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UNICITY THEOREMS FOR MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES

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This paper studies the problem of the uniqueness of meromorphic functions that share three values. The results in this paper improve some theorems given by H. Ueda, Shou-Zhen Ye and Hong-Xun Yi. Examples are provided to show that our results are sharp.

1. Introduction and main results

Let f and g be two nonconstant meromorphic functions in the complex plane. If f and g have the same a-points with the same multiplicities, we say f and g share the value a CM. (see [1]). It is assumed that the reader is familiar with the basic notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [2]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r, f)$ denotes any quantity satisfying

$$S(r, f)=o(T(r, f)) \qquad (r\to\infty,\ r\notin E).$$

H. Ueda proved the following theorem.

THEOREM A (see [3]). *Let f and g be two distinct nonconstant entire functions such that f and g share $0, 1$ $CM.$, and let a be a finite complex number, and $a\neq 0, 1$. If a is lacunary for f, then $1-a$ is lacunary for g, and*

$$(f-a)(g+a-1)\equiv a(1-a).$$

In [4] the present author proved the following result which is an improvement of the above result.

THEOREM B. *Let f and g be two distinct nonconstant entire functions such that f and g share $0, 1$ $CM.$, and let a be a finite complex number, and $a\neq 0, 1$.*

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If $\delta(a, f)>1/3$, *then* a *and* $1-a$ *are Picard exceptional values of* f *and* g *respectively, and*

$$(f-a)(g+a-1)\equiv a(1-a).$$

Recently Shou-Zhen Ye extended the above theorems to meromorphic functions, and proved the following theorems.

THEOREM C (see [5]). *Let* f *and* g *be two distinct nonconstant meromorphic functions such that* f *and* g *share* $0, 1, \infty$ *CM., and let* a *be a finite complex number, and* $a\neq 0, 1$. *If*

$$\delta(a, f)+\delta(\infty, f)>\frac{4}{3},$$

then a *and* $1-a$ *are Picard values of* f *and* g *respectively, and also* ∞ *is so, and*

$$(f-a)(g+a-1)\equiv a(1-a).$$

THEOREM D (see [5]). *Let* f *and* g *be two distinct nonconstant meromorphic functions such that* f *and* g *share* $0, 1, \infty$ *CM., and let* $a_1, a_2, \cdots, a_p$ *be* p $(\geqq 1)$ *distinct finite complex numbers, and* $a_j\neq 0, 1$ $(j=1, 2, \cdots, p)$. *If*

$$\sum_{j=1}^{p}\delta(a_j, f)+\delta(\infty, f)>\frac{2(p+1)}{p+2},$$

then there exists one and only one a_k *in* $a_1, a_2, \cdots, a_p$ *such that* a_k *and* $1-a_k$ *are Picard values of* f *and* g *respectively, and also* ∞ *is so, and*

$$(f-a_k)(g+a_k-1)\equiv a_k(1-a_k).$$

In this paper we improve the above theorems and obtain the following results.

THEOREM 1. *Let* f *and* g *be two distinct nonconstant meromorphic functions such that* f *and* g *share* $0, 1, \infty$ *CM., and let* a *be a finite complex number, and* $a\neq 0, 1$. *If*

$$N\left(r, \frac{1}{f-a}\right)\neq T(r, f)+S(r, f)$$

and

$$N(r, f)\neq T(r, f)+S(r, f),$$

then a *and* $1-a$ *are Picard values of* f *and* g *respectively, and also* ∞ *is so, and*

$$(f-a)(g+a-1)\equiv a(1-a).$$

By Theorem 1 we immediately obtain the following result which is an improvement of Theorems A, B, C and D.

