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GEOMETRY AND TOPOLOGY OF SUBMANIFOLDS IMMERSED IN SPACE FORMS AND ELLIPSOIDS

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Abstract

Let M^m be a compact submanifold of a simply connected space form $N^n(c)$ with $c \geqq 0$. Denote by s and H the square length of the second fundamental form and the mean curvature vector field of M respectively. By introducing a selfadjoint linear operator Q^A associated with the shape operator of M, we show that there are no stable currents in M and topologically, M is a sphere if $s<H^2/(m-1)$. For an immersed submanifold of the ellipsoid we show that appropriate assumption on Q^A implies the vanishing of a given homology group.

1. Introduction

Let M^m be a submanifold immersed in a Riemannian manifold N^n. Denote by $V(N, M)$ the normal bundle of M in N. For a smooth section $\nu \in C(V(N, M))$, the shape operator A_ν determined by ν is given by

$$\langle A_\nu X, Y\rangle = \langle h(X, Y), \nu\rangle ,$$

where $X, Y \in C(TM)$ and h is the second fundamental form of M.

In 1973, by using techniques of the calculus of variations in geometric measure theory, H. B. Lawson and J. Simons [4] showed the following

THEOREM LS. *Let M^m be a compact submanifold of S^n and p a given integer, $p \in (0, m)$. If for any $x \in M$ and any orthonormal basis $\{e_i, e_\alpha\}$ $(i=1, \cdots, p; \alpha=p+1, \cdots, m)$ of T_xM the following condition is satisfied*

$$\sum_{i,\alpha} [2\|h(e_i, e_\alpha)\|^2 - \langle h(e_i, e_i), h(e_\alpha, e_\alpha)\rangle] < p(m-p),$$

then there is no stable p-current in M and hence

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$$H_p(M, Z)=H_{m-p}(M, Z)=0.$$

Theorem LS has been extended to the submanifolds of E^n and $S^{n_1}\times S^{n_2}$ by Y. L. Xin [7] and the author [8], respectively.

On the other hand, as an extension of the well-known gap theorem in the minimal submanifolds, M. Okumura [5] proved that

THEOREM O. *Let M^m be a compact, connected submanifold immersed in a Riemannian manifold of non-negative constant curvature. Suppose that*

(c) *the connection of the normal bundle is flat and the mean curvature vector field H is parallel with respect to the connection of the normal bundle.*

If $s=\sum_{\lambda}\operatorname{tr} A_\lambda^2$, the square length of the second fundamental form, satisfies $s<H^2/(m-1)$, then M is totally umbilical.

In this paper, we shall cancel the condition (c) in Theorem O and prove that

THEOREM 1. *Let $\phi: M^m\to N^n(c)$ be an isometric immersion of a Riemannian manifold M^m into a simply connected space form $N^n(c)$, $m=\dim M\geqq 3$. If one of the following is satisfied:*

C1. *M is compact $c\geqq 0$ and $s<H^2/(m-1)$ on M,*

C2. *M is complete, $c>0$ and $s\leqq H^2/(m-1)$ on M,*

then

i) *there exist no stable p-currents in M and hence*

$$H_p(M, Z)=0 \qquad \text{for } p=1, 2, \cdots, m-1;$$

ii) *M is homeomorphic to a sphere when $m\geqq 4$.*

Remark. Just as Okumura [5] indicated, the condition $s<H^2/(m-1)$ is the best possible when $N^n=E^n$. For example, let $M^m=S^{m-1}\times E^1\hookrightarrow E^{m+1}$, then $s=H^2/(m-1)$ on M.

Now we give an example for Theorem 1. Consider the ellipsoid

$$M^m: x_1^2+x_2^2+\cdots+x_m^2+(x_{m+1}^2/c^2)=1 \qquad (c>0).$$

Denote by r the position vector of the point $x\in M^m$ in E^{m+1}. Then M can also be expressed by

$$r=(\sin\theta_m \sin\theta_{m-1}\cdots\sin\theta_2\sin\theta, \cdots, \cos\theta_2\sin\theta, c\cos\theta).$$

A calculation indicates that the shape operator A of M can be given by

$$AX=aX+b\langle X, t\rangle t, \tag{1.1}$$

where $X\in C(TM)$ and

$$t=(\partial r/\partial\theta)/\|\partial r/\partial\theta\|\,,\qquad a=c/(\cos^2\theta+c^2\sin^2\theta)^{1/2}\,,$$
$$b=c(1-c^2)\sin^2\theta/(\cos^2\theta+c^2\sin^2\theta)^{3/2}\,. \tag{1.2}$$

