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ENTIRE FUNCTIONS THAT SHARE ONE VALUE WITH THEIR DERIVATIVES

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Abstract

The paper generalizes a result of [2] and makes an example which shows that the generalization is precise. Also we get similar conclusions in other cases.

§ 1. Introduction

We say that nonconstant meromorphic functions f and g share the value a provided that $f(z)=a$ if and only if $g(z)=a$. We will state whether the shared value is by CM (counting mulitiplicities) or by IM (ignoring multiplicities).

L. Rubel and C. C. Yang proved the following theorem:

THEOREM A[1]. *Let $f(z)$ be a nonconstant entire function. If f and f' share two distinctive valves a and b IM, then $f\equiv f'$.*

1986, Jank, Mues and Volkman proved:

THEOREM B[2]. *Let $f(z)$ be a nonconstant entire function. If f and f' share the value a $(a\neq 0)$, and $f''(z)=a$ when $f(z)=a$, then $f\equiv f'$.*

It is asked naturally whether the f'' of Theorem B can be simply replaced by $f^{(k)}$ $(k\geqq 3)$. We make an example which shows that the answer of this question is negative.

Let k be a positive integer $(k\geqq 3)$ and let $\omega(\neq 1)$ be a $(k-1)$-th root of unity. Set $g(z)=e^{\omega z}+\omega-1$. It is easy to know that g, g' and $g^{(k)}$ share the value ω CM, but $g\not\equiv g'$ and $g\not\equiv g^{(k)}$.

Between the example and Theorem B we will prove the following results.

THEOREM 1. *Let $f(z)$ be a nonconstant entire function. If f and f' share the value a $(a\neq 0)$ CM, and $f^{(n)}(z)=f^{(n+1)}(z)=a$ $(n\geqq 1)$ when $f(z)=a$, then $f\equiv f^{(n)}$.*

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It is obvious that Theorem B is a special case of Theorem 1.

Before stating Theorem 2, we need a slight general notation about the "share". We assume that the reader is familiar with the usual notations and fundamental results of Nevanlinna's theory of meromorphic functions (see, e.g. [3], [8]). In particular, $S(r, f)$ will denote any quantity that satisfies $S(r, f) = o(1)T(r, f)$ as $r \to +\infty$, possibly outside a set e of r of finite linear measure.

Let $E = E\{f_1, f_2, \cdots, f_n; (a_1, a_2, \cdots, a_n)\} = (\bigcup_{j=1}^{n} f_j^{-1}(a_j)) \setminus (\bigcap_{j=1}^{n} f_j^{-1}(a_j))$ and let $N_E(r, 1/(f_i - a_i))$ denote the counting function of those a_i-points of f_i which belong to the set E, $i=1, 2, \cdots, n$. We introduce the following definitions:

DEFINITION 1. Two nonconstant meromorphic functions f and g share the value a CMN (or IMN) if f and g share the value a CM (or IM) outside the set E where $E(=E\{f, g; (a, a)\})$ satisfies that $N_E(r, 1/(f-a)) = S(r, f)$ and $N_E(r, (1/g-a)) = S(r, g)$.

THEOREM 2. *Let $f(z)$ be a nonconstant entire function, f, $f^{(n)}$ $(n \geq 1)$ share the value a $(\neq 0)$ IMN, and $f'(z) = f^{(n+1)}(z) = a$ when $f(z) = a$. If for any set e of finite linear measure*

$$\lim_{\substack{r \to \infty \\ r \notin e}} \frac{N_E\left(r, \frac{1}{f^{(n+1)} - a}\right)}{N\left(r, \frac{1}{f^{(n+1)} - a}\right)} \neq \frac{1}{2} \tag{1.1}$$

where $E = E\{f, f^{(n+1)}; (a, a)\}$, then $f \equiv f^{(n)}$.

We don't know whether it is possible to get rid of the condition (1.1) in Theorem 2.