THEOREM 2. *Let f and g be two distint nonconstant meromorphic functions such that f and g share 0, 1, ∞ CM., and let a be a finite complex number, and* $a\neq 0, 1$. *If* $\delta(a, f)>0$ *and* $\delta(\infty, f)>0$, *then a and* $1-a$ *are Picard values of f and g respectively, and also ∞ is so, and*

$$(f-a)(g+a-1)\equiv a(1-a).$$

Example 1. Let $f(z)=(e^{2z}+1)/(e^{z}+1)$, $g(z)=(e^{-2z}+1)/(e^{-z}+1)$, $a=2$. It is easy to see that f and g share 0, 1, ∞ *CM.*, and

$$N(r, f)\neq T(r, f)+S(r, f)$$

and $\delta(\infty, f)=1/2>0$. Noting

$$f(z)-a=\frac{e^{2z}-2e^{z}-1}{e^{z}+1},$$

we have

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f)$$

and $\delta(a, f)=0$. $(f-a)(g+a-1)\not\equiv a(1-a)$ is evident.

Example 2. Let $f(z)=2/(1+e^{z})$, $g(z)=2/(1+e^{-z})$, $a=2$. It is easy to see that f and g share 0, 1, ∞ *CM.*, and

$$N(r, f)=T(r, f)+S(r, f)$$

and $\delta(\infty, f)=0$. Noting

$$f(z)-a=-\frac{2e^{z}}{1+e^{z}},$$

we have

$$N\left(r, \frac{1}{f-a}\right)\neq T(r, f)+S(r, f)$$

and $\delta(a, f)=1>0$. $(f-a)(g+a-1)\not\equiv a(1-a)$ is evident.

Example 3. Let $f(z)=1/(e^{z}+1)$, $g(z)=1/(e^{-z}+1)$, $a=2$. It is easy to see that f and g share 0, 1, ∞ *CM.*, and

$$N(r, f)=T(r, f)+S(r, f)$$

and $\delta(\infty, f)=0$. Noting

$$f(z)-a=-\frac{2e^{z}+1}{e^{z}+1},$$

we have

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f)$$

and $\delta(a, f)=0$. $(f-a)(g+a-1)\not\equiv a(1-a)$ is evident.

The above examples show that Theorem 1 and Theorem 2 are sharp.

2. Some lemmas

The following lemmas will be needed in the proof of our theorems.

LEMMA 1. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞ CM., then*

$$f=\frac{e^q-1}{e^p-1}, \qquad g=\frac{e^{-q}-1}{e^{-p}-1},$$

where p and q are entire functions such that $e^p\not\equiv 1$, $e^q\not\equiv 1$ and $e^{q-p}\not\equiv 1$, and

$$T(r, g)+T(r, e^p)+T(r, e^q)=O(T(r, f)) \qquad (r\notin E).$$

Proof. By assumption we have with two entire functions α and β,

$$f=e^\alpha\cdot g \quad \text{and} \quad f-1=e^\beta\cdot(g-1).$$

Since $f\not\equiv g$, then $e^\alpha\not\equiv 1$, $e^\beta\not\equiv 1$ and $e^{\beta-\alpha}\not\equiv 1$. Setting $p=\beta-\alpha$ and $q=\beta$, we have $e^p\not\equiv 1$, $e^q\not\equiv 1$ and $e^{q-p}\not\equiv 1$. Thus from this we get

$$f=\frac{e^q-1}{e^p-1} \quad \text{and} \quad g=\frac{e^{-q}-1}{e^{-p}-1}.$$

By the second fundamental theorem, we have

$$\begin{aligned} T(r, g)&<N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N(r, g)+S(r, g)\\ &=N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f-1}\right)+N(r, f)+S(r, g)\\ &<3T(r, f)+S(r, g). \end{aligned}$$

Thus from this we get

$$\begin{aligned} T(r, e^q)&=T(r, e^\beta)\leqq T(r, f-1)+T(r, g-1)+O(1)\\ &<4T(r, f)+S(r, g),\\ T(r, e^{q-p})&=T(r, e^\alpha)\leqq T(r, f)+T(r, g)+O(1)\\ &<4T(r, f)+S(r, g),\\ T(r, e^p)&=T(r, e^{q-(q-p)})\leqq T(r, e^q)+T(r, e^{q-p})+O(1)\\ &<8T(r, f)+S(r, g). \end{aligned}$$

Hence

$$T(r, g)+T(r, e^p)+T(r, e^q)=O(T(r, f)) \qquad (r \notin E).$$

This completes the proof of Lemma 1.

LEMMA 2. *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If*

$$c_1 f + c_2 g \equiv c_3,$$

then

$$T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, f).$$

Proof. By the second fundamental theorem, we have

$$T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \left(f - \frac{c_3}{c_1}\right)^{-1}\right) + \bar{N}(r, f) + S(r, f)$$

$$= \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, f),$$

which proves Lemma 2.

LEMMA 3 (see [6, Lemma 4]). *Let $p(z)$ be a nonconstant entire function, then*

$$T(r, p') = o(T(r, e^p)) \qquad (r \notin E).$$

In order to state Lemma 4, we introduce the following notations.

Let f be a meromorphic function. We denote by $n_1(r, f)$ the number of simple poles of f in $|z| \leqq r$. $N_1(r, f)$ is defined in terms of $n_1(r, f)$ in the usual way. We further define

$$N_2(r, f) = N(r, f) - N_1(r, f).$$

In the same way we can define $N_2(r, 1/f)$ and $N_2(r, 1/(f-1))$.

LEMMA 4. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty$ CM., then*

$$N_2(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{f-1}\right) = S(r, f).$$

Proof. By Lemma 1 we have

$$f = \frac{e^q - 1}{e^p - 1},$$

where p and q are entire functions such that $e^p \not\equiv 1$, $e^q \not\equiv 1$ and $e^{q-p} \not\equiv 1$, and

$$T(r, g)+T(r, e^p)+T(r, e^q)=O(T(r, f)) \qquad (r \notin E).$$

If $e^q \equiv c$, where c $(\neq 0, 1)$ is a constant, then we have $N(r, 1/f)=0$. If e^q

is not a constant, let $\{z_n\}$ be all the roots of $f=0$ with multiplicity$\geqq 2$, then $\{z_n\}$ are the roots of $(e^q-1)'=q'e^q=0$. Thus

$$N_2\left(r, \frac{1}{f}\right)\leqq 2N\left(r, \frac{1}{q'}\right)\leqq 2T(r, q')+O(1).$$

Again by Lemma 3, we deduce

$$N_2\left(r, \frac{1}{f}\right)=S(r, f).$$

If $e^p\equiv c$, where c $(\neq 0, 1)$ is a constant, then we have $N(r, f)=0$. If e^p is not a constant, let $\{z_n\}$ be all the roots of $1/f=0$ with multiplicity$\geqq 2$, then $\{z_n\}$ are the roots of $(e^p-1)'=p'e^p=0$. Thus

$$N_2(r, f)\leqq 2N\left(r, \frac{1}{p'}\right)\leqq 2T(r, p')+O(1).$$

Again by Lemma 3, we have

$$N_2(r, f)=S(r, f).$$

Note that

$$f(z)-1=\frac{e^p(e^{q-p}-1)}{e^p-1}.$$

If $e^{q-p}\equiv c$, where c $(\neq 0, 1)$ is a constant, then $N(r, 1/(f-1))=0$. If e^{q-p} is not a constant, let $\{z_n\}$ be all the roots of $f-1=0$ with multiplicity $\geqq 2$, then $\{z_n\}$ are the roots of $(e^{q-p}-1)'=(q'-p')e^{q-p}=0$. Thus

$$N_2\left(r, \frac{1}{f-1}\right)\leqq 2N\left(r, \frac{1}{q'-p'}\right)\leqq 2T(r, q')+2T(r, p')+O(1).$$

Again by Lemma 3, we have

$$N_2\left(r, \frac{1}{f-1}\right)=S(r, f).$$

From the above three equalities, we obtain

$$N_2(r, f)+N_2\left(r, \frac{1}{f}\right)+N_2\left(r, \frac{1}{f-1}\right)=S(r, f),$$

which proves Lemma 4.

3. Proof of Theorem 1

By the assumption, from Lemma 1 we have

$$(1) \qquad f=\frac{e^q-1}{e^p-1}, \qquad g=\frac{e^{-q}-1}{e^{-p}-1},$$

where p and q are entire functions such that $e^p \not\equiv 1$, $e^q \not\equiv 1$ and $e^{q-p} \not\equiv 1$, and

$$T(r, g)+T(r, e^p)+T(r, e^q)=O(T(r, f)) \qquad (r \notin E). \tag{2}$$

We discuss the following four cases.

a) Suppose that $e^p \equiv c(\neq 0, 1)$.

By (1) we have

$$f=\frac{e^q-1}{c-1} \tag{3}$$

and

$$f-a=\frac{e^q-1-a(c-1)}{c-1}. \tag{4}$$

If $-1-a(c-1)\neq 0$, from (4),

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f),$$

which contradicts the assumption of Theorem 1. Then $-1-a(c-1)=0$ and $c=(a-1)/a$. Again by (1), we obtain

$$f=a-ae^q$$

and

$$g=(1-a)-(1-a)e^{-q}.$$

Thus a and $1-a$ are Picard values of f and g respectively, and also ∞ is so, and

$$(f-a)(g+a-1)\equiv a(1-a).$$

b) Suppose that $e^q \equiv c(\neq 0, 1)$.

By (1) we have

$$f=\frac{c-1}{e^p-1}.$$

Thus

$$N(r, f)=T(r, f)+S(r, f),$$

which contradicts the assumption of Theorem 1.

c) Suppose that $e^{q-p} \equiv c\ (\neq 0, 1)$.

By (1) we have

$$f=\frac{ce^p-1}{e^p-1}=c+\frac{c-1}{e^p-1}.$$

Thus

$$N(r, f)=T(r, f)+S(r, f),$$

which is again a contradiction.

d) Suppose that none of e^p, e^q and e^{q-p} are constants.

It is clear that $p'\not\equiv 0$, $q'\not\equiv 0$ and $p'\not\equiv q'$. By Lemma 1 and Lemma 3, we have

$$T(r, p')+T(r, q')=S(r, f). \tag{5}$$

Set

$$h=\frac{q'}{p'}. \tag{6}$$

From (5) and (6) we obtain $h\not\equiv 0$, 1, and

$$T(r, h)=S(r, f).$$

If

$$q'(h-1)-h'\equiv 0,$$

by integration, we have

$$h-1=c_1e^q, \tag{7}$$

where c_1 is a constant, and $c_1\neq 0$. From (6) and (7), we obtain

$$\frac{q'}{c_1e^q+1}=p'.$$

Again by integration, we get

$$c_1+e^{-q}=c_2e^{-p},$$

where c_2 is a constant, and $c_2\neq 0$. Thus

$$c_2e^{-p}-e^{-q}=c_1.$$

By Lemma 2, we obtain

$$T(r, e^{-p})=S(r, e^{-p}),$$

which is impossible. Hence

$$q'(h-1)-h'\not\equiv 0.$$

From (1), we have

$$f-h=\frac{e^q-he^p+h-1}{e^p-1}. \tag{8}$$

Set

$$F=(f-h)(e^p-1)=e^q-he^p+h-1,$$

then

$$\frac{F'}{F}-q'=\frac{(e^q-he^p+h-1)'-q'(e^q-he^p+h-1)}{(f-h)(e^p-1)}$$

$$=\frac{q'(h-1)-h'}{f-h}$$

and hence

$$\frac{1}{f-h}=\frac{(F'/F)-q'}{q'(h-1)-h'}. \tag{9}$$

From (9) we get

$$m\left(r, \frac{1}{f-h}\right)\leqq m\left(r, \frac{F'}{F}\right)+S(r, f)=S(r, f) \tag{10}$$

and

$$N_2\left(r, \frac{1}{f-h}\right)=S(r, f). \tag{11}$$

Again from (1), we have

$$\frac{f-g}{g-1}=e^q-1$$

and

$$\frac{g'}{g}=\frac{q'e^p-p'e^q+(p'-q')}{(e^q-1)(e^p-1)}.$$

Thus

$$\frac{g'(f-g)}{g(g-1)}=\frac{q'e^p-p'e^q+(p'-q')}{e^p-1}. \tag{12}$$

From (6) and (8), we obtain

$$-p'(f-h)=\frac{q'e^p-p'e^q+(p'-q')}{e^p-1}. \tag{13}$$

By (12) and (13), we get

$$-p'(f-h)=\frac{g'(f-g)}{g(g-1)}. \tag{14}$$

Again by Lemma 4 and (11), we have

$$N\left(r, \frac{1}{f-h}\right)=N\left(r, \frac{1}{g'}\right)+N_0(r)+S(r, f), \tag{15}$$

where $N_0(r)$ denotes the counting function of the zeros of $f-g$ that are not zeros of g and $g-1$. From (10) and (15), we obtain

$$T(r, f)=T(r, f-h)+S(r, f)$$

$$=m\left(r, \frac{1}{f-h}\right)+N\left(r, \frac{1}{f-h}\right)+S(r, f)$$

$$=N\left(r, \frac{1}{g'}\right)+N_0(r)+S(r, f).$$

Thus

(16) $$T(r, f)-N\left(r, \frac{1}{g'}\right)=N_0(r)+S(r, f).$$

In the same manner as above, we have

(17) $$T(r, g)-N\left(r, \frac{1}{f'}\right)=N_0(r)+S(r, f).$$

By the second fundamental theorem, and using (16), we obtain

$$T(r, f)+T(r, g)\leqq T(r, f)+N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N(r, g)$$

$$-N\left(r, \frac{1}{g'}\right)+S(r, f)$$

$$=N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N(r, g)+N_0(r)+S(r, f)$$

$$\leqq N\left(, r\frac{1}{f-g}\right)+N(r, g)+S(r, f)$$

$$\leqq T(r, f-g)+N(r, g)+S(r, f)$$

$$\leqq m(r, f)+m(r, g)+N(r, f-g)+N(r, g)+S(r, f)$$

$$\leqq m(r, f)+m(r, g)+N(r, f)+N(r, g)+S(r, f)$$

$$=T(r, f)+T(r, g)+S(r, f).$$

Thus

(18) $$T(r, f)+T(r, g)=N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N(r, g)+N_0(r)+S(r, f).$$

Again by the second fundamental theorem, and using (17) and (18), we have

$$2T(r, f)\leqq N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f-1}\right)+N\left(r, \frac{1}{f-a}\right)$$

$$+N(r, f)-N\left(r, \frac{1}{f'}\right)+S(r, f)$$

$$\leqq N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N\left(r, \frac{1}{f-a}\right)$$

$$+N(r, g)+N_0(r)-T(r, g)+S(r, f)$$

$$=T(r, f)+N\left(r, \frac{1}{f-a}\right)+S(r, f)$$

$$\leqq 2T(r, f)+S(r, f).$$

Thus

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f),$$

which contradicts the assumption of Theorem 1.

This completes the proof of Theorem 1.

4. Applications of Theorem 1

For any set S and any meromorphic function f let

$$E_f(S)=\bigcup_{a\in S}\{z\mid f(z)-a=0\},$$

where each zero of $f-a$ with multiplicity m is repeated m times in $E_f(S)$ (see [9]).

Recently the present author corrected a result of Gross and Yang [11] and proved the following theorems.

THEOREM E (see [10]). *Let* $S_1=\{a_1, a_2\}$ *and* $S_2=\{b_1, b_2\}$ *be two pairs of distinct elements with* $a_1+a_2=b_1+b_2$ *but* $a_1a_2\neq b_1b_2$, *and let* $S_3=\{\infty\}$. *Suppose that* f *and* g *are two nonconstant meromorphic functions satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1, 2, 3$. *Then* $N(r, f)=S(r, f)$ *and*

$$T(r, f)=T(r, g)+S(r, f).$$

THEOREM F (see [10]). *If, in addition to the assumptions of Theorem E,* $\delta(c/2, f)>1/5$, *where* $c=a_1+a_2$, *then* f *and* g *must satisfy exactly one of the following relations*:

(i) $f\equiv g$,

(ii) $f+g\equiv a_1+a_2$,

(iii) $(f-c/2)(g-c/2)\equiv\pm((a_1-a_2)/2)^2$. *This occurs only for* $(a_1-a_2)^2+(b_1-b_2)^2=0$.

THEOREM G (see [10]). *If, in addition to the assumptions of Theorem E,*

$$N\left(r, \frac{1}{f-b_1}\right)+N\left(r, \frac{1}{f-b_2}\right)=2T(r, f)+S(r, f)$$

and $\delta(c/2, f)>0$, *where* $c=a_1+a_2$, *then the conclusions of Theorem F hold.*

Applying Theorem 1, we immediately obtain the following result which is an improvement of Theorem F and Theorem G.

THEOREM 3. *If, in addition to the assumptions of Theorem E,* $\delta(c/2, f)>0$, *where* $c=a_1+a_2$, *then the conclusions of Theorem F hold.*

Proof. By Theorem E we have

(19) $$N(r, f)\neq T(r, f)+S(r, f).$$

Again by $\delta(c/2, f)>0$, we also have

(20) $$N\left(r, \frac{1}{f-c/2}\right)\neq T(r, f)+S(r, f).$$

Let

$$F=\frac{(f-c/2)^2-((a_1-a_2)/2)^2}{((b_1-b_2)/2)^2-((a_1-a_2)/2)^2}, \qquad G=\frac{(g-c/2)^2-((a_1-a_2)/2)^2}{((b_1-b_2)/2)^2-((a_1-a_2)/2)^2}.$$

If $F\equiv G$, it is obvious that $f\equiv g$ or $f+g\equiv a_1+a_2$. Next, assume that $F\not\equiv G$.

By $E_f(S_j)=E_g(S_j)\ (j=1, 2, 3)$, we know that F and G share $0, 1, \infty$ CM. From (19) and (20), we have

$$N(r, F)\neq T(r, F)+S(r, F)$$

and

$$N\left(r, \frac{1}{F-a}\right)\neq T(r, F)+S(r, F),$$

where

$$a=\frac{-((a_1-a_2)/2)^2}{((b_1-b_2)/2)^2-((a_1-a_2)/2)^2}\neq 0, 1.$$

By Theorem 1, it follows that a is a Picard value of F, and hence $c/2$ is a Picard value of f. Thus $\delta(c/2, f)=1>1/5$. Again by Theorem F, we obtain

$$(f-c/2)(g-c/2)\equiv\pm((a_1-a_2)/2)^2,$$

this occurs only for $(a_1-a_2)^2+(b_1-b_2)^2=0$.

This also completes the proof of Theorem 3.

Example 4. Let $f(z)=1-4e^z$, $g(z)=1-e^{-z}$, $a_1=-1$, $a_2=1$, $b_1=-\sqrt{3}i$, $b_2=\sqrt{3}i$, $S_1=\{a_1, a_2\}$, $S_2=\{b_1, b_2\}$, $S_3=\{\infty\}$. It is easy to verify that

$$\frac{(f-a_1)(f-a_2)}{(g-a_1)(g-a_2)}=-8e^{3z}, \qquad \frac{(f-b_1)(f-b_2)}{(g-b_1)(g-b_2)}=4e^{2z},$$

which show $E_f(S_j)=E_g(S_j)$ for $j=1, 2, 3$. Thus, f and g satisfy the conditions of Theorem E. Noting that $c=a_1+a_2=0$ and $f(z)-c/2=1-4e^z$, we have $\delta(c/2, f)=0$. $f\not\equiv g$, $f+g\not\equiv a_1+a_2$ and $(f-c/2)(g-c/2)\not\equiv\pm((a_1-a_2)/2)^2$ are evident. This shows that Theorem 3 is sharp.

