THE LOGARITHMIC DERIVATIVE AND A HOMOGENEOUS DIFFERENTIAL POLYNOMIAL OF A MEROMORPHIC FUNCTION

BY KAZUYA TOHGE

1. Introduction

In this note, by a meromorphic function we mean a function meromorphic in the complex plane \( \mathbb{C} \). We shall here assume that the reader is familiar with the standard notation and terminology of value distribution theory (see for example, Hayman [1] or [3]). For a meromorphic function \( g(z) \), which does not vanish identically, we can consider the logarithmic derivative \( g'(z)/g(z) \). It plays an important role in Nevanlinna’s theory of meromorphic functions. The following occupies the main part.

**Lemma.** Let \( g(z) \) be a meromorphic function. If \( g(z) \) is transcendental, we have

\[
m(r, \frac{g'}{g}) = O(\log^* T(r, g) + \log r)
\]

as \( r \to \infty \) through all values if \( g(z) \) has finite order and as \( r \to \infty \) outside a set of \( r \) of finite linear measure otherwise. If \( g(z) \) is a rational function and not identically equal to zero,

\[
m(r, \frac{g'}{g}) = o(1)
\]

as \( r \to \infty \) through all values.

For the sake of simplicity, we shall use the symbol “n.e. (nearly everywhere)” instead of tediously saying that possibly outside a set of \( r \) of finite linear measure.

W. K. Hayman pointed out the necessity of treating a homogeneous differential polynomial \( g^r g - 2g^r \) of an entire function \( g(z) \) in his famous book [1; §3.6, p. 77]. Concerning this proposal E. Mues [4] studied an influence of the zeros of \( g^r g - a g^r \) with a complex number \( a \) on the entire function \( g(z) \) itself.

Received April 28, 1992.
He proved that if $g''g - ag'^2$ has no zero, then $g(z) = \exp(az + \beta)$ are the only transcendental functions with this property if $a \neq 1$. This result settled a question of Hayman made in [1]. Our purpose of this note is to give an estimate of the zeros of $g(z)$ by those of the homogeneous differential polynomial $g''g - ag'^2$. With the equation

$$T(r, \frac{g'}{g}) = m(r, \frac{g'}{g}) + N(r, 0, g) + N(r, g)$$

for a meromorphic function $g(z) \neq 0$ and the above lemma, it gains our purpose to estimate the characteristic function of a logarithmic derivative by the counting function with respect to the zeros of a homogeneous differential polynomial. Our method to obtain such a result is based on arriving at a homogeneous linear equation in $g'$ and $g$ after a linearization of $g''g - ag'^2$. The term “linearization” was introduced by M. Ozawa [5], and Mues [4], Ozawa, G. Frank and others have made frequent use of this method. We now represent the differential polynomial $g''g - ag'^2$ by means of a Wronskian determinant

$$W(f_1, f_2) = f_1f_2' - f_1'f_2.$$

We have indeed for a constant $a \in \mathbb{C}$

$$W((a-1)zg'(z) + g(z), g'(z)) = g''(z)g(z) - ag'(z)^2,$$

which we denote by $W_a(z)$. That is a reason why we can treat this homogeneous differential polynomial $W_a(z)$.

We shall naturally consider only the case where $W_a(z) \neq 0$ and the above lemma, because if $W_a(z) \equiv 0$, two functions $(a-1)zg'(z) + g(z)$ and $g'(z)$ are linearly dependent over $\mathbb{C}$. Then there exist two constants $C_1$ and $C_2$, at least one of which is different from zero, such that an equation

$$C_1(a-1)z + C_2g(z) + C_1g(z) = 0$$

holds. If $C_1(a-1)z + C_2 = 0$, we have $C_2 = 0$ and $a = 1$. By (1.4) it thus follows $g(z) \equiv 0$, which is a contradiction. Hence unless $g(z) \equiv 0$, it is equal to $\exp(-C_1z/C_2 + \text{const.})$ if $a = 1$ and thus $C_2 \neq 0$, and $g(z) = C \exp(C_1(a-1)z + C_2)^{-1/(a-1)}$, $C_2 \equiv -C - \{0\}$, if $a \neq 1$. If $C_1 \neq 0$ in the latter case, the exponent $-1/(a-1)$ must be an integer $m$ ($\neq 0$), say. The following is a summary of this trivial observation:

The meromorphic functions $g(z)$ with the property $W_a(z) \equiv 0$ are reduced to the next three: for $a \neq 0$, $\beta \in \mathbb{C}$,

1°. $g(z) \equiv \beta$, when $a$ is any complex number;
2°. $g(z) = \exp(\alpha z + \beta)$, when $a = 1$;
3°. $g(z) = (\alpha z + \beta)^m$, when $a = (m-1)/m$ with a non-zero integer $m$. 

The meromorphic functions $g(z)$ with the property $W_a(z) \equiv 0$ are reduced to the next three: for $a \neq 0$, $\beta \in \mathbb{C}$,
2. Results

We shall prove the following theorem which gives a desired estimate of the logarithmic derivative \( g'(z)/g(z) \).

**Theorem.** Let \( g(z) \) be a non-constant meromorphic function and define a homogeneous differential polynomial \( W_a(z) \) in \( g(z) \) for a complex number \( a \) by (1.3). If \( W_a(z) \) does not vanish identically, then an inequality

\[
T(r, \frac{g'}{g}) \leq A_a m(r, \frac{g'}{g}) + B_a m(r, \frac{W_a'}{W_a}) + C_a \{N(r, 0, W_a) + N(r, g)\} + U_a(r)
\]

holds as \( r \to \infty \), except for two cases (i), (ii) below. Here the constants \( A_a, B_a, C_a \) depend only on the number \( a \) and satisfy

\[
0 \leq A_a \leq \begin{cases} 4, & \text{if } a \neq 1, 1/2, \\ 2, & \text{if } a = 1, \\ 1, & \text{if } a = 1/2, \end{cases} \quad 0 \leq B_a \leq \begin{cases} 5, & \text{if } a \neq 1, 1/2, 0, \\ 4, & \text{if } a = 1/2, \\ 1, & \text{if } a = 0, \end{cases} \quad 0 \leq C_a \leq 5
\]

for any \( a \), and also \( U_a(r) \) is a real-valued function on \([0, \infty)\) such that if we fix the number \( a \), then it satisfies

\[
U_a(r) = \begin{cases} O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W_a'}{W_a}\right) + \log^+ \{N(r, 0, W_a) + N(r, g)\} + \log r\right], & \text{if } a \neq 1/2, 0, \\ O(1), & \text{if } a = 1/2, \end{cases}
\]

as \( r \to \infty \) possibly outside a set \( E_a \) of \( r \) of finite linear measure depending on the number \( a \).

(i) When \( a = 1/2 \), \( g(z) = az^{1/2} + \beta z + \gamma \), where \( \alpha, \beta, \gamma \) are complex constants with \( \beta^2 - 4\alpha \gamma \neq 0 \); and

(ii) when \( a = 1 \), \( g(z) = C_1 e^{i\lambda_1 z} + C_2 e^{i\lambda_2 z} \), where \( \lambda_1, \lambda_2, C_1, C_2 \) are complex constants with \( \lambda_1 \neq \lambda_2 \) and \( C_1, C_2 \neq 0 \);

are the exceptions as mentioned above.

**Remark.** It is easy to see that \( g(z) \) as in the cases (i) and (ii) indeed fails to satisfy the inequality (2.1). In fact:

(i). \( g(z) = az^2 + \beta z + \gamma \) gives \( W_{1/2}(z) = -(1/2)(\beta^2 - 4\alpha \gamma) \neq 0 \). Then we deduce

\[
m(r, g'/g) = O(1),
\]

\[
m(r, W_{1/2}'/W_{1/2}) = N(r, 0, W_{1/2}) = N(r, g) = 0,
\]

and

\[
U_{1/2}(r) = O(1)
\]
as \( r \to \infty \), while

\[
T(r, \frac{g'}{g}) = m(r, \frac{g'}{g}) + N(r, \frac{g'}{g}) = \varepsilon \log r + o(1)
\]

as \( r \to \infty \), with \( \varepsilon = 1 \) if \( \alpha = 0 \) and \( \varepsilon = 2 \) if \( \alpha \neq 0 \).

(ii). In this case, \( W_i(z) = (\lambda_i - \lambda_0) C_i C_2 e^{(\lambda_i - \lambda_0) z} \), and that \( g(z) \) is an entire function of order 1. Thus we have

\[
m(r, \frac{g'}{g}) = O(\log r),
\]

\[
m(r, \frac{W_i'}{W_i}) = O(\log r),
\]

\[
N(r, 0, W_i) = N(r, g) = 0,
\]

and

\[
U_i(r) = O(\log r) \quad \text{as } r \to \infty.
\]

Using an expression \( g(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} \), we are led to

\[
T(r, \frac{g'}{g}) = N(r, 0, g) + O(\log r) = \frac{1}{\pi} |\lambda_1 - \lambda_2| r + O(\log r)
\]

as \( r \to \infty \) (see for example, Hayman [1: p. 7]), however.

Our way to prove this theorem also applies to the following

**Corollary.** Besides the hypothesis of our theorem we assume that \( g(z) \) is an entire function and that as \( r \to \infty \), n.e.,

\[
(2.2) \quad m(r, W_a) = o\{m(r, g)\}.
\]

Then \( W_a(z) \) must be a constant \((\neq 0)\) and \( g(z) \) is at least one of the following:

(i) when \( a = 1/2 \), \( g(z) = \alpha z^2 + \beta z + \gamma \), where \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \beta^2 - 4\alpha \gamma \neq 0 \);

(ii) when \( a = 1 \), \( g(z) = C_1 e^{\lambda_1 z} + C_2 e^{-\lambda_2 z} \), where \( \lambda, C_1, C_2 \in \mathbb{C} \) with \( \lambda \neq 0 \); and

(iii) when \( a \neq 0, 1/2 \), \( g(z) = \alpha z + \beta \), where \( \alpha \neq 0, \beta \in \mathbb{C} \).

**Remarks.** If we further suppose that \( g(z) \) is of finite order \( \rho \) and \( W_a(z) \) has the order \( \lambda \) satisfying \( \lambda < \rho \), the case (ii) is the only possible one and then \( \rho = 1 \) and \( \lambda = 0 \) (in particular, \( W_i(z) \) is a constant). We may regard it as a partial answer to a problem of A. Edrei (see Hayman [2: Problem 2.25]) when \( f = g' \) there.

2°. Replacing the condition (2.2) by

\[
T(r, W_a) = o\{T(r, g)\} \quad \text{as } r \to \infty, \text{ n.e.},
\]

we can prove this result for meromorphic functions with the property such that

\[
N(r, g) = o\{T(r, g)\} \quad \text{as } r \to \infty, \text{ n.e.}.
\]
3. Proof of Theorem: A Preparation

Because of $W_a(z) \neq 0$, we can consider a second order linear ordinary differential equation

\[ w'' + G_a(z)w' + H_a(z)w = 0 \]  

with the coefficients

\[ G_a(z) = -\frac{W'_a(z)}{W_a(z)}, \]

and

\[ H_a(z) = \frac{W_a[(a - l)zg(z) + g(z)]'}{W_a(z)}. \]

Since (3.1) is written as $(W[w, (a - l)zg'(z) + g(z)]'/W_a)=0$, two functions $(a - l)zg'(z) + g(z)$ and $g'(z)$ form a fundamental system of this equation. Firstly for $w = (a - l)zg'(z) + g(z)$ Equation (3.1) gives

\[ (a-l)z\{g''(z)+G_a(z)g''(z)+H_a(z)g'(z)\} \]
\[ +(2a-l)g''(z)+aG_a(z)g'(z)+H_a(z)g(z)=0. \]

Also for $w = g'(z)$,

\[ g'''(z)+G_a(z)g''(z)+H_a(z)g'(z)=0. \]

Together with (3.4) the first equation is reduced to

\[ (2a-1)g''(z)+aG_a(z)g'(z)+H_a(z)g(z)=0. \]

The two, (3.4) and (3.5), are called a linearization of the differential polynomial $W_a(z)$. Eliminating $g''(z)$ and $g'''(z)$ from them we shall obtain an equation in $g'(z)$ and $g(z)$. To do this, we differentiate both sides of (3.5) with respect to $z$ and get

\[ (2a-1)g''(z)+aG_a(z)g'(z)+H_a(z)g(z)=0. \]

Using (3.5) and (3.6) we reduce Equation (3.4) to

\[ \{a(2a-1)G_a'(z)+a(a-1)G_a(z)\} - 2(a-1)(2a-1)H_a(z)g'(z) \]
\[ = - \{(2a-1)H_a'(z)+(a-1)G_a(z)H_a(z)\} g(z), \]

which is the homogeneous linear equation in $g'(z)$ and $g(z)$ as desired. For the sake of simplicity, we denote the coefficients by $\phi_a(z)$ and $\phi_a(z)$, i.e.,

\[ \phi_a(z) := a(2a-1)G_a'(z)+a(a-1)G_a(z)^2-2(a-1)(2a-1)H_a(z), \]
(3.9) \( \phi_a(z) := -(2a-1)H'_a(z)-(a-1)G_a(z)H_a(z). \)

Then the above equation (3.7) is of the form

(3.10) \( \phi_a \cdot g' = \phi_a \cdot g. \)

Now it makes all the difference in methods whether or not \( \phi_a \equiv 0. \)

4. Proof of Theorem: Case I where \( \phi_a \not\equiv 0 \)

Since \( g \not\equiv 0, \) Equation (3.10) gives an expression of the logarithmic derivative, i.e.,

(4.1) \( \frac{g'}{g} = \frac{\phi_a}{\phi_a}. \)

We shall now distinguish the cases about the value of \( a \) to study the value distribution of meromorphic functions \( G_a, H_a, \phi_a, \) and \( \phi_a. \)

Subcase i. \( a \) is different from \( 0, 1/2, 1. \)

Using Equation (3.5) we have

(4.2) \( H_a = -(2a-1)\frac{g'}{g} - aG_a \frac{g'}{g} \)

\( = -(2a-1) \left\{ \left( \frac{g'}{g} \right)' + \left( \frac{g'}{g} \right)^2 \right\} - aG_a \frac{g'}{g}. \)

Thus we apply Lemma to (4.2) and obtain

(4.3) \( m(r, G_a) = m(r, \frac{W'_a}{W_a}), \)

and

(4.4) \( m(r, H_a) = m(r, \frac{g'}{g} \{ -(2a-1) \frac{(g'/g)'}{(g'/g)} - (2a-1) \frac{g'}{g} - aG_a \} \) \)

\( \leq 2m(r, \frac{g'}{g}) + m(r, G_a) + m(r, \frac{(g'/g)'}{(g'/g)}) + O(1) \)

\( \leq 2m(r, \frac{g'}{g}) + m(r, \frac{W'_a}{W_a}) + O \left\{ \log^* T(r, \frac{g'}{g}) + \log r \right\}, \)

as \( r \to \infty, \) n.e.. Since the poles of \( W_a \) occur possibly at those of \( g, \) we have

\( \bar{N}(r, W_a) \leq \bar{N}(r, g), \)

and thus

(4.5) \( N(r, G_a) = N\left(r, \frac{W'_a}{W_a}\right) = \bar{N}(r, W_a) + \bar{N}(r, 0, W_a) \)

\( \leq \bar{N}(r, 0, W_a) + \bar{N}(r, g). \)
Whenever $H_a$ as well as $G_a$ has a pole, $W_a$ has a zero or $g$ has a pole. Its multiplicity is at most two as we see from (4.2). Thus

\begin{equation}
N(r, H_a) \leq 2 \{N(r, 0, W_a) + N(r, g)\}.
\end{equation}

If $G_a$ does not vanish identically, an application of Lemma to (3.8) implies

\begin{equation}
m(r, \phi_a) \leq m\left(r, aG_a\left(2a-1\frac{G_a'}{G_a}+(a-1)G_a\right)\right) + m(r, H_a) + O(1)
\end{equation}

\begin{equation}
\leq 2m(r, G_a) + m\left(r, \frac{G_a'}{G_a}\right) + m(r, H_a) + O(1)
\end{equation}

\begin{equation}
\leq 2m(r, G_a) + m(r, H_a) + O\{\log^{+}T(r, G_a) + \log r\}
\end{equation}

as $r \to \infty$, n.e.. This is also valid when $G_a \equiv 0$ so that $\phi_a = -2(a-1)(2a-1)H_a$. We see that $H_a$ is not constantly equal to zero. In fact otherwise, $\phi_a \equiv 0$ by (3.9) and therefore $g'(z) \equiv 0$ by (4.1), which is a contradiction. It follows from (3.9)

\begin{equation}
m(r, \phi_a) = m\left(r, -H_a\left(2a-1\frac{H_a'}{H_a}+(a-1)G_a\right)\right)
\end{equation}

\begin{equation}
\leq m(r, H_a) + m(r, G_a) + O\{\log^{+}T(r, H_a) + \log r\}
\end{equation}

as $r \to \infty$, n.e.. We can observe the poles of $\phi_a$ and $\psi_a$ similarly to those of $G_a$ and $H_a$ as in (4.5) and (4.6), respectively, i.e.,

\begin{equation}
N(r, \phi_a) \leq 2 \{N(r, 0, W_a) + N(r, g)\},
\end{equation}

and

\begin{equation}
N(r, \phi_a) \leq 3 \{N(r, 0, W_a) + N(r, g)\}.
\end{equation}

From the estimates (4.7), (4.8), (4.9) and (4.10) we arrive at

\begin{equation}
T(r, \phi_a) \leq 2m(r, G_a) + m(r, H_a) + 2 \{N(r, 0, W_a) + N(r, g)\}
\end{equation}

\begin{equation}
+ O\{\log^{+}T(r, G_a) + \log r\}
\end{equation}

and

\begin{equation}
T(r, \phi_a) \leq m(r, G_a) + m(r, H_a) + 3 \{N(r, 0, W_a) + N(r, g)\}
\end{equation}

\begin{equation}
+ O\{\log^{+}T(r, H_a) + \log r\}
\end{equation}

as $r \to \infty$, n.e.. Also from (4.3), (4.4), (4.5) and (4.6),

\begin{equation}
T(r, G_a) \leq m(r, W_a'/W_a) + N(r, 0, W_a) + N(r, g),
\end{equation}

and

\begin{equation}
T(r, H_a) \leq 2m(r, g'/g) + m(r, W_a'/W_a) + 2 \{N(r, 0, W_a) + N(r, g)\}
\end{equation}

\begin{equation}
+ O\{\log^{+}T(r, g'/g) + \log r\}
\end{equation}
as \( r \to \infty, \text{ n.e.} \). Combining them with the former two, we obtain the following two estimates:

\[
T(r, \phi_a) \leq 2m(r, \frac{g'}{g}) + 3m(r, \frac{W_{a'}}{W_a}) + 2\{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} + O\left[\log^+ T(r, \frac{g'}{g}) + \log^+ m(r, \frac{W_{a'}}{W_a}) + \log^+ \{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} \right]
\]

and

\[
T(r, \phi_a) \leq 2m(r, \frac{g'}{g}) + 2m(r, \frac{W_{a'}}{W_a}) + 3\{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} + O\left[\log^+ T(r, \frac{g'}{g}) + \log^+ m(r, \frac{W_{a'}}{W_a}) + \log^+ \{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} \right]
\]

as \( r \to \infty, \text{ n.e.} \). By virtue of (4.1) the characteristic function of \( g'/g \) is now given by

\[
T\left(r, \frac{g'}{g}\right) \leq T(r, \phi_a) + T(r, \phi_a) + O(1).\]

Hence from (4.11) and (4.12) we conclude that

\[
T\left(r, \frac{g'}{g}\right) \leq 4m\left(r, \frac{g'}{g}\right) + 5m\left(r, \frac{W_{a'}}{W_a}\right) + 5\{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} + O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W_{a'}}{W_a}\right) + \log^+ \{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} \right]
\]

as \( r \to \infty, \text{ n.e.} \), which is the inequality as claimed.

Subcase ii. \( a = 0 \).

Then (3.8) becomes \( \phi_a = -2H_0 \). Refining (4.4) in this case we have

\[
m(r, H_0) = m\left(r, \frac{g'}{g} \left\{ \frac{(g'/g)' + g'}{g} \right\} \right) \leq 2m\left(r, \frac{g'}{g}\right) + O\{\log^+ T\left(r, \frac{g'}{g}\right) + \log r\}
\]

as \( r \to \infty, \text{ n.e.} \). Therefore this together with (4.6) leads to an estimate

\[
T(r, \phi_a) \leq 2m\left(r, \frac{g'}{g}\right) + 2\{\mathcal{N}(r, 0, W_a) + \mathcal{N}(r, g)\} + O\{\log^+ T\left(r, \frac{g'}{g}\right) + \log r\}
\]
as $r \to \infty$, n.e. On the other hand (3.9) becomes $\phi = H_0' + G_0 H_0$. It gives
\[ m(r, \phi) \leq m(r, H_0) + m(r, G_0) + m\left(r, \frac{H_0'}{H_0}\right) + O(1) \]
\[ \leq 2m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_0'}{W_0}\right) + O\left[ \log^* T\left(r, \frac{g'}{g}\right) \right] \]
\[ + \log^* \{ \overline{N}(r, 0, W_0) + \overline{N}(r, g) \} + \log r \],

as $r \to \infty$, n.e. by (4.3) and (4.13), because of $H_0 \not\equiv 0$. Estimation of (4.10) is now valid as well, so we have
\[ T(r, \phi) \leq 2m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_0'}{W_0}\right) + 3 \{ \overline{N}(r, 0, W_0) + \overline{N}(r, g) \} \]
\[ + O\left[ \log^* T\left(r, \frac{g'}{g}\right) + \log^* \{ \overline{N}(r, 0, W_0) + \overline{N}(r, g) \} + \log r \right] \]
as $r \to \infty$, n.e.. Hence we can estimate the logarithmic derivative in terms of $W_0$ by an inequality
\[ T\left(r, \frac{g'}{g}\right) \leq 4m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_0'}{W_0}\right) + 5 \{ \overline{N}(r, 0, W_0) + \overline{N}(r, g) \} \]
\[ + O\left[ \log^* T\left(r, \frac{g'}{g}\right) + \log^* \{ \overline{N}(r, 0, W_0) + \overline{N}(r, g) \} + \log r \right] \]
as $r \to \infty$, n.e..

Subcase iii. $a = 1/2$. 
Then (3.8) becomes $\phi = - (1/2)G_1/2$. Since we have
\[ m(r, \phi_{1/2}) \leq 2m(r, G_{1/2}) = 2m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right) \]
and
\[ N(r, \phi_{1/2}) = 2N(r, G_{1/2}) \leq 2 \{ \overline{N}(r, 0, W_{1/2}) + \overline{N}(r, g) \} \]
by (4.3) and (4.5), it follows
\[ T(r, \phi_{1/2}) \leq 2m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right) + 2 \{ \overline{N}(r, 0, W_{1/2}) + \overline{N}(r, g) \}. \]

Similarly (3.9) becomes $\phi = (1/2)G_{1/2}H_{1/2}$. Reconsidering (4.4) as $a = 1/2$ we refine it by
\[ m(r, H_{1/2}) \leq m\left(r, \frac{g'}{g}\right) + m(r, G_{1/2}) \]
\[ = m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right). \]
By this and (4.3),
\[ m(r, \varphi_{1/3}) \leq m(r, G_{1/3}) + m(r, H_{1/3}) \leq m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W_{1/3}'}{W_{1/3}}\right), \]
which together with (4.10) gives
\[ T(r, \varphi_{1/3}) \leq m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W_{1/3}'}{W_{1/3}}\right) + 3 \{ \overline{N}(r, 0, W_{1/2}) + \overline{N}(r, g) \}. \]

Hence we are led to
\[ T\left(r, \frac{g'}{g}\right) \leq T(r, \varphi_{1/3}) + T(r, \varphi_{1/3}) + O(1) \]
\[ \leq m\left(r, \frac{g'}{g}\right) + 4m\left(r, \frac{W_{1/3}'}{W_{1/3}}\right) + 5 \{ \overline{N}(r, 0, W_{1/2}) + \overline{N}(r, g) \} + O(1). \]

**Subcase iv.** \( a = 1. \)

In this case Estimates (3.8) and (3.9) become \( \varphi_{1} = G_{1}' \) and \( \varphi_{1} = -H_{1}' \), respectively. Because of \( G_{1}' \neq 0 \) and \( H_{1}' \neq 0 \) we deduce that
\[ m(r, \varphi_{1}) \leq m(r, G_{1}) + O\left[ \log^{+}T(r, G_{1}) + \log r \right], \]
\[ m(r, \varphi_{1}) \leq m(r, H_{1}) + O\left[ \log^{+}T(r, H_{1}) + \log r \right], \]
as \( r \to \infty \), n.e., and
\[ N(r, \varphi_{1}) = N(r, G_{1}) + \overline{N}(r, G_{1}) = 2 \overline{N}(r, G_{1}), \]
\[ N(r, \varphi_{1}) = N(r, H_{1}) + \overline{N}(r, H_{1}) = 3 \overline{N}(r, H_{1}). \]

Using (4.3), (4.4), (4.5) and (4.6) we get
\[ m(r, \varphi_{1}) \leq m\left(r, \frac{W_{1}'}{W_{1}}\right) + O\left[ \log^{+}m\left(r, \frac{W_{1}'}{W_{1}}\right) + \log^{+} \{ \overline{N}(r, 0, W_{1}) + \overline{N}(r, g) \} + \log r \right], \]
\[ m(r, \varphi_{1}) \leq 2m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_{1}'}{W_{1}}\right) + O\left[ \log^{+}T\left(r, \frac{g'}{g}\right) + \log^{+}m\left(r, \frac{W_{1}'}{W_{1}}\right) \right. \]
\[ + \log^{+} \{ \overline{N}(r, 0, W_{1}) + \overline{N}(r, g) \} + \log r \]
as \( r \to \infty \), n.e., and
\[ N(r, \varphi_{1}) \leq 2 \{ \overline{N}(r, 0, W_{1}) + \overline{N}(r, g) \}, \]
\[ N(r, \varphi_{1}) \leq 3 \{ \overline{N}(r, 0, W_{1}) + \overline{N}(r, g) \}. \]
Then our desired estimate is of an inequality

\[ T\left(r, \frac{g'}{g}\right) \leq 2m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W'_i}{W_i}\right) + 5\{N(r, 0, W_i) + N(r, g)\} + O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W'_i}{W_i}\right) + \log^+ \{N(r, 0, W_i) + N(r, g)\} + \log r\right]. \]

as \( r \to \infty, \) n. e..

Hence in Case I the inequality (2.1) never fails to hold for any fixed number \( a. \)

5. Proof of Theorem: Case II where \( \phi_a = 0 \)

Since \( g \not\equiv 0, \) (3.10) is reduced to \( \phi_a \equiv 0. \) Therefore the following two equations are given:

(5.1) \[ a(2a-1)G_a'(z) + a(a-1)G_a(z)^2 - 2(a-1)(2a-1)H_a(z) \equiv 0, \]

(5.2) \[ a(2a-1)H_a'(z) + (a-1)G_a(z)H_a(z) \equiv 0. \]

We now distinguish the cases with respect to the value of \( a \) and determine all the forms of \( g(z) \) to satisfy the two equations above.

Subcase i. \( a = 0. \)

Then these become the equations \( H_0(z) \equiv 0 \) and \( H_0'(z) + G_0(z)H_0(z) \equiv 0, \) which we can reduce to \( H_0(z) \equiv 0. \) Applying this to (3.5) we see that \( g^*(z) \equiv 0. \) This is however the \( g(z) \) listed in §1, 3°, so that \( W_0(z) \equiv 0. \) Hence \( \phi_0 \) cannot vanish identically if \( a = 0. \)

Subcase ii. \( a = 1/2. \)

Then Equations (5.1) and (5.2) become \( G_{1/2}(z) \equiv 0 \) and \( G_{1/2}(z)H_{1/2}(z) \equiv 0. \) Using Equation (3.5) with \( a = 1/2 \) and \( G_{1/2}(z) \equiv 0 \) we get \( H_{1/2}(z) \equiv 0 \) by \( g \equiv 0. \) Applying these to (3.4) we see that \( g^*(z) \equiv 0, \) so that

(1) \[ g(z) = az^2 + \beta z + \gamma, \quad \text{where } a, \beta, \gamma \in C. \]

For this \( g(z) \) we find

\[ W_{1/2}(z) = W\left(\frac{1}{2} \beta z + \gamma, 2\alpha z + \beta\right) = -\frac{1}{2} (\beta^2 - 4\alpha \gamma). \]

Further the constants \( a, \beta, \gamma \) must be taken as \( \beta^2 - 4\alpha \gamma \neq 0 \) in (1) (and then clearly \( G_{1/2} = H_{1/2} \equiv 0. \)) It is such a condition that immediately follows from the negation of that in §1, 3° with \( m = 2 \) as well.

Subcase iii. \( a = 1. \)

Then Equations (5.1) and (5.2) become \( G_1'(z) \equiv 0 \) and \( H_1'(z) \equiv 0. \) Therefore
we can obtain the functions \( g(z) \) to be determined as entire solutions of a second order linear differential equation

\[
(5.3) \quad w'' + k_1 w' + k_0 w = 0
\]

with the constant coefficients \( k_1 \) and \( k_0 \). Let \( \lambda_1 \) and \( \lambda_2 \) be the roots of its characteristic equation \( \lambda^2 + k_1 \lambda + k_0 = 0 \). If \( \lambda_1 \neq \lambda_2 \), a general solution of this (5.3) is given by

\[
(2) \quad w = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}
\]

for arbitrary constants \( C_1 \), \( C_2 \). In order that \( W_1(z) = W(w, w') = C_1 C_2 (\lambda_1 - \lambda_2)^2 \exp{(\lambda_1 + \lambda_2)z} \) does not vanish identically, both \( C_1 \) and \( C_2 \) should differ from zero. If \( \lambda_1 = \lambda_2 = \lambda \), say, a general solution to (5.3) has a form

\[
(3) \quad w = (C_2 z + C_1) e^{\lambda z},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants. For the \( w \) we have \( W(w, w') = -C_2^2 \exp(2\lambda z) \). Hence it is sufficient for our purpose to choose a non-zero constant \( C_2 \) in (3). In this subcase \( g(z) \) must be of the form either (2) or (3) for suitable constants \( C_1 \) and \( C_2 \).

**Subcase iv. a is different from 0, 1/2, 1.**

Firstly suppose that \( H_a(z) \equiv 0 \). Then Equation (3.5) gives

\[
(5.4) \quad (2a - 1)g''(z) = -ag'(z)G_a(z)
\]

and (5.1) also gives

\[
(5.5) \quad (2a - 1)G_a'(z) + (a - 1)G_a(z)^2 = 0.
\]

If \( G_a(z) \equiv 0 \), \( g''(z) \equiv 0 \) by (5.4), so that

\[
(4) \quad g(z) = az + \beta
\]

for \( a \neq 0 \), \( \beta \in \mathbb{C} \). Then \( W_a(z) = -a \alpha^2 \neq 0 \). Unless \( G_a \equiv 0 \), Equation (5.5) leads us to

\[
G_a(z) = \frac{2a - 1}{a - 1} \cdot \frac{1}{z - z_0}, \quad z_0 \in \mathbb{C},
\]

and then Equation (5.4) gives

\[
\frac{g''(z)}{g'(z)} = -\frac{a}{2a - 1} G_a(z) = -\frac{a}{a - 1} \cdot \frac{1}{z - z_0}.
\]

Therefore it follows

\[
g'(z) = C(z - z_0)^{-a/(a - 1)}, \quad C \in \mathbb{C} - \{0\},
\]

so that \(-a/(a - 1) = m\), say, is an integer different from 0, \(-1\), and
Noting \( \alpha = \frac{m}{m+1} \) we see that \( \zeta \neq 0 \) in order that \( W_a(z) = -aC^2(z-z_a)^{m-1} \) should not vanish identically.