DEFINITION 2. Meromorphic functions $f_1, f_2, \cdots, f_n$ share the array $(a_1, a_2, \cdots, a_n)$ CM provided that $f_1^{-1}(a_1) = f_j^{-1}(a_j)$ $(1 \leq j \leq n)$ by counting mulitiplicities. Similarly we say $f_1, f_2, \cdots, f_n$ share the array $(a_1, a_2, \cdots, a_n)$ CMN (or IMN) if $f_1, f_2, \cdots, f_n$ share the array $(a_1, a_2, \cdots, a_n)$ CM (or IM) outside the set $E = E\{f_1, f_2, \cdots, f_n, (a_1, a_2, \cdots, a_n)\}$ where E satisfies that $N_E(r, 1/(f_j - a_j)) = S(r, f_j)$, $j=1, 2, \cdots, n$.

There is an example which shows that the condition of Theorem B can't be replaced by the condition that f, f' and f'' share the array (a, a, b) CM. Let $g(z) = e^{2z} + 1$. It is easy to know that g, g', g'' share $(2, 2, 4)$ and $g \not\equiv g'$. However we can make the following conclusion.

THEOREM 3. *Let $f(z)$ be a nonconstant entire function. If f, f', f'' share the array (a, b, b) IMN, where $ab \neq 0$, then $f - a \equiv f' - b$.*

§ 2. Lemmas

LEMMA 1. *Let $f(z)$ be a nonconstant meromorphic function and let b_j $(j=1, 2, \cdots, q)$ be q distinctive finite numbers. Then for any integer $k \geq 0$,*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f^{(k)}-b_j}\right) \leqq m\left(r, \frac{1}{f^{(k+1)}}\right)+S(r, f)$$

where $S(r, f)=o(1)T(r, f)$ *as* $r\to\infty$, *possibly outside a set e of r of fiınte linear measure.*

This lemma can be simply derived from p. 16 of [3].

LEMMA 2. *Suppose that entire functions* f, f' *share the value* a $(\neq 0)$ *CMN. Then*

$$T(r, f)\leqq 2N\left(r, \frac{1}{f-a}\right)+S(r, f).$$

Proof. Since, from Lemma 1, we know that

$$m\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{1}{f'-a}\right)\leqq m\left(r, \frac{1}{f'}\right)+m\left(r, \frac{1}{f'-a}\right)+S(r, f)$$

$$\leqq m\left(r, \frac{1}{f''}\right)+S(r, f)\leqq T(r, f'')+S(r, f),$$

it follows that

$$T(r, f)+T(r, f')\leqq 2N\left(r, \frac{1}{f-a}\right)+T(r, f'')+S(r, f).$$

By using $T(r, f'')\leqq T(r, f')+S(r, f)$, we get

$$T(r, f)\leqq 2N\left(r, \frac{1}{f-a}\right)+S(r, f).$$

LEMMA 3. *Suppose that nonconstant entire functions* f, $f^{(k)}$ $(k\geqq 1)$ *share the array* (a, b) *IMN,* $f'(z)=f^{(k+1)}(z)=b$ *when* $f(z)=a$, *where* $ab\neq 0$. *If* $f^{(k)}-b\not\equiv f-a$, *then the following statements hold*:

$$2N\left(r, \frac{1}{f-a}\right)\leqq T(r, f)+S(r, f) \tag{2.1}$$

$$T(r, f)\leqq 2N\left(r, \frac{1}{f-a}\right)+S(r, f). \tag{2.2}$$

$$N\left(r, \frac{1}{f^{(k+1)}}\right)\leqq S(r, f). \tag{2.3}$$

Proof. From the conditions of the lemma, we see that if $z_0\in f^{-1}(a)\cap(f^{(k)})^{-1}(b)$, then z_0 must be a common zero-point of $f-a$, $f^{(k)}-b$ and $(f^{(k)}-b)-(f-a)$ with multiplicities 1, 1 and $\gamma(\geqq 2)$ respectively. Then

$$(2.4)\qquad 2N\Big(r,\frac{1}{f-a}\Big)\leqq N\Big(r,\frac{1}{(f^{(k)}-b)-(f-a)}\Big)+S(r,f)$$
$$\leqq T(r,f^{(k)}-f)+S(r,f)\leqq T(r,f)+S(r,f).$$

Since, by using Lemma 1, we get that

$$(2.5)\quad m\Big(r,\frac{1}{f-a}\Big)+m\Big(r,\frac{1}{f^{(k)}-b}\Big)\leqq m\Big(r,\frac{1}{f^{(k)}}\Big)+m\Big(r,\frac{f^{(k)}}{f-a}\Big)+\log 2+m\Big(r,\frac{1}{f^{(k)}-b}\Big)$$
$$\leqq m\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f),$$

it is deduced that

$$(2.6)\quad T(r,f)+T(r,f^{(k)})\leqq N\Big(r,\frac{1}{f-a}\Big)+N\Big(r,\frac{1}{f^{(k)}-b}\Big)+m\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f)$$
$$\leqq 2N\Big(r,\frac{1}{f-a}\Big)+T(r,f^{(k+1)})-N\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f).$$

So

$$T(r,f)\leqq 2N\Big(r,\frac{1}{f-a}\Big)+S(r,f).$$

By using (2.4) and (2.6), we get (2.3).

LEMMA 4[3]. *Let $f(z)$ be an entire function. Then*

$$T(r,f)\leqq \bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}\Big(r,\frac{1}{f-a}\Big)+S(r,f)\quad \text{where } a\neq 0.$$

LEMMA 5. *If entire functions f, f' and f'' share the array (a, b, b) IMN $(ab\neq 0)$, $z_0\in f^{-1}(a)\cap(f')^{-1}(b)\cap(f'')^{-1}(b)$,* set

$$H(z)=\frac{f'}{f-a}-\frac{f''}{f'-b}\qquad K(z)=\frac{f''-f'}{f-a},$$

then

$$(2.7)\qquad H(z)=\frac{1}{2}\Big(1-\frac{f''(z_0)}{b}\Big)+O\{(z-z_0)\}$$

$$(2.8)\qquad K(z)=\frac{f''(z_0)}{b}-1+O\{(z-z_0)\}.$$

Proof. Since the Taylor expansion of $f(z)$ about z_0 is

$$f(z)=a+b(z-z_0)+\frac{b}{2!}(z-z_0)^2+\frac{f'''(z_0)}{3!}(z-z_0)^3+\cdots,$$

by an elementary calculation, we can obtain (2.7) and (2.8).

§ 3. Proof of Theorem 1

From Theorem B, we know that Theorem 1 is valid for $n=1$. Next we suppose that $n\geqq 2$. Set

$$\varphi(z)=\frac{f^{(n+1)}(z)-f^{(n)}(z)}{f(z)-a}, \qquad \psi(z)=\frac{f^{(n+1)}(z)-f^{(n)}(z)}{f'(z)-a}.$$

Then

$$m(r, \varphi)=S(r, f), \qquad m(r, \psi)=S(r, f). \tag{3.1}$$

Suppose z_1 is an a-point of $f(z)$. Then the Taylor expansion of $f(z)$ about z_1 is

$$f(z)=a+a(z-z_1)+\frac{f''(z_1)}{2!}(z-z_1)^2+\cdots+\frac{f^{(n-1)}(z_1)}{(n-1)!}(z-z_1)^{n-1} \tag{3.2}$$

$$+\frac{a}{n!}(z-z_1)^n+\frac{a}{(n+1)!}(z-z_1)^{n+1}+\cdots$$

Hence

$$\frac{f^{(n)}}{f-a}=\frac{1}{z-z_1}+\left(1-\frac{f''(z_1)}{2a}\right)+O\{(z-z_1)\} \tag{3.3}$$

$$\frac{f^{(n+1)}}{f-a}=\frac{1}{z-z_1}+\frac{1}{a}\left[f^{(n+2)}(z_1)-\frac{f''(z_1)}{2}\right]+O\{(z-z_1)\}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\varphi(z)=\frac{f^{(n+2)}(z_1)-a}{a}+O\{(z-z_1)\}. \tag{3.5}$$