5. Improvement of Theorem 1

Proceeding as in the proof of Theorem 1, we can prove the following result, which is an improvement of Theorem 1.

THEOREM 4. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty$ CM., and let a be a finite complex number, and $a \neq 0, 1$. If*

$$N\left(r, \frac{1}{f-a}\right) \neq T(r, f)+S(r, f),$$

then a is a Picard exceptional value of f, and f and g must satisfy exactly one of the following relations:

(i) $(f-a)(g+a-1) \equiv a(1-a)$. *This occurs only for ∞ be a Picard exceptional value of f. In this case, $1-a$ and ∞ are Picard exceptional values of g.*

(ii) $f+(a-1)g \equiv a$. *This occurs only for 0 be a Picard exceptional value of f. In this case, $a/(a-1)$ and 0 are Picard exceptional values of g.*

(iii) $f \equiv ag$. *This occurs only for 1 be a Picard exceptional value of f. In this case, $1/a$ and 1 are Picard exceptional values of g.*

Proof. Proceeding as in the proof of Theorem 1, we can obtain (1).

We discuss the following four cases.

a) Suppose that $e^p \equiv c(\neq 0, 1)$.

Proceeding as in the proof of Theorem 1, we can obtain the relation (i), and a and ∞ are Picard exceptional values of f, $1-a$ and ∞ are Picard exceptional values of g.

b) Suppose that $e^q \equiv c(\neq 0, 1)$.

By (1) we have

$$f=\frac{c-1}{e^p-1}$$

and

$$f-a=\frac{(c-1+a)-ae^p}{e^p-1}. \tag{21}$$

If $c-1+a \neq 0$, from (21),

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f),$$

which contradicts the assumption of Theorem 4. Then $c-1+a=0$ and $c=1-a$. Again by (1), we obtain

$$f=-\frac{a}{e^p-1}$$

and

$$g=\frac{ae^p}{(a-1)(e^p-1)}.$$

Thus, we get the relation (ii), and a and 0 are Picard exceptional values of f, $a/(a-1)$ and 0 are Picard exceptional values of g.

c) Suppose that $e^{q-p}\equiv c(\neq 0, 1)$.

By (1) we have

$$f=\frac{ce^{p}-1}{e^{p}-1}$$

and

$$f-a=\frac{(c-a)e^{p}-(1-a)}{e^{p}-1}. \tag{22}$$

If $c-a\neq 0$, from (22),

$$N\left(r, \frac{1}{f-a}\right)=T(r, f)+S(r, f),$$

which contradicts the assumption of Theorem 4. Then $c=a$. Again by (1), we obtain

$$f=\frac{ae^{p}-1}{e^{p}-1}$$

and

$$g=\frac{ae^{p}-1}{a(e^{p}-1)}.$$

Thus, we get the relation (iii), and a and 1 are Picard exceptional values of f, $1/a$ and 1 are Picard exceptional values of g.

d) Suppose that none of e^{p}, e^{q} and e^{q-p} are constants.

Proceeding as in the proof of Theorem 1, we can arrive at a contradiction. This completes the proof of Theorem 4.

By Theorem 4 we immediately obtain the following corollary.

COROLLARY. *Let f and g be two nonconstant meromorphic functions such that f and g share* 0, 1, ∞ *CM., and let a be a finite complex number, and* $a\neq 0$, 1. *If*

$$N\left(r, \frac{1}{f-a}\right)\neq T(r, f)+S(r, f),$$

and none of 0, 1, ∞ *are Picard exceptional values of f, then $f\equiv g$.*

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