Choose an orthonormal basis $\{e_i\}$ of T_xM such that e_i is parallel to $\partial r/\partial\theta_i$ ($i=1, 2, \cdots, m$, $\theta_1=\theta$). Then from (1.1),

$$s=\operatorname{tr} A^2=\sum_i\langle A^2e_i, e_i\rangle=(a+b)^2+(m-1)a^2\,,$$

$$H^2=(\operatorname{tr} A)^2=(\sum_i\langle Ae_i, e_i\rangle)^2=[(a+b)+(m-1)a]^2\,.$$

And thus

$$H^2/(m-1)-s=(a+b)[ma+(2-m)b]/(m-1)\,.$$

It is easy to verify that $a+b>0$ and $ma+(2-m)b>0$ when $c^2>(m-2)/2(m-1)$. Therefore, $s<H^2/(m-1)$ on the ellipsoid with $c^2>(m-2)/2(m-1)$.

The above example tells us, just as on S^n [4, p. 438], there is no stable p-currents on the ellipsoid with $c^2>(m-2)/2(m-1)$. Besides, if the condition (c) in Theorem O is canceled, then the submanifolds in Theorem O do not necessarily have to be totally umbilical.

Furthermore, we shall prove the following

THEOREM 2. *Let $\phi: M^m\to N^n$ be an isometric immersion of a compact Riemannian manifold M in the ellipsoid $N^n: x_1^2+\cdots+x_n^2+x_{n+1}^2/c^2=1$, $c\leqq 1$ and p a given integer, $p\in(0, m)$. If for any $x\in M$ and any p-subspace V of T_xM*

$$\operatorname{tr} Q^A<p(m-p)c^2\,,$$

Then there is no stable p-current in M and

$$H_p(M, Z)=H_{m-p}(M, Z)=0\,.$$

Remark. When $c=1$, Theorem 2 is due to Theorem LS.

2. Rectifiable currents

In this section we shall give a brief description of rectifiable currents (ref. [3, 4, 8]).

Let M^m be an m-dimensional compact Riemannian manifold with Riemannian metric $\langle\,,\rangle$ and Levi-Civita connection ∇. Denote by $\mathcal{H}^p$ Hausdorff p-measure on M. A subset S of M is called a p-rectifiable set if S is a countable union of disjoint p-dimensional C^1 submanifolds, up to sets of $\mathcal{H}^p$-measure zero. Consider over S an $\mathcal{H}^p$-measurable section $\xi: S\to\wedge^pTM$ with the property that for $\mathcal{H}^p$-almost all $x\in S$, ξ_x is a simple vector of unit length which represents T_xS. Such a pair (S, ξ) is called an oriented, p-rectifiable set.

The set of rectifiable p-currents is defined by

$$\mathcal{R}_p(M)=\left\{\mathfrak{S}=\sum_{n=1}^{\infty} n\mathfrak{S}_n\,;\ \mathfrak{S}_n=(S_n,\ \xi_n),\ M(\mathfrak{S})=\sum_{n=1}^{\infty} n\mathcal{H}^p(S_n)<\infty\right\}.$$

In the case that $\mathfrak{S}$ and $\partial\mathfrak{S}$ are both rectifiable currents, $\mathfrak{S}$ is called an integral p-current. The space of integral p-currents is denoted by $\mathcal{I}_p(M)$. The direct sum $\mathcal{I}_*(M)=\bigoplus_p \mathcal{I}_p(M)$ together with $\partial: \mathcal{I}_*(M)\to\mathcal{I}_*(M)$ forms a differential chain complex. For this complex there are the following results due to Federer and Fleming [3].

THEOREM FF. *For each $p\geqq 0$ there is a natural isomorphism*

$$H_p(\mathcal{I}_*(M))\cong H_p(M,\ Z)\,.$$

And for each $\alpha\in H_p(\mathcal{I}_*(M))$ there exists a current $\mathfrak{S}\in\alpha$ of "least area", that is,

$$M(\mathfrak{S})\leqq M(\mathfrak{S}')$$

for all $\mathfrak{S}'\in\alpha$.

For a smooth vector field $X\in C(TM)$, let $\phi_t: M\to M$ be the 1-parameter group of diffeomorphisms generated by X. A current $\mathfrak{S}\in\mathcal{R}_p(M)$ is said to be stable if for each vector field X there is an $\varepsilon>0$ such that

$$M(\phi_{t*}\mathfrak{S})\geqq M(\mathfrak{S})$$

for $|t|<\varepsilon$.

Lawson and Simons [4] derived the following formulae:

$$\frac{d}{dt}M(\phi_{t*}\mathfrak{S})\Big|_{t=0}=\int\langle a^X(\vec{\mathfrak{S}}),\ \vec{\mathfrak{S}}\rangle d\|\mathfrak{S}\|\,,$$

$$\text{(2.1)}\qquad \frac{d^2}{dt^2}M(\phi_{t*}\mathfrak{S})\Big|_{t=0}=\int\{-\langle a^X(\vec{\mathfrak{S}}),\ \vec{\mathfrak{S}}\rangle^2+\langle a^X a^X(\vec{\mathfrak{S}}),\ \vec{\mathfrak{S}}\rangle$$

$$+\|a^X(\vec{\mathfrak{S}})\|^2+\langle\nabla_{X,\vec{\mathfrak{S}}}X,\ \vec{\mathfrak{S}}\rangle\}\,d\|\mathfrak{S}\|\,,$$

where $a^X: \wedge^p T_xM\to\wedge^p T_xM$ is a linear map given by

$$a^X(X_1\wedge\cdots\wedge X_p)=\sum_j X_1\wedge\cdots\wedge a^X(X_j)\wedge\cdots\wedge X_p\,,$$

$$a^X(X_j)=\nabla_{X_j}X\,,$$

and $\nabla_X,\ X: \wedge^p T_xM\to\wedge^p T_xM$ is another linear map defined by

$$\nabla_{X,\,X_1\wedge\cdots\wedge X_p}X=\sum_j X_1\wedge\cdots\wedge(\nabla_{X,\,X_j}X)\wedge\cdots\wedge X_p\,,$$

$$\nabla_{X,\,X_j}X=\nabla_X\nabla_{X_j}X-\nabla_{\nabla_X X_j}X\,.$$

To any simple p-vector $\xi\in\wedge^p T_xM$ and $X\in C(TM)$, let ϕ_t be the flow generated by X, and define

$$Q_\xi(X)=\frac{d^2}{dt^2}\|\phi_{t*}\xi\|\Big|_{t=0}.$$

Then the expression (2.1) can be denoted by

$$\frac{d^2}{dt^2}M(\phi_{t*}\mathfrak{S})\Big|_{t=0}=\sum_n n\int_{S_n}Q_{\xi_n}(X)d\mathcal{H}^p(x). \tag{2.2}$$

If $X=\nabla f$ for some $f\in C^3(M)$ and $\{e_i, e_\alpha\}$ $(i=1, \cdots, p; \alpha=p+1, \cdots, m)$ is an orthonormal basis of T_xM with $\xi=e_1\wedge e_2\wedge\cdots\wedge e_p$, then we can obtain

$$Q_\xi(X)=[\sum_j\langle a^X(e_j), e_j\rangle]^2+2\sum_{j,\alpha}\langle a^X(e_j), e_\alpha\rangle^2+\sum_j\langle\nabla_{X,e_j}X, e_j\rangle. \tag{2.3}$$

3. Linear operator Q^A

For a p-rectifiable set S in M, we know that at $\mathcal{H}^p$-almost all point $x\in S$, there exists an approximate p-space $T_xS\subset T_xM$, to S. In this section we shall introduce a selfadjoint linear operator Q^A on T_xS and prove some lemmas.

Let $\phi: M^m\to N^n$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N. The Levi-Civita connections of M and N are denoted by ∇ and $\bar{\nabla}$ respectively.

For a given integer $p\in(0, m)$ let V be a p-dimensional subspace in T_xM. And for $\nu\in C(V(N, M))$, let A_ν be the shape operator determined by ν. Define a map $B_\nu: V\to V$ associated with A_ν by

$$B_\nu X=\text{orthogonal projection of } A_\nu X \text{ onto } V,$$

where $X\in V$. If $\{e_i\}$ is an orthonormal basis of V, we have

$$B_\nu X=\sum_i\langle A_\nu X, e_i\rangle e_i. \tag{3.1}$$

Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_x(N, M)$ and $A_\lambda=A_{\nu_\lambda}$. Define a selfadjoint linear map $Q^A: V\to V$ associated with the immersion ϕ by

$$Q^AX=\sum_\lambda[2(\sum_i\langle A_\lambda^2X, e_i\rangle e_i-B_\lambda^2X)-(\operatorname{tr}A_\lambda-\operatorname{tr}B_\lambda)B_\lambda X], \tag{3.2}$$

where $X\in V$ and $\{e_i\}$ is an orthonormal basis of V. Let $\{e_\alpha\}$ be an orthonormal basis of $V^\perp$ which is the orthogonal complement of V in T_xM. Then $\{e_i, e_\alpha\}$ is an orthonormal basis of T_xM and from [8] the trace of Q^A is

$$\operatorname{tr}Q^A=\sum_i\langle Q^Ae_i, e_i\rangle=\sum_\lambda[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2-(\operatorname{tr}A_\lambda-\operatorname{tr}B_\lambda)\operatorname{tr}B_\lambda]. \tag{3.3}$$

LEMMA O [5]. *Let $a_1, a_2, \cdots, a_m$ and b be $m+1$ $(m\geqq 2)$ real numbers satisfying the inequality*

$$\sum_{s=1}^m(a_s)^2+b<\frac{1}{m-1}\Big(\sum_{s=1}^m a_s\Big)^2 \quad (\textit{resp.} \leqq),$$

then $2a_s a_t > b$ (*resp.* $\geqq$) *for any* $s \neq t$.

Let $s = \sum_\lambda \operatorname{tr} A_\lambda{}^2$, $H = \sum_\lambda (\operatorname{tr} A_\lambda)\nu_\lambda$, Now we shall prove

LEMMA 1. *Let* $m \geqq 3$ *and* $p \in (0, m)$. *If* $s < H^2/(m-1)$ (*resp.* $\leqq$), *then for any* $x \in M$ *and any* p-*subspace* V *in* $T_x M$,

$$\operatorname{tr} Q^A < 0 \qquad (\textit{resp.} \ \leqq).$$

Proof. If $s < H^2/(m-1)$, because $s \geqq 0$ we see that $H \neq 0$. So we can choose an orthonormal basis $\{\nu_\lambda\}$ of $V_x(N, M)$ such that $H = (\operatorname{tr} A_1)\nu_1$. Hence $\operatorname{tr} A_\lambda = 0$ for $\lambda \geqq 2$.

Because the maps $B_1: V \to V$ and $B_1{}^\perp: V^\perp \to V^\perp$ associated with A_1 are selfadjoint linear, we can choose orthonormal basises $\{e_i\}$ of V and $\{e_\alpha\}$ of $V^\perp$ respectively such that

$$B_1 e_i = \mu_i e_i, \qquad B_1^\perp e_\alpha = \mu_\alpha e_\alpha.$$

So from (3.1) we have

$$\langle A_1 e_i, e_j \rangle = \langle B_1 e_i, e_j \rangle = \mu_i \delta_{ij}, \qquad \langle A_1 e_\alpha, e_\beta \rangle = \langle B_1^\perp e_\alpha, e_\beta \rangle = \mu_\alpha \delta_{\alpha\beta},$$

and then

$$\operatorname{tr} A_1 = \sum_i \langle A_1 e_i, e_i \rangle + \sum_\alpha \langle A_1 e_\alpha, e_\alpha \rangle = \sum_i \mu_i + \sum_\alpha \mu_\alpha,$$

$$\operatorname{tr} B_1 = \sum_i \mu_i, \qquad \operatorname{tr} A_1 - \operatorname{tr} B_1 = \sum_\alpha \mu_\alpha, \tag{3.4}$$

$$s = \sum_i \mu_i^2 + \sum_\alpha \mu_\alpha^2 + 2\sum_{i,\alpha} (A_{i\alpha}^1)^2 + \sum_{\lambda \geqq 2} \Big[\sum_{i,j} (A_{ij}^\lambda)^2 + 2\sum_{i,\alpha} (A_{i\alpha}^\lambda)^2 + \sum_{\alpha,\beta} (A_{\alpha\beta}^\lambda)^2 \Big],$$

where $A_{ij}^\lambda = \langle A_\lambda e_i, e_j \rangle$, $A_{i\alpha}^\lambda = \langle A_\lambda e_i, e_\alpha \rangle$, $A_{\alpha\beta}^\lambda = \langle A_\lambda e_\alpha, e_\beta \rangle$. Hence the condition $s < H^2/(m-1)$ becomes

$$\sum_i \mu_i^2 + \sum_\alpha \mu_\alpha^2 + b < \Big(\sum_i \mu_i + \sum_\alpha \mu_\alpha\Big)^2/(m-1), \tag{3.5}$$

where

$$b = 2\sum_{i,\alpha} (A_{i\alpha}^1)^2 + \sum_{\lambda \geqq 2} \Big[2\sum_{i,\alpha} (A_{i\alpha}^\lambda)^2 + \sum_{i,j} (A_{ij}^\lambda)^2 + \sum_{\alpha,\beta} (A_{\alpha\beta}^\lambda)^2 \Big].$$

Using Lemma O to (3.5), we get that

$$2\mu_k \mu_\gamma > b = 2\sum_{\lambda,i,\alpha} (A_{i\alpha}^\lambda)^2 + \sum_{\lambda \geqq 2} \Big[\sum_{i,j} (A_{ij}^\lambda)^2 + \sum_{\alpha,\beta} (A_{\alpha\beta}^\lambda)^2 \Big]. \tag{3.6}$$

Combining (3.3), (3.4) with $\operatorname{tr} A_\lambda = 0$ ($\lambda \geqq 2$) and (3.6) gives

$$\begin{aligned}\operatorname{tr} Q^A &= 2\sum_{i,\alpha}\langle A_1 e_i, e_\alpha\rangle^2 - \sum_{i,\alpha}\mu_i\mu_\alpha + \sum_{\lambda\geq 2}[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 + (\operatorname{tr} B_\lambda)^2]\\ &< 2\sum_{\lambda,i,\alpha}(A^\lambda_{i\alpha})^2 - p(m-p)\Big\{\sum_{\lambda,i,\alpha}(A^\lambda_{i\alpha})^2 + \frac{1}{2}\sum_{\lambda\geq 2}[\sum_{i,j}(A^\lambda_{ij})^2 + \sum_{\alpha,\beta}(A^\lambda_{\alpha\beta})^2\Big\}\\ &\quad + \sum_{\lambda\geq 2}(\sum_i\langle A_\lambda e_i, e_i\rangle)^2\\ &= [2-p(m-p)]\sum_{\lambda,i,\alpha}(A^\lambda_{i\alpha})^2 - \frac{1}{2}p(m-p)\sum_{\lambda\geq 2}[\sum_{i\neq j}(A^\lambda_{ij})^2 + \sum_{\alpha\neq\beta}(A^\lambda_{\alpha\beta})^2]\\ &\quad + \sum_{\lambda\geq 2}\Big\{(\sum_i A^\lambda_{ii})^2 - \frac{1}{2}p(m-p)[\sum_i(A^\lambda_{ii})^2 + \sum_\alpha(A^\lambda_{\alpha\alpha})^2]\Big\}.\end{aligned}$$

Because $m\geqq 3$ and $0<p<m$, $2-p(m-p)\leqq 0$. Therefore,

$$\operatorname{tr} Q^A < \sum_{\lambda\geq 2}\Big\{(\sum_i A^\lambda_{ii})^2 - \frac{1}{2}p(m-p)[\sum_i(A^\lambda_{ii})^2 + \sum_\alpha(A^\lambda_{\alpha\alpha})^2]\Big\}. \tag{3.7}$$

Noting that for $\lambda\geqq 2$, $\operatorname{tr} A_\lambda=0$, that is

$$\sum_i A^\lambda_{ii} + \sum_\alpha A^\lambda_{\alpha\alpha} = \sum_i\langle A_\lambda e_i, e_i\rangle + \sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle = 0\,,$$

we have

$$(\sum_i A^\lambda_{ii})^2 = \frac{1}{2}(\sum_i A^\lambda_{ii})^2 + \frac{1}{2}(\sum_\alpha A^\lambda_{\alpha\alpha})^2 \leqq \frac{p}{2}\sum_i(A^\lambda_{ii})^2 + \frac{m-p}{2}\sum_\alpha(A^\lambda_{\alpha\alpha})^2. \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain $\operatorname{tr} Q^A<0$.

Repeating the above, if $s\leqq H^2/(m-1)$ and $H\neq 0$ we have $\operatorname{tr} Q^A\leqq 0$. If $s\leqq H^2/(m-1)$ and $H=0$, then $A_\lambda=0$ and hence $\operatorname{tr} Q^A=0$. Q.E.D.

Let the ambient space N be of constant curvature c, and $s<H^2/(m-1)$ on M. Choose an orthonormal basis $\{\nu_\lambda\}$ of $V_x(N, M)$ such that $H=(\operatorname{tr} A_1)\nu_1$ and hence $\operatorname{tr} A_\lambda=0$ when $\lambda\geqq 2$. And choose an orthonormal basis $\{E_s\}$ of T_xM so that $A_1E_s=\lambda_sE_s$ $(s=1, \cdots, m)$. Then the condition $s<H^2/(m-1)$ gives

$$\sum_s\lambda_s{}^2 + \sum_{\lambda\geq 2}\sum_{s,t}\langle A_\lambda E_s, E_t\rangle^2 < (\sum_s\lambda_s)^2/(m-1)\,.$$

From Lemma O, for any $s\neq t$ we have

$$2\lambda_s\lambda_t > \sum_{\lambda\geq 2}\sum_{q,r}\langle A_\lambda E_q, E_r\rangle^2\,. \tag{3.9}$$

Let v be a unit vector in T_xM and $m\geqq 3$. Applying (3.9) and the equation of Gauss we can obtain

$$Ric\,(v, v) > (m-1)c\,.$$

If $s\leqq H^2/(m-1)$, we can get that $Ric\,(v, v)\geqq(m-1)c$. Hence from Myers' theorem (ref. [1, p. 28]) we have

LEMMA 2. *Let M^m be a submanifold immersed in a Riemannian manifold of constant curvature c and $m\geqq 3$. If M is compact, $c\geqq 0$ and $s<H^2/(m-1)$ on M, then the fundamental group of M is finite. If M is complete, $c>0$ and $s\leqq H^2/(m-1)$ on M, then the fundamental group of M is finite and M is compact.*

Now assume $\psi: N^n\to E^l$ is an isometric immersion of the Riemannian manifold N in the Euclidean space E^l. Let D be the Levi-Civita connection on E^l. Associated with the isometric immersion $x=\psi\circ\phi: M^m\to E^l$, the shape operator A'_ν determined by $\nu\in C(V(E^l, M))$ is given by

$$A'_\nu Y=-(D_Y\nu)^T ,$$

where $Y\in C(TM)$. Especially, if $\nu\in C(V(N, M))$,

$$A'_\nu Y=-(D_Y\nu)^T=-[\bar{\nabla}_Y\nu+\bar{h}(\nu, Y)]^T=-(-A_\nu Y+\nabla^\perp_Y\nu)^T=A_\nu Y , \tag{3.10}$$

where $\bar{h}$ is the second fundamental form of the immersion ψ. And if $\nu\in C(V(E^l, N))$,

$$A'_\nu Y=(\bar{A}_\nu Y)^T . \tag{3.11}$$

Let S be a p-rectifiable set. At $x\in S$, associate a tangent p-space $V=T_xS\subset T_xM$. Choose an orthonormal basis $\{e_i, e_\alpha\}$ of T_xM such that $\{e_i\}$ is a basis of V and $\xi=e_1\wedge\cdots\wedge e_p$. Let $Q^{A'}$ be the selfadjoint linear operator on V associated with the immersion $\psi\circ\phi: M^m\to E^l$ defined by (3.2). At $x\in M$ let $\{\nu_\sigma\}$ be an orthonormal basis of $V_x(E^l, M)$ and $A'_\sigma=A'_{\nu_\sigma}$. Then there is the following relation between $Q^{A'}$ and Q_ξ given by (2.3) from [8]

LEMMA 3. $\operatorname{tr} Q_\xi=\operatorname{tr} Q^{A'}$, *where*

$$\operatorname{tr} Q^{A'}=\sum_\sigma [2\sum_{i,\alpha}\langle A'_\sigma e_i, e_\alpha\rangle^2-(\operatorname{tr} A'_\sigma-\operatorname{tr} B'_\sigma)\operatorname{tr} B'_\sigma] . \tag{3.12}$$

At a point $x\in M$, we take an orthonormal basis $\{\nu_\lambda, \eta_a\}$ of $V_x(E^l, M)$ so that $\{\nu_\lambda\}$ and $\{\eta_a\}$ are bases of $V_x(N, M)$ and $V_x(E^l, N)$ respectively. From (3.10) and (3.11) we obtain

$$\operatorname{tr} Q^{A'}=\operatorname{tr} Q^A+\bar{A}(V) , \tag{3.13}$$

where $\operatorname{tr} Q^A$ is given by (3.3) and

$$\bar{A}(V)=\sum_{a,i,\alpha} [2\langle\bar{A}_a e_i, e_\alpha\rangle^2-\langle\bar{A}_a e_\alpha, e_\alpha\rangle\langle\bar{A}_a e_i, e_i\rangle] . \tag{3.14}$$

4. Proof of Theorem 1

Let $\theta=\{\nabla f\,;\, f: E^{n+1}\to R$ is linear$\}$ and $\mathfrak{S}\in\mathfrak{R}_p(M)$. For $X\in\theta$, let ϕ_t be the flow generated by X and set

(4.1) $$Q_{\mathfrak{S}}(X)=\frac{d^2}{dt^2}M(\phi_{t*}\mathfrak{S})\Big|_{t=0}.$$

Then from (2.2) and (2.3), $Q_{\mathfrak{S}}$ can be considered as a quadratic form on θ and

(4.2) $$\operatorname{tr} Q_{\mathfrak{S}}=\sum_n n\int_{S_n}\operatorname{tr} Q_{\xi_n} d\mathcal{H}^p(X).$$

According to the assumption in Theorem 1, N can be considered as a totally umbilical hypersurface of E^{n+1} (ref. [1, p. 41]). In this case, (3.14) becomes

$$\bar{A}(V)=-p(m-p)c.$$

Thus from (3.13) we obtain

$$\operatorname{tr} Q^{A'}=\operatorname{tr} Q^A-p(m-p)c.$$

From Lemma 3 and Lemma 1, the condition C1 or C2 in Theorem 1 gives that $\operatorname{tr} Q_{\xi_n}<0$ for any n. Therefore $\operatorname{tr} Q_{\mathfrak{S}}<0$. This implies that there is no stable p-current in M for $p=1, \cdots, m-1$. By using Theorem FF we have $H_p(M, Z)=0$ $(p=1, \cdots, m-1)$. The proof of the conclusion i) is completed.

As for ii), from i) we have $H_1(M, Z)=\cdots=H_{m-1}(M, Z)=0$ and so M is a homology sphere. From Lemma 2, M and its universal covering space $\tilde{M}$ are compact. So $\tilde{M}$ is also a homology sphere and from the Hurewicz isomorphism theorem $\tilde{M}$ is $(m-1)$-connected, and thus it is a homotopy sphere. By the generalized Poincare conjecture, we know that $\tilde{M}$ is homeomorphic to a sphere. Now the homology sphere M is covered by a sphere $\tilde{M}$ and hence by a theorem of D. Sjerve [6] we have $\pi_1(M)=0$. Using Hurewicz's theorem and the generalized Poincaré conjecture again we get that M is homeomorphic to a sphere.

5. Proof of Theorem 2

Let $\{e_i, e_\alpha\}$ be an orthonormal basis of T_xM so that $\{e_i\}$ is a basis of the p-subspace V. Denote by $\bar{A}$ the shape operator of the ellipsoid $N^n\to E^{n+1}$. Then for any $X\in C(TM)$, from (1.1)

$$\bar{A}X=aX+b\langle X, t\rangle t,$$

where a, b and t are given by (1.2). Thus

$$\langle \bar{A}e_i, e_\alpha\rangle=b\langle e_i, t\rangle\langle e_\alpha, t\rangle,$$

$$\langle \bar{A}e_i, e_i\rangle=a+b\langle e_i, t\rangle^2, \qquad \langle \bar{A}e_\alpha, e_\alpha\rangle=a+b\langle e_\alpha, t\rangle^2.$$

Substituting these into (3.14) we get

(5.1) $$\bar{A}(V)=\sum_{i,\alpha}[b^2\langle e_i, t\rangle^2\langle e_\alpha, t\rangle^2-ab(\langle e_i, t\rangle^2+\langle e_\alpha, t\rangle^2)]-p(m-p)a^2.$$

