a 1-point z_1, then the right hand side of (4.9) has a pole of order 3 at z_1. This is impossible. Therefore $\Theta(1, g/f)=1$, so that $\Theta(0, g/f)+\Theta(1, g/f)+\Theta(\infty, g/f)\geqq 7/3$. This is also a contradiction. Thus we conclude that g/f is a constant. ■

LEMMA 5. *If g/f is a constant C, then $\{a, b\}=\{\omega, \omega^2\}$ and $C^3=1$.*

Proof. Since f and g are distinct, all the 1-, a-, b-points of f and g are of order $\geqq 3$. Hence f maps 1, a, b on a, b, 1 (or b, 1, a) respectively. Therefore $C^3=1$ and $\{a, b\}=\{\omega, \omega^2\}$. ■

5. Proof of Theorem 1

Assume that $f\not\equiv g$. From (3.3)′, (3.16), (3.17) and (3.6)′ we see that $\overline{N}(r, 0, f; 3)+\overline{N}(r, \infty, f; 3)=S(r)$. Hence Lemma 3 holds, and so that from Lemmas 4 and 5 it follows that $\{a, b\}=\{\omega, \omega^2\}$ and $f^3\equiv g^3$. This completes the proof of Theorem 1. ■

6. Elementary estimates on meromorphic functions satisfying $E_1(a_j, f)=E_1(a_j, g)$ for six distinct values a_j ($j=1, 2, 3, 4, 5, 6$)

In this section, we assume that f and g are distinct nonconstant meromorphic functions satisfying $E_1(a_j, f)=E_1(a_j, g)$ for six distinct values a_j ($j=1, 2, 3, 4, 5, 6$) in $\boldsymbol{C}\cup\{\infty\}$. Under these assumptions we write $\overline{N}(r, a_j, f; 1)=\overline{N}(r, a_j, g; 1)=\overline{N}(r, a_j; 1)$.

THEOREM 4. *If $a_j\in\boldsymbol{C}$ ($j=1, 2, 3, 4, 5, 6$), then we have the following estimates:*

$$T(r, f)=T(r)+S(r),\quad T(r, g)=T(r)+S(r); \tag{6.1}$$

(6.2) $\sum_{j=1}^{6}\bar{N}(r, a_j; 1)=2T(r)+S(r)$;

(6.3) $N(r, 0, f-g)=\bar{N}(r, 0, f-g)+S(r)=\sum_{j=1}^{6}\bar{N}(r, a_j; 1)+S(r)$;

(6.4) *For any* $c\neq a_j$ $(j=1, 2, 3, 4, 5, 6)$ *in* $\boldsymbol{C}\cup\{\infty\}$

$N(r, c, f)=\bar{N}(r, c, f)+S(r)=T(r)+S(r)$, *and*

$N(r, c, g)=\bar{N}(r, c, g)+S(r)=T(r)+S(r)$;

(6.5) $N(r, a_j, f)=\bar{N}(r, a_j, f; 1)+2\bar{N}(r, a_j, f; 2)+S(r)=T(r)+S(r)$,

$N(r, a_j, g)=\bar{N}(r, a_j, g; 1)+2\bar{N}(r, a_j, g; 2)+S(r)=T(r)+S(r)$ $(j=1, 2, 3, 4, 5, 6)$;

(6.6) $m(r, 0, f-g)=S(r)$;

(6.7) $T(r, f-g)=2T(r)+S(r)$;

(6.8) *If* $N_1'(r, f)$ *refers only to those multiple points of* f *such that* $f\neq a_j$ $(j=1, 2, 3, 4, 5, 6)$ *and if* $N_1'(r, g)$ *is similarly defined, then* $N_1'(r, f)=S(r)$ *and* $N_1'(r, g)=S(r)$.

The proof is much the same as the proof of Theorem 3.

THEOREM 4′. *If* $a_1=\infty$, *then we have the following estimates*:

(6.1)′ $T(r, f)=T(r)+S(r)$, $T(r, g)=T(r)+S(r)$;

(6.2)′ $\sum_{j=1}^{6}\bar{N}(r, a_j; 1)=2T(r)+S(r)$;

(6.3)′ $N(r, 0, f-g)=\bar{N}(r, 0, f-g)+S(r)=\sum_{j=2}^{6}\bar{N}(r, a_j; 1)+S(r)$;

(3.4)′ *For any* $c\neq a_j$ $(j=1, 2, 3, 4, 5, 6)$ *in* $\boldsymbol{C}$

$N(r, c, f)=\bar{N}(r, c, f)+S(r)=T(r)+S(r)$, *and*

$N(r, c, g)=\bar{N}(r, c, g)+S(r)=T(r)+S(r)$;