Since the pole points of $\varphi(z)$ come from the a-points of $f(z)$, we see from (3.5) that

$$N(r, \varphi)=0. \tag{3.6}$$

Also we know from (3.2) that

$$\frac{f^{(n)}}{f'-a}=\frac{a}{f''(z_1)}\cdot\frac{1}{(z-z_1)}+\frac{1}{f''(z_1)}\left[f^{(n+1)}(z_1)-\frac{af'''(z_1)}{2f''(z_1)}\right]+O\{(z-z_1)\}$$

and that

$$\frac{f^{(n+1)}}{f'-a}=\frac{a}{f''(z_1)}\cdot\frac{1}{(z-z_1)}+\frac{1}{f''(z_1)}\left[f^{(n+2)}(z_1)-\frac{af'''(z_1)}{2f''(z_1)}\right]+O\{(z-z_1)\}.$$

Therefore

$$\psi(z)=\frac{f^{(n+2)}(z_1)-a}{f''(z_1)}+O\{(z-z_1)\}. \tag{3.7}$$

Thus

(3.8) $$N(r, \psi)=0 .$$

Combining (3.6), (3.8) with (3.1), we see that

(3.9) $$T(r, \varphi)=S(r, f), T(r, \psi)=S(r, f) .$$

From (3.5) and (3.7), it follows that

(3.10) $$\frac{\varphi(z)}{\psi(z)}=\frac{f''(z_1)}{a}+O\{(z-z_1)\} .$$

Set $H(z)=\dfrac{f^{(n)}(z)-f'(z)}{f(z)-a}$. It is easy to know that

(3.11) $$T(r, H)\leqq S(r, f) .$$

From (3.2), we deduce that

(3.12) $$H(z)=\left[1-\frac{f''(z_1)}{a}\right]+O\{(z-z_1)\} .$$

Hence

(3.13) $$H(z)+\frac{\varphi(z)}{\psi(z)}=1+O\{(z-z_1)\} .$$

Next we consider the following two cases:

(i) Suppose that $H(z)+\varphi(z)/\psi(z)\not\equiv 1$. Set $L(z)=H(z)+\varphi(z)/\psi(z)-1$. Combining (3.9), (3.11) and (3.13), we deduce that

(3.14) $$N\left(r, \frac{1}{f-a}\right)\leqq N\left(r, \frac{1}{L}\right)\leqq T(r, L)\leqq S(r, f) .$$

By using Lemma 2, we get $T(r, f)\leqq S(r, f)$. It follows that $f(z)$ is a polynomial. So f and f' can not share the value a CM which contradicts the condition of Theorem 1.

(ii) Let $H(z)+(\varphi(z)/\psi(z))\equiv 1$. Then

$$\frac{f^{(n)}(z)-f'(z)}{f(z)-a}+\frac{f'(z)-a}{f(z)-a}\equiv 1 .$$

Thus $f\equiv f^{(n)}$.

§ 4. Proof of Theorem 2

Assume that $f\not\equiv f^{(n)}$. From Lemma 3, we see that

(4.1) $$T(r, f)\leqq 2m\left(r, \frac{1}{f-a}\right)+S(r, f) .$$

Hence

(4.2) $$T(r, f^{(n)})\leqq T(r, f)+S(r, f)\leqq 2m\left(r, \frac{1}{f-a}\right)+S(r, f)\leqq 2m\left(r, \frac{1}{f^{(n)}}\right)+S(r, f) .$$

Thus

$$2N\left(r, \frac{1}{f^{(n)}}\right) \leqq T(r, f^{(n)}) + S(r, f). \tag{4.3}$$

Now suppose that $f^{(n)} \equiv f^{(n+1)}$. Then $f^{(n)}(z) = ce^z$ and $f'(z) - f(z) = c_0 z^{n-1} + \cdots + c_{n-2}z + c_{n-1}$ where c and c_i $(i=0, 1, \cdots, n-1)$ are constants. If $c \neq 0$, then $f^{(n)}$ has infinitely many a-points and there exist infinitely many points z such that $f(z) = f'(z)$. It is deduced that $c_i = 0$ $(i=0, \cdots, n-1)$ and that $f \equiv f'$, which contradicts the assumption of $f \not\equiv f^{(n)}$. If $c=0$, then f is a polynomial whose degree is at most $n-1$. Since f is nonconstant, it has necessarily an a-point. By the assumption, it is also an a-point of $f^{(n+1)} (\equiv 0)$. This is a contradiction. So we assume that $f^{(n)} \not\equiv f^{(n+1)}$, then

$$N\left(r, \frac{1}{f-a}\right) \leqq N\left(r, \frac{1}{\frac{f^{(n+1)}}{f^{(n)}} - 1}\right) + S(r, f) \leqq T\left(r, \frac{f^{(n+1)}}{f^{(n)}}\right) + S(r, f) \tag{4.4}$$

$$\leqq N\left(r, \frac{1}{f^{(n)}}\right) + S(r, f).$$

Combining (4.3), (4.4) and (2.2), we get

$$T(r, f) \leqq T(r, f^{(n)}) + S(r, f). \tag{4.5}$$

On the other hand, it follows from (2.1) that

$$2N\left(r, \frac{1}{f^{(n)} - a}\right) = 2N\left(r, \frac{1}{f-a}\right) + S(r, f) \leqq T(r, f) + S(r, f) \tag{4.6}$$

$$\leqq T(r, f^{(n)}) + S(r, f).$$

So

$$T(r, f^{(n)}) \leqq 2m\left(r, \frac{1}{f^{(n)} - a}\right) + S(r, f). \tag{4.7}$$

Combining with (4.2), we deduce from Lemma 1 that

$$m\left(r, \frac{1}{f^{(n+1)}}\right) \geqq m\left(r, \frac{1}{f^{(n)}}\right) + m\left(r, \frac{1}{f^{(n)} - a}\right) - S(r, f) \geqq T(r, f^{(n)}) - S(r, f). \tag{4.8}$$

Hence

$$T(r, f^{(n)}) \leqq T(r, f^{(n+1)}) + S(r, f). \tag{4.9}$$

By using Lemma 3 and Lemma 4, we see that

$$T(r, f^{(n+1)}) \leqq \overline{N}\left(r, \frac{1}{f^{(n+1)}}\right) + \overline{N}\left(r, \frac{1}{f^{(n+1)} - a}\right) + S(r, f) \tag{4.10}$$

$$\leqq \overline{N}\left(r, \frac{1}{f^{(n+1)} - a}\right) + S(r, f).$$

Hence

$$N\left(r, \frac{1}{f^{(n+1)}-a}\right)=\bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right)+S(r, f). \tag{4.11}$$

From (4.10), (4.9), (4.5) and (2.1) we get

$$2N\left(r, \frac{1}{f-a}\right) \leqq \bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right)+S(r, f). \tag{4.12}$$

Also we know that

$$\bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right) \leqq T(r, f^{(n+1)}) \leqq T(r, f)+S(r, f) \leqq 2N\left(r, \frac{1}{f-a}\right)+S(r, f) \tag{4.13}$$

and

$$\begin{aligned} 2N\left(r, \frac{1}{f-a}\right) &= 2\bar{N}\left(r, \frac{1}{f-a}\right)+S(r, f) \\ &= 2\left[\bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right)-\bar{N}_E\left(r, \frac{1}{f^{(n+1)}-a}\right)\right]+S(r, f), \end{aligned} \tag{4.14}$$

where $E=E\{f, f^{(n+1)}; (a, a)\}$.

We deduce from (4.12) and (4.14) that

$$\bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right) \leqq 2\bar{N}_E\left(r, \frac{1}{f^{(n+1)}-a}\right)+S(r, f), \tag{4.15}$$

and from (4.13), (4.14) that

$$\bar{N}\left(r, \frac{1}{f^{(n+1)}-a}\right) \geqq 2\bar{N}_E\left(r, \frac{1}{f^{(n+1)}-a}\right)-S(r, f). \tag{4.16}$$

Combining (4.15), (4.16) with (4.11), we get

$$\lim_{\substack{r\to\infty \\ r\notin e}} \frac{N_E\left(r, \frac{1}{f^{(n+1)}-a}\right)}{N\left(r, \frac{1}{f^{(n+1)}-a}\right)}=\frac{1}{2}$$

for some finite linear measure e of r which contradicts the assumption of the theorem.

§ 5. Proof of Theorem 3

Assume that $f'-b \not\equiv f-a$. By using the same methods as those in proof of Lemma 3, we get that

$$N\left(r, \frac{1}{f''}\right)=S(r, f), \tag{5.1}$$

and that

$$T(r, f) \leqq 2N\left(r, \frac{1}{f-a}\right)+S(r, f). \tag{5.2}$$

Using an argument similar to that in the proof of Theorem 2, we can assume that $f'\not\equiv f''$. Then

$$N\Big(r, \frac{1}{f-a}\Big)\leqq N\Big(r, \frac{1}{\frac{f''}{f'}-1}\Big)+S(r, f)\leqq T\Big(r, \frac{f''}{f'}\Big)+S(r, f)\leqq N\Big(r, \frac{1}{f'}\Big)+S(r, f).$$

Hence

$$T(r, f)\leqq 2N\Big(r, \frac{1}{f'}\Big)+S(r, f). \tag{5.3}$$

Let $z_0\in f^{-1}(a)\cap(f')^{-1}(b)\cap(f'')^{-1}(b)$. From Lemma 5 and the assumption that f, f', f'' share (a, b, b) IMN, we get that

$$2H(z)+K(z)=O\{(z-z_0)\}, \tag{5.4}$$

$$T(r, H)=S(r, f),\ T(r, K)=S(r, f). \tag{5.5}$$

Now we suppose that $2H(z)+K(z)\not\equiv 0$. From (5.4) we have

$$N\Big(r, \frac{1}{f-a}\Big)\leqq N\Big(r, \frac{1}{2H-K}\Big)+S(r, f). \tag{5.6}$$

Combining with (5.2) and (5.5), we get $T(r, f)\leqq S(r, f)$ which contradicts the assumption of the theorem.

Next let $2H(z)+K(z)\equiv 0$. Then

$$\frac{f''(z)+f'(z)}{f(z)-a}=\frac{2f''(z)}{f'(z)-b}. \tag{5.7}$$

From (5.1) and (5.3), we choose z_1 satisying $f'(z_1)=0$, $f''(z_1)\neq 0$. It follows from (5.7) that

$$f''(z_1)\Big(\frac{1}{f(z_1)-a}+\frac{2}{b}\Big)=0.$$

Hence

$$f(z_1)=a-\frac{b}{2}.$$

By the derivative calculation on both sides of (5.7), combining with $f'(z_1)=0$ and $f(z_1)=a-b/2$, we get that $f''(z_1)=b$. Therefore

$$\frac{1}{2}T(r, f)-S(r, f)\leqq N\Big(r, \frac{1}{f'}\Big)-N\Big(r, \frac{1}{f''}\Big)\leqq N_{E_1}\Big(r, \frac{1}{f''-b}\Big)\leqq N_E\Big(r, \frac{1}{f''-b}\Big)$$

where $E_1=E\{f', f''; (b, b)\}$, $E=\{f, f', f''; (a, b, b)\}$ which contradicts the condition that f, f' and f'' share the array (a, b, b) IMN.

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