For each pair of fixed indices i, α, let

$$f_{i\alpha}=b^2\langle e_i, t\rangle^2\langle e_\alpha, t\rangle^2-ab(\langle e_i, t\rangle^2+\langle e_\alpha, t\rangle^2)\,.$$

If $c=1$, then $a=1$, $b=0$ from (1.2) and hence $f_{i\alpha}=0$. If $c<1$, then $a>0$, $b>0$. In this case, $f_{i\alpha}\leqq 0$. In fact, let

$$\langle e_i, t\rangle=e_{it}\,, \qquad \langle e_\alpha, t\rangle=e_{\alpha t}\,.$$

Then

$$f_{i\alpha}=b^2e_{it}{}^2e_{\alpha t}{}^2-ab(e_{it}{}^2+e_{\alpha t}{}^2)\,, \tag{5.2}$$

where

$$0\leqq e_{it}{}^2\leqq 1\,, \qquad 0\leqq e_{\alpha t}{}^2\leqq 1\,.$$

Partially differentiating (5.2) with respect to each variable and equating to zero, we obtain

$$2b^2e_{it}e_{\alpha t}{}^2-2abe_{it}=0\,, \qquad 2b^2e_{it}{}^2e_{\alpha t}-2abe_{\alpha t}=0\,.$$

If $e_{it}=0$ or $e_{\alpha t}=0$, then $f_{i\alpha}=-ab(e_{\alpha t}{}^2$ or $e_{it}{}^2)\leqq 0$. If $e_{it}\neq 0$ and $e_{\alpha t}\neq 0$, we have $e_{\alpha t}{}^2=e_{it}{}^2=a/b$. And hence $f_{i\alpha}=-a^2<0$. Note that $e_{it}{}^2=1$ and $e_{\alpha t}{}^2=1$ can not hold simultaneously because $\langle e_i, e_\alpha\rangle=0$. Thus $f_{i\alpha}\leqq 0$ when $c<1$.

Since $f_{i\alpha}\leqq 0$, from (3.13) and (5.1) we have

$$\operatorname{tr} Q^{A'}\leqq \operatorname{tr} Q^A-p(m-p)a^2\,.$$

Because $c^2\leqq a^2\leqq 1$ for $c\leqq 1$, $\operatorname{tr} Q^{A'}<0$ when $\operatorname{tr} Q^A<p(m-p)c^2$. Therefore, from Lemma 3 the trace of the quadratic form $Q_{\mathfrak{S}}$ defined by (4.1) is less than zero when $\operatorname{tr} Q^A<p(m-p)c^2$. This means that there is no stable p-current. The proof is completed.

If the immersion $\phi: M\to N^n$ is minimal, then $\operatorname{tr} A_\lambda=0$, from (3.3)

$$\begin{aligned}\operatorname{tr} Q^A&=\sum_\lambda\,[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2+(\operatorname{tr} B_\lambda)^2]\\ &=\sum_\lambda\,[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2+(\sum_i\langle A_\lambda e_i, e_i\rangle)^2]\,.\end{aligned}$$

But

$$\begin{aligned}(\sum_i\langle A_\lambda e_i, e_i\rangle)^2&=\frac{1}{2}(\sum_i\langle A_\lambda e_i, e_i\rangle)^2+\frac{1}{2}(\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle)^2\\ &\leqq\frac{p}{2}\sum_i\langle A_\lambda e_i, e_i\rangle^2+\frac{m-p}{2}\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle^2\,,\end{aligned}$$

because $\sum_i\langle A_\lambda e_i, e_i\rangle+\sum\langle A_\lambda e_\alpha, e_\alpha\rangle=\operatorname{tr} A_\lambda=0$. Thus

$$\begin{aligned}\operatorname{tr} Q^A&\leqq\sum_\lambda\Big[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2+\frac{p}{2}\sum_i\langle A_\lambda e_i, e_i\rangle^2+\frac{m-p}{2}\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle^2\Big]\\ &\leqq\frac{1}{2}\max\{p, m-p\}s\,,\end{aligned}$$

where $s=\mathrm{tr}\,A_\lambda{}^2$.

COROLLARY. *Let M^m be a compact minimal submanifold immersed in the ellipsoid with $c\leqq 1$ and $p\in(0, m)$. If the square length of the second fundamental form of M satisfies $s<2\min\{p, m-p\}c^2$, then*

$$H_p(M, Z)=H_{m-p}(M, Z)=0\,.$$

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