(6.5)′ $N(r, a_j, f)=\bar{N}(r, a_j, f; 1)+2\bar{N}(r, a_j, f; 2)+S(r)=T(r)+S(r)$,

$N(r, a_j, g)=\bar{N}(r, a_j, g; 1)+2\bar{N}(r, a_j, g; 2)+S(r)=T(r)+S(r)$ $(j=1, 2, 3, 4, 5, 6)$;

(6.6)′ $m(r, 0, f-g)=S(r)$;

(6.7)′ $T(r, f-g)+\bar{N}(r, \infty; 1)=2T(r)+S(r)$;

(6.8)′ *If* $N_1'(r, f)$ *refers only to those multiple points of* f *such that* $f\neq a_j$ $(j=1, 2, 3, 4, 5, 6)$ *and if* $N_1'(r, g)$ *is similarly defined, then* $N_1'(r, f)=S(r)$ *and* $N_1'(r, g)=S(r)$.

The proof is much the same as the proof of Theorem 3′.

7. Outline of the proof of Theorem 2

Let $a_1=\infty$, $a_2=0$, $a_3=1$, $a_4=a$, $a_5=b$ and $a_6=c$. In this section, for these six distinct values $\{a_j\}$ we assume that two distinct nonconstant meromorphic functions f and g satisfy $E_1(a_j, f)=E_1(a_j, g)$.

LEMMA 6. *The function*

$$\Lambda=\frac{(f')^2(g')^2(f-g)^4}{f^2g^2\{(f-1)(g-1)(f-a)(g-a)(f-b)(g-b)(f-c)(g-c)\}}$$

satisfies

$$m(r, \infty, \Lambda)=S(r) \quad \text{and} \quad N(r, \infty, \Lambda)$$
$$=2\{\bar{N}(r, 0, f; 2)+\bar{N}(r, 0, g; 2)+\bar{N}(r, \infty, f; 2)+\bar{N}(r, \infty, g; 2)\}+S(r).$$

In what follows, for the sake of simplicity we write

$$[f]_3=2\frac{f''}{f'}-4\frac{f'}{f}-\left\{\frac{f'}{f-1}+\frac{f'}{f-a}+\frac{f'}{f-b}+\frac{f'}{f-c}\right\},$$

$$[f]_4=2\frac{f''}{f'}+4\frac{f'}{f}-\left\{\frac{f'}{f-1}+\frac{f'}{f-a}+\frac{f'}{f-b}+\frac{f'}{f-c}\right\},$$

$$\Omega_1=\{[f]_3-[g]_3\}^4-a^2b^2c^2(1+a^{-1}+b^{-1}+c^{-1})^4\Lambda \text{ and}$$

$$\Omega_2=\{[f]_4-[g]_4\}^4-(1+a+b+c)^4\Lambda.$$

LEMMA 7. (i) *For* $[f]_j-[g]_j$ $(j=3, 4)$ *we have*

$$N(r, \infty, [f]_j-[g]_j)\leqq\bar{N}(r, 0, f; 2)+\bar{N}(r, 0, g; 2)+\bar{N}(r, \infty, f; 2)$$
$$+\bar{N}(r, \infty, g; 2)+S(r).$$

(ii) *If* z_0 *denotes a simple zero of* f *which is also a simple zero of* g, *then* $\Omega_1(z_0)=0$. *Similarly, if* z_∞ *is a common simple pole of* f *and* g, *then* $\Omega_2(z_\infty)=0$.

LEMMA 8. *If there is a constant* $\tau'\in[0, 1/10)$ *such that*

$$\bar{N}(r, 0, f; 2)+\bar{N}(r, \infty, f; 2)\leqq\tau'T(r)+S(r),$$

then both $\Omega_1(z)\equiv0$ *and* $\Omega_2(z)\equiv0$ *hold.*

LEMMA 9. *Assume that* f *and* g *share* 0 *and* ∞ *CM. If both* $\Omega_1(z)\equiv0$ *and* $\Omega_2(z)\equiv0$ *hold, then* g/f *is a constant.*

LEMMA 10. *If* g/f *is a constant* C, *then* $\{a, b, c\}=\{\alpha, -1, -\alpha\}$ *with* $\alpha\neq0$, ±1 *and* $C^4=1$.

The proofs of Lemmas 6-10 are similar to the one of Lemmas 1-5. Combining these we easily obtain Theorem 2.

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Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan