THE COMPLEX OSCILLATION THEORY OF $f'' + Af' + Bf = F$, WHERE $A, B, F \neq 0$ ARE TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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Abstract

In this paper, we investigate the complex oscillation of the differential equation

$$f'' + Af' + Bf = F$$

where $A$, $B$, $F \neq 0$ are finite order transcendental meromorphic functions. In some cases we obtain estimates of the order of growth and the exponent of convergence of the zero-sequence of solutions for above equation. Theorem 3 and Theorem 4 are the main results among the Theorems in this paper.

§ 1. Introduction and results

In this paper, we will use the standard notations of the Nevanlinna theory (e.g. see [9]). In addition, we will also use the same notations as in [1], i.e. we will use, $\lambda(f)$ and $\bar{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$. The individual notations will be shown when they appear.

G. Gundersen proved in [8]:

THEOREM A. If $f \neq 0$ is a solution of

$$(1.1) \quad f'' + Af' + Bf = 0,$$

where $A$, $B$ are entire such that

(i) $\sigma(B) < \sigma(A) < 1/2$

or

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(ii) \( A \) is transcendental with \( \sigma(A) = 0 \) and \( B \) is a polynomial, then \( \sigma(f) = \infty \).

Gao Shi-an proved in [6]

**Theorem B.** For the equation

\[
f'' + a_0 f = p_1 e^{p_0}
\]

where \( a_0, p_0, p_1 \) are polynomials, \( \deg a_0 = n, \deg p_0 < 1 + (n/2) \)

(a) If \( n > 1 \) and \( \deg p_1 < n \), then every solution \( f \) of (1.2) satisfies

\[
\lambda(f) = \lambda(f) = \sigma(f) = 1 + \left( \frac{n}{2} \right) > \deg p_0.
\]

(b) If \( \deg p_1 \geq n \geq 0 \), then the solution \( f \) of (1.2) either satisfies \( \lambda(f) = \lambda(f) = \sigma(f) = 1 + \left( \frac{n}{2} \right) > \deg p_0 \), or is of the form \( f = Q e^{p_0} \), where \( Q \) is a polynomial. And if (1.2) has a solution of the form \( Q e^{p_0} \) with \( Q \) polynomial, then (1.2) must have solutions which satisfy \( \lambda(f) = \lambda(f) = \sigma(f) = 1 + \left( \frac{n}{2} \right) > \deg p_0 \).

Chen Zong-xuan and Gao Shi-an investigated the complex oscillation of non-homogeneous linear differential equations with rational coefficients in [4].

In this paper, we will investigate the complex oscillation of the second order non-homogeneous linear differential equation

\[
f'' + A f' + B f = F
\]

where \( A, B, F \equiv 0 \) are transcendental meromorphic functions. We will prove the following four theorems:

**Theorem 1.** Suppose that \( A, B, F \equiv 0 \) are finite order meromorphic functions, that either (i) or (ii) below holds:

(i) \( \lim_{r \to \infty} \frac{\log m(r, A)}{\log r} < \lim_{r \to \infty} \frac{\log m(r, B)}{\log r} \)

(ii) \( \lim_{r \to \infty} m(r, B)/\log r = \infty \), and \( A \) is rational.

If non-homogeneous linear differential equation (1.3) has meromorphic solution \( f(z) \), then

(a) All meromorphic solutions of (1.3) satisfy

\[
\lambda(f) = \lambda(f) = \sigma(f) = \infty
\]

with at most one possible finite order meromorphic solution \( f_0 \). If all solutions of (1.3) are meromorphic, then (1.3) must have solutions which satisfy (1.4).

(b) If there exists a finite order meromorphic solution of in case (a), then \( f_0 \) satisfies

\[
\sigma(f_0) \leq \max \{ \lambda(f_0), \sigma(F), \sigma(A), \sigma(B) \}.
\]

If \( \lambda(f_0) < \sigma(f_0) \), and \( \sigma(F), \sigma(A), \sigma(B) \), are unequal each other, then

\[
\sigma(f_0) = \max \{ \sigma(F), \sigma(A), \sigma(B) \}.
\]
THEOREM 2. Suppose that $A, B, F \neq 0$ are finite order meromorphic functions having only finitely many poles, that either (i) or (ii) below holds:

(i) $\sigma(A) < \sigma(B)$,

(ii) $B$ is transcendental, and $A$ is rational.

If the equation (1.3) has meromorphic solutions $f(z)$, then

(a) All meromorphic solutions of (1.3) satisfy (1.4) with at most one possible finite order meromorphic solution $f_0$. If all solutions of (1.3) are meromorphic, then (1.3) must have solutions which satisfy (1.4).

(b) If there exists a finite order meromorphic solution $f_0$ in case (a), then $f_0$ satisfies

$$\sigma(f_0) \leq \max \{\tilde{\sigma}(f_0), \sigma(B), \sigma(F)\}.$$ 

If $\tilde{\sigma}(f_0) < \sigma(f_0)$, $\sigma(F) \neq \sigma(B)$, then $\sigma(f_0) = \max \{\sigma(B), \sigma(F)\}$.

THEOREM 3. Suppose that $A, B, F \neq 0$ are meromorphic functions having only finitely many poles, $F \equiv cB$ (c is a constant), that either (i) or (ii) below holds:

(i) $\sigma(B) < \sigma(A) < 1/2$, and $\sigma(F) < \sigma(A)$

(ii) $A$ is transcendental and $\sigma(A) = 0$, $B$ and $F$ are rational.

If $f(z)$ is a meromorphic solution of (1.3) then $f$ satisfies (1.4).

THEOREM 4. Suppose that $A, B, F \neq 0$ are finite order meromorphic functions having only finitely many poles, that either (i) or (ii) below holds:

(i) $\sigma(B) < \sigma(A) < 1/2$ and $\sigma(A) \leq \sigma(F)$.

(ii) $A, F$ are transcendental and $\sigma(A) = 0$, $B$ is rational.

If the equation (1.3) has meromorphic solution $f(z)$, then:

(a) If $B \equiv 0$, then all meromorphic solutions of (1.3) satisfy (1.4) with some possible finite order solutions $f_c = f_0 + c$ ($f_0$ is some finite order meromorphic solution, $C$ is an arbitrary constant).

(b) If $B \equiv 0$, then all meromorphic solutions of (1.3) satisfy (1.4) with at most one finite order meromorphic solution $f_0$.

(c) The finite order meromorphic solution $f_c$ of (1.3) satisfies

$$\sigma(f_c) \leq \max \{\sigma(F), \tilde{\sigma}(f_c)\}.$$ 

If $\sigma(A) < \sigma(F)$, $\tilde{\sigma}(f_c) < \sigma(f_c)$, then $\sigma(f_c) = \sigma(F)$

(d) If all solutions of (1.3) are meromorphic, then (1.3) must have solutions which satisfy (1.4).

§ 2. Lemmas

LEMMA 1. Suppose that $f(z) = g(z)/h(z)$ is transcendental meromorphic function having only finitely many poles, where $g(z)$ is a transcendental entire function, $h$ is a polynomial. Let $z$ be a point with $|z| = r$ at which $|g(z)| = M(r, g)$, $h(z) \neq 0$, $\nu(r)$ denote the centralindex of the entire function $g(z)$, then
holds for all $|z|=r$ outside a subset $E$ of $r$ of finite logarithmic measure.

Proof. By $f=g/h$, we have

$$f'(z)=(g'(z)/h(z)) - \frac{g(z)}{h(z)} \cdot \frac{h'(z)}{h(z)}.$$  

On the other hand, from the Wiman-Valiron theory (see [10, 11, 12]), let $z$ be a point with $|z|=r$, at which $|g(z)|=M(r, g)$, $h(z)\neq 0$, then we have

$$g'(z)=(\frac{\nu(r)}{z}) g(z)(1+o(1)) \quad r \in E$$

where $E \subset (0, \infty)$ has finite logarithmic measure.

Substituting (2.3) into (2.2), we have

$$f'(z)=(\frac{\nu(r)}{z}) \cdot f(z) \cdot (1+o(1)) \quad r \in E.$$  

Since $g(z)$ is transcendental, we have $(\nu(r)/z)^{-1} \rightarrow o(r \rightarrow \infty)$. And $h(z)$ is a polynomial, $|z \cdot h'(z)/h(z)|=O(1)$ $(r \rightarrow \infty)$, so

$$g'(z)=(\nu(r)/z) \cdot f(z) \cdot (1+o(1)) \quad r \in E.$$  

Therefore, by (2.4) and (2.5), we obtain

$$f'(z)=(\nu(r)/z) \cdot \cdot f(z) \cdot (1+o(1)) \quad r \in E.$$  

This proves Lemma 1.

LEMMA 2. Suppose that $A, B$ satisfy the hypotheses of Theorem 1. If $g(z) \neq 0$ is a meromorphic solution of the homogeneous linear differential equation

$$g'' + Ag' +Bg=0$$

then $\sigma(g)=\infty$.

Proof. If $\sigma(g)<\infty$, then we have from (2.6)

$$m(r, A) \leq m(r, A)+m(r, g'/g)+m(r, g'/g)=m(r, A)+O(\log r)$$

If $A$ is transcendental, then

$$\lim_{r \to \infty} \log m(r, B)/\log r \leq \lim_{r \to \infty} \log m(r, A)/\log r ;$$

if $A$ is rational, then $\lim_{r \to \infty} m(r, B)/\log r \leq \lim_{r \to \infty} m(r, A)/\log r < M(M>0$ is some constant), this contradict on the hypotheses $A, B$.

LEMMA 3. Suppose that $A, B$ satisfy hypotheses of Theorem 3 or Theorem 4. If $g(z) \neq 0$ is a meromorphic solution of (2.6), then: if $B \neq 0$, then $\sigma(g)=\infty$ ;
if \( B = Q \), then either \( g(z) \) is a constant, or \( \sigma(g) = \infty \).

Proof. Assume that \( g(z) \) is a transcendental meromorphic solution and \( \sigma(g) = \sigma < \infty \). By (2.6) and fact that \( A, B \) have only finitely many poles, it is easy to see that \( g(z) \) has only finitely many poles.

Now set
\[
g(z) = u(z)/p(z), \quad A(z) = u_A/p_A(z), \quad B(z) = u_B/z/p_B(z)
\]
where \( p, p_A, p_B \) are polynomials, \( u, u_A, u_B \) are entire functions, \( u, u_A \) are transcendental, and \( \sigma(u_A) = \sigma(A) < 1/2 \), \( \sigma(u_B) = \sigma(B) \), \( \sigma(u) = \sigma(g) = \sigma \)

From Lemma 1, let \( z \) be a point with \( |z| = r \) at which \( |u(z)| = M(r, u) \), then
\[
g'(z)/g(z) = ((\nu(r)/z) (1 + o(1)) \tag{2.8}
\]
holds for all \( |z| = r \) outside a set \( E_1 \) of \( r \) of finite logarithmic measure, where, \( \nu(r) \) denotes the central-index of the entire function \( u(z) \).

On the other hand, by \( \sigma(g) = \sigma < \infty \), and Corollary 2 of [7], we have
\[
g''(z)/g(z) \leq |z|^2 \gamma - 1 \tag{2.9}
\]
for all \( |z| = r \in E_2 \cup [0, 1] \), \( E_2 \subseteq (1, \infty) \) has finite logarithmic measure.

From (2.6) and (2.7), we have
\[
|u_A g'/g| \leq |p_A g'/g| + |p_A u_B|/|p_B| \tag{2.10}
\]

Now divide the discussion into two cases.

**CASE I.** Suppose that \( \sigma(u_A) = \sigma(A) > 0 \). Then we take \( \rho, \tau \) such that
\[
\sigma(u_B) = \sigma(B) < \rho < \tau < \sigma(u_A) < 1/2.
\]

From Theorem of \( \cos(\pi \sigma) \) type in [2, 3], it is easy to know that there exists a subset \( H \subseteq (1, +\infty) \) with infinite logarithmic measure, such that if \( |z| = r \in H \), then
\[
\log |u_A(z)| > r^\tau, \quad \log |u_B(z)| < r^\rho \tag{2.11}
\]

By (2.9)-(2.11), for \( |z| = r \in H - (E_1 \cup E_2 \cup [0, 1]) \), \( H - (E_1 \cup E_2 \cup [0, 1]) \) has infinite logarithmic measure) we have as \( r \to \infty \),
\[
|z^s g'(z)/g(z)| \leq |z|^s [ |p_A \cdot p_B \cdot g''(z)/g(z)| + |p_A u_B|/|p_B| ] < 0, \tag{2.12}
\]
\[
|z^s g'/g| \leq O(r^M_1 \cdot \exp(r^\rho)/\exp(r^\gamma)) \to 0, \tag{2.13}
\]
where \( M_1 > 0 \) is a constant.

**CASE II.** Suppose that \( \sigma(u_A) = \sigma(A) = 0 \), \( u_A \) is transcendental, then also from Theorem of \( \cos(\pi \sigma) \) type, there exists a subset \( H_i \subseteq (1, \infty) \) with infinite logarithmic measure such that if \( |z| = r \in H_i \), then
\[
\min \{|\log|u_A(z)| : |z| = r|/\log r \to \infty \quad (r \to \infty). \tag{2.14}
\]
By (2.9), (2.12) and (2.14), for $|z|=r \in H_1 - (E_1 \cup E_2 \cup [0, 1])$ ($H_1 - (E_1 \cup E_2 \cup [0, 1])$ has infinite logarithmic measure), we have as $r \to \infty$

$$(2.15) \quad |z^4 \cdot g'(z)/g(z)| \leq 0(r^M) / \min |u_A(z)| \to 0.$$ 

Therefore, for both cases above, by (2.13) or (2.15),

$$(2.16) \quad |z^4 \cdot g'(z)/g(z)| \to 0 \quad (r \to \infty)$$

holds for $r \in H - (E_1 \cup E_2 \cup [0, 1])$, or $r \in H_1 - (E_1 \cup E_2 \cup [0, 1])$.

But by (2.8), for such $z$ satisfying $|z|=r \in H - (E_1 \cup E_2 \cup [0, 1])$ or $r \in H_1 - (E_1 \cup E_2 \cup [0, 1])$ and $|u(z)|=M(r, u)$, $r \to \infty$, we have

$$(2.17) \quad z^4 \cdot g'(z)/g(z)=z \cdot \nu(r) (1+o(1)).$$

By (2.16) and (2.17), we have $\nu(r) \to 0 \quad (r \to \infty)$. This contradicts the fact that $u$ is a transcendental entire function if and only if $\nu(r) \to \infty$ (as $r \to \infty$). Therefore, $u(z)$ either is a polynomial, or satisfies $\sigma(u)=\infty$, i.e. $g(z)$ either is a rational function, or satisfies $\sigma(g)=\infty$.

By (2.6), it is easy to know that if $g(z) \equiv 0$ is a nonconstant rational function, then $g^*+Ag'+Bg$ is a transcendental function with $\sigma(g^*+Ag'+Bg)=\sigma(A)$, this is a contradiction; if $B \equiv 0$ and $g(z)$ is a constant $C \neq 0$, then $g^*+Ag'+Bg=CB \equiv 0$, this contradicts (2.6).

**Lemma 4.** Suppose that $A, B, F \equiv 0$ are finite order meromorphic functions. If $f(z)$ is a meromorphic solution of equation (1.3) with $\sigma(f)=\infty$, then $\tilde{\lambda}(f)=\sigma(f)=\infty$.

**Proof.** We can write from (1.3)

$$(2.18) \quad 1/f = (1/F)(f^*/f) + A(f'/f) + B),$$

hence

$$(2.19) \quad N(r, 1/f) \leq 2N(r, 1/f) + N(r, 1/F) + N(r, A) + N(r, B).$$

Applying the Lemma of the logarithmic derivative, from (2.18), we have

$$(2.20) \quad m(r, 1/f) \leq m(r, 1/F) + m(r, A) + m(r, B) + 0\{\log T(r, f) + \log r\} \quad (r \in E)$$

where a subset $E \subseteq (0, \infty)$ has finite linear measure, (2.19) and (2.20) give

$$T(r, f) = T(r, 1/f) + O(1)$$

$$\leq 2N(r, 1/f) + T(r, 1/F) + T(r, A) + T(r, B) + O\{\log T(r, f) + \log r\} \quad (r \in E).$$

Since $\sigma(f)=\infty$, there exists $\{r_n\} (r_n \to \infty)$ such that

$$\lim_{r_n \to \infty} \log T(r_n, f)/\log r_n = \infty.$$
Setting the linear measure of $E$, $mE=\delta<\infty$, then there exists a point $r_n \in [r'_n, r'_n+\delta+1]-E$. From

$$
\log T(r_n, f)/\log r_n \geq \log T(r'_n, f)/\log(r'_n+\delta+1)
$$

$$
= \log T(r'_n, f)/[\log r'_n + \log[1+(\delta+1)/r'_n]],
$$

we have

$$
\lim_{r_n \to \infty} \log T(r_n, f)/\log r_n \geq \lim_{r_n \to \infty} \log T(r'_n, f)/[\log r'_n + \log(1+(\delta+1)/r'_n)] = \infty.
$$

For a given arbitrary large $\beta (\beta > \delta = \max\{\sigma(A), \sigma(B), \sigma(F)\})$, by (2.22),

$$
T(r_n, f) \geq r_n^\beta,
$$

hold for sufficiently large $r_n$.

On the other hand, for a given $\varepsilon (0 < \varepsilon < \beta - \delta)$, for sufficiently large $r_n$, we have

$$
T(r_n, A) < r_n^{\varepsilon+\delta}, \quad T(r_n, B) < r_n^{\varepsilon+\delta}, \quad T(r_n, F) < r_n^{\varepsilon+\delta}.
$$

By (2.23) as $r_n \to \infty$, we have

$$
T(r_n, A)/T(r_n, f) < r_n^{\varepsilon+\delta-\beta} \to 0
$$

$$
T(r_n, B)/T(r_n, f) < r_n^{\varepsilon+\delta-\beta} \to 0
$$

$$
T(r_n, F)/T(r_n, f) < r_n^{\varepsilon+\delta-\beta} \to 0
$$

Therefore,

$$
T(r_n, A) < (1/5)T(r_n, f)
$$

$$
T(r_n, B) < (1/5)T(r_n, f)
$$

$$
T(r_n, F) < (1/5)T(r_n, f)
$$

hold for sufficiently large $r_n$. From

$$
O\{\log T(r_n, f)+\log r_n\} = o\{T(r_n, f)\},
$$

we obtain that

$$
O\{\log T(r_n, f)+\log r_n\} \leq (1/5)T(r_n, f)
$$

also holds for sufficiently large $r_n$. Substituting (2.24)-(2.27) into (2.21), we obtain

$$
T(r_n, f) < 10\bar{N}(r, 1/f).
$$

By (2.22) and (2.28), we have

$$
\lim_{r_n \to \infty} \log T(r_n, f)/\log r_n \leq \lim_{r_n \to \infty} \log \bar{N}(r_n, 1/f)/\log r_n \leq \tilde{\lambda}(f)
$$
therefore, $\lambda(f) = \lambda(f) = \sigma(f) = \infty$.

§ 3. Proofs of theorems

Proof of Theorem 1. (a) Assume that $f_0$ is a meromorphic solution of (1.3) with $\sigma(f_0) = \sigma < \infty$. If $f_i(\equiv f_0)$ is second finite order meromorphic solution of (1.3), then $\sigma(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a meromorphic solution of the corresponding homogeneous equation (2.6) of (1.3). But $\sigma(f_1 - f_0) = \infty$ from Lemma 2, this is a contradiction.

Now assume that $f(z)$ is an infinite order meromorphic solution of (1.3), then $\lambda(f) = \lambda(f) = \sigma(f) = \infty$ from Lemma 4.

If all solutions of (1.3) are meromorphic functions, then all solutions of the corresponding homogeneous equation (2.6) of (1.3) are meromorphic functions. Assume $\{f_1, f_2\}$ is fundamental solution set of (2.6). By [5, p. 412], we have

$$m(r, B) = O\{\log[\max(T(r, f_1), s=1, 2)] + O(\log r)\}.$$ 

Since $B$ is transcendental, there exists at least $f_1$ or $f_2$ with infinite order of growth. If $f_0$ is a solution of (1.3), then every solution $f$ of (1.3) can be written in the form

$$f = c_1 f_1 + c_2 f_2 + f_0$$

where $c_1, c_2$ are arbitrary constants. Hence (1.3) must have infinite order solutions, and all infinite order solutions satisfy (1.4) from Lemma 4.

(b) For the finite order meromorphic solution $f_0$ of (1.3), using the analogous proof as in Lemma 4, and remarking $m(r, f^{(j)}/f) = O\{\log r\} (j=1, 2)$ from $\sigma(f_0) = \sigma < \infty$, we easily know that

$$T(r, f_0) \leq 2N(r, 1/f_0) + T(r, F) + T(r, A) + T(r, B) + O\{\log r\}$$

holds for all $r$. Hence

$$\sigma(f_0) \leq \max\{\lambda(f_0), \sigma(F), \sigma(A), \sigma(B)\}$$

If $\lambda(f_0) < \sigma(f_0)$, and $\sigma(F), \sigma(A), \sigma(B)$ are different from each other, then from (1.3), we have

$$\sigma(f_0) \leq \max\{\sigma(F), \sigma(A), \sigma(B)\}.$$ 

Therefore, (3.2) and (3.3) give

$$\sigma(f_0) = \max\{\sigma(F), \sigma(A), \sigma(B)\}.$$ 

Proof of Theorem 2. Theorem 2 immediately follows from Theorem 1.

Proof of Theorem 3. From $F \equiv cB$, we know that (1.3) has no constant solutions. If $f$ is a nonconstant rational function, then for case (i), we have $\sigma(f^2 + Af' + Bf) = \sigma(A) > \sigma(F)$; for case (ii), we have $f^2 + Af' + Bf$ is trans-
F is a rational function. Hence (1.3) has no rational solutions, i.e. $f$ must be a transcendental meromorphic solution.

Now assume that $f$ is a transcendental meromorphic solution with $\sigma(f) = \sigma < \infty$. From (1.3) and fact that $A, B, F$ have only finitely many poles, we know that $f$ has only finitely many poles.

Set

\begin{align}
\tag{3.5}
& f(z) = u(z)/p(z), \ A(z) = u_A/p_A, \ B(z) = u_B/p_B, \ F = u_F/p_F \\
& \text{where } u, u_A, u_B, u_F \text{ are entire and } u, u_A \text{ are transcendental } p, p_A, p_B, p_F \text{ are polynomials, } \sigma(u) = \sigma(f) = \sigma, \ \sigma(u_A) = \sigma(u), \ \sigma(u_B) = \sigma(B), \ \sigma(u_F) = \sigma(F).
\end{align}

For $f$, using the same reasoning as in Lemma 3, by Lemma 1, we have

\begin{align}
& \tag{3.6} f'(z)/f(z) = (v(r)/r)(1 + o(1)) \quad r \in E_1, \\
& \text{where } |z| = r, \ |u(z)| = M(r, u), \ E_1 \subset (1, \infty) \text{ has finite logarithmic measure, } v(r) \text{ denotes the centralindex of } u(z). \text{ From Corollary 2 of [7], we have}
\end{align}

\begin{align}
& \tag{3.7} |f''(z)/f(z)| \leq |z|^{\sigma + 1} \quad r \in E_3 \cup [0, 1] \\
& \text{where } E_3 \subset (1, \infty) \text{ has finite logarithmic measure. By (3.5) and (1.3), we obtain}
\end{align}

\begin{align}
& \tag{3.8} |u_A f'/f| \leq \left[ |p_A p_B f''/f| + |p_A u_B|/|p_B| + |u_F p_A p_B p_F u_B|/|p_F u| \right] \\
& \text{From } u(z) \text{ is a transcendental entire function, we take } z \text{ satisfying } |z| = r, \ |u(z)| = M(r, u), \text{ then for sufficiently large } |z|, \text{ we have } |u(z)| > 1 \text{ and } |u_F p_A p_B p_F u_B|/|p_F u| < |u_F p_A p_B p_F|/|p_F| \text{. By (3.8), we have}
\end{align}

\begin{align}
& \tag{3.9} |u_A f'/f| \leq \left[ |p_A p_B f''/f| + |p_A u_B|/|p_B| + |u_F p_A p_B p_F u_B|/|p_F| \right] \\
& \text{for sufficiently large } |z|, \text{ and } z \text{ satisfying } |z| = r, \ |u(z)| = M(r, u). \text{ Divide the discussion into two cases.}
\end{align}

**CASE I.** Suppose that $\sigma(u_A) = \sigma(A) > 0$, then we take $\rho, \tau$, such that

\begin{equation}
\max\{\sigma(u_B), \sigma(u_F)\} < \rho < \tau < \sigma(u_A) < 1/2.
\end{equation}

From theorem of cos($\pi \sigma$) type [2, 3], it is easy to know that there exists a subset $H \subset (1, \infty)$ with infinite logarithmic measure such that if $|z| = r \in H$, then

\begin{equation}
\tag{3.10} \log|u_A(z)| > r^\rho, \ \log|u_B(z)| < r^\rho, \ \log|u_F(z)| < r^\rho.
\end{equation}

By (3.6)-(3.10), for $|z| = r \in H \setminus (E_1 \cup E_2 \cup [0, 1])$ and $z$ satifying $|u(z)| = M(r, u)$, $r \to \infty$, we have

\begin{align}
& |z^2 f''(z)/f(z)| \leq |z^2 p_A p_B p_F f''/f| + |z^2 p_A p_F u_B| \\
& \tag{3.11} + |z^2 u_F p_A p_B p_F u_B|/|p_F p_B u_B| < O(r^{M_1}) \exp(r^\rho) \exp(r^\tau) \to 0
\end{align}

where $M_1 > 0$ is a constant.

**CASE II.** Suppose that $\sigma(u_A) = \sigma(A) = 0$, $u_A$ is transcendental, then also from
Theorem of cos(πσ) type, there exists a subset \( H_1 \subset (1, \infty) \) with infinite logarithmic measure such that if \( |z|=r \in H_1 \), then

\[
\min \{ \log |u_1(z)| : |z|=r \} / \log r \to \infty \quad (r \to \infty).
\]

By (3.6)-(3.9), (3.12), and the fact that \( B, F \) are rational function, for \( |z|=r \in H_1-(E_1 U E_2 U [0.1]) \), and \( z \) satisfying \( |u(z)|=M(r, u) \), \( r \to \infty \), we have

\[
\left| z^2 f'(z)/f(z) \right| \leq O \left( r^{M+1} / \min |u_1(z)| \right) \to 0.
\]

Therefore, for both cases above, by (3.11) or (3.13), for \( r \in H-(E_1 U E_2 U [0.1]) \) (or \( r \in H_1-(E_1 U E_2 U [0, 1]) \)) and \( z \) satisfying \( |u(z)|=M(r, u) \), \( r \to \infty \), we have

\[
\left| z^2 f'(z)/f(z) \right| \to 0.
\]

On the other hand, for \( r \in H-(E_1 U E_2 U [0, 1]) \) (or \( r \in H_1-(E_1 U E_2 U [0, 1]) \)) and \( z \) satisfying \( |z|=r, |u(z)|=M(r, u) \), by (3.6) as \( r \to \infty \), we have

\[
\left| z^2 f'(z)/f(z) \right| \sim z \cdot \nu(r).
\]

(3.15) and (3.14) give \( \nu(r) \to 0 (r \to \infty) \), this contradicts the fact that \( u \) is a transcendental entire function if and only if \( \nu(r) \to \infty (r \to \infty) \). Therefore, we have \( \sigma(f)=\infty \). From Lemma 4, we know that \( f \) satisfies (1.4).

Proof of Theorem 4. (a) If \( B=0 \), then arbitrary constant \( c \) is a solution of the corresponding homogeneous equation (2.6) of (1.3). Assume \( f_0 \) is a finite order meromorphic solution of (1.3), then \( f_c=f_0+c \) are also solutions of (1.3). If \( f_1(\equiv f_0) \) is second finite order meromorphic solution of (1.3), then \( f_1-f_0 \) is a constant solution of the corresponding homogeneous equation (2.6) of (1.3). From Lemma 3 and \( \sigma(f_1-f_0)<\infty \), all finite order meromorphic solutions of (1.3) are of the form \( f_c=f_0+c \).

If \( f \) is a meromorphic solution of (1.3) with \( \sigma(f)=\infty \), then \( \lambda(f)=\lambda(f)=\sigma(f) \to \infty \) from Lemma 4.

(b) If \( B=0 \), using the same reasoning as in Theorem 1 by Lemma 3, we know that (1.3) has at most one finite order meromorphic solution \( f_0 \). If \( f \) is a meromorphic solution of (1.3) with \( \sigma(f)=\infty \), then \( \lambda(f)=\lambda(f)=\sigma(f)=\infty \) from Lemma 4.

(c) For the finite order meromorphic solution \( f_c \) of (1.3), using the same reasoning as in Theorem 1, and remarking \( \sigma(A) \leq \sigma(F) \), we can obtain

\[
\sigma(f_c) \leq \max \{ \lambda(f_c), \sigma(F) \}
\]

If \( \lambda(f_c)<\sigma(f_c) \) and \( \sigma(A) \leq \sigma(F) \), then \( \sigma(f_c) \leq \sigma(F) \) from (1.3), combining (3.4), we have \( \sigma(f_c)=\sigma(F) \).

(d) We can use the same proof as in Theorem 1 (a).
§4. Examples for having finite order solutions

Example 1. The equation

\[ f'' - 2zf' + (\sin z - 2)f = \exp(x^2) \cdot \sin z \]

satisfies hypotheses of Theorem 1 or Theorem 2, it has a finite order solution \( f = \exp(x^2) \).

Example 2. Suppose \( A \) is a transcendental meromorphic function satisfying the additional hypothesis of \( A \) in Theorem 4, then the equation

\[ f'' + Af' + zf = (A + z + 1)e^z \]

has finite order solution \( f_0 = e^z \).

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References