After this we may suppose that \( H_a(z) \neq 0 \). By Equation (5.2) we obtain

\[
G_a(z) = -\frac{2a-1}{a-1} \cdot \frac{H_a'(z)}{H_a(z)}.
\]

Then (5.1) gives

\[
2(a-1)(2a-1)H_a(z) = -\frac{a(2a-1)}{a-1} \left( \frac{H_a'(z)}{H_a(z)} \right) + \frac{a(2a-1)}{a-1} \left( \frac{H_a''(z)}{H_a(z)} \right) = -\frac{a(2a-1)}{a-1} \cdot \frac{H_a''(z)H_a(z) - 2H_a'(z)^2}{H_a(z)^2},
\]

and thus

\[
\left( \frac{1}{H_a(z)} \right)' = -\frac{H_a''(z)H_a(z) - 2H_a'(z)^2}{H_a(z)^3} = \frac{2(a-1)^2}{a(2a-1)},
\]

Hence we deduce

\[
H_a(z) = \frac{a(2a-1)}{(a-1)^{k+1}} \cdot \frac{1}{(z-\alpha)(z-\beta)}, \quad \alpha, \beta \in \mathbb{C}
\]

In virtue of this expression we get

\[
G_a(z) = \frac{2a-1}{a-1} \left\{ \frac{1}{z-\alpha} + \frac{1}{z-\beta} \right\}
\]

from Equation (5.6) and thus it follows from (3.2) that

\[
W_a(z) = C \left\{ (z-\alpha)(z-\beta) \right\}^{-\frac{2a-1}{(a-1)}}
\]

for a non-zero constant \( C \). Here \(-\frac{2a-1}{(a-1)}=m\), say, is a number different from \(-2\), \(-1\), and \(0\), which is equal to an integer if \( \alpha \neq \beta \) and to half an integer if \( \alpha = \beta \). Then we can transform (3.5) into an equation

\[
g^*(z) = (m+1) \left( \frac{1}{z-\alpha} + \frac{1}{z-\beta} \right) g'(z) + \frac{(m+1)(m+2)}{(z-\alpha)(z-\beta)} g(z) = 0.
\]

Therefore \( g(z) \) is able to possess the poles possibly at \( z=\alpha \) or \( \beta \). Let \( f(z) \) be an entire function with \( f(\alpha) = 0 \) and \( f(\beta) = 0 \), and both \( k \) and \( l \) be integers if \( \alpha \neq \beta \) and half integers with \( k=l \) if \( \alpha = \beta \), such that

\[
g(z) = (z-\alpha)^k(z-\beta)^l f(z).
\]

Using Equation (5.8) we write
\[
g^\alpha(z) = (m+1) \left( \frac{1}{z-\alpha} + \frac{1}{z-\beta} \right) g'(z) \frac{(m+1)g(z)}{(z-\alpha)(z-\beta)}
\]

and thus an expression
\[
\frac{W_\alpha(z)}{g(z)} = \frac{g^\alpha(z)}{g(z)} \frac{m+1}{m+2} \left( \frac{g'(z)}{g(z)} \right)^2
\]

On the other hand, (5.7) gives
\[
\frac{W_\alpha(z)}{g(z)} = \frac{g^\alpha(z)}{g(z)} \frac{m+1}{m+2} \left( \frac{g'(z)}{g(z)} \right)^2
\]

In the case where \( \alpha \neq \beta \), \( g(z) \) is expressed by
\[
g(z) = c_\alpha (z-\alpha)^k \{1+O(z-\alpha)\}, \quad c_\alpha \in C - \{0\}
\]
in a neighborhood of \( z=\alpha \). Substituting this into the equation (5.8) and comparing the coefficients of the term \((z-\alpha)^{k-2}\), we get a characteristic equation
\[k(k-m-2)=0,\]
so that \( k \) can be of the value 0 or \( m+2 \). Then it immediately follows that \( m=-3 \) and \( k=-1 \), when we compare the behavior of two expressions above for \( W_\alpha(z)/g(z)^2 \) in a neighborhood of \( z=\alpha \). In fact we see that \( 2k-m-2=0 \) in both cases of \( k=0 \) and \( k=m+2 \). We have \( m=-3 \) when \( k=m+2 \), while \( m=-1 \) when \( k=0 \). The latter is now excluded. The same is true of the number \( l \). Hence \( a=2 \) and
\[
g(z) = \frac{f(z)}{(z-\alpha)(z-\beta)}
\]
if \( \alpha \neq \beta \). Concerning \( f(z) \) we have
\[
\frac{C}{(z-\alpha)(z-\beta)f(z)} = -2 \left( \frac{1}{z-\beta} + \frac{f'(z)}{f(z)} \right) \left( \frac{1}{z-\alpha} + \frac{f'(z)}{f(z)} \right)
\]
and thus
\[
\{(z-\beta)f'(z)-f(z)\} \{(z-\alpha)f'(z)-f(z)\} \equiv -\frac{C}{2} \quad (\neq 0).
\]

Differentiating both sides of this, we get an identity
\[
(z-\beta)f''(z) \{(z-\alpha)f'(z)-f(z)\} = -(z-\alpha)f''(z) \{(z-\beta)f'(z)-f(z)\}.
\]

Unless \( f''(z)=0 \),
\[(z-\beta)[(z-\alpha)f'(z)-f(z)] = -(z-\alpha)\{(z-\beta)f'(z)-f(z)\}\]

and therefore
\[(\alpha-\beta)f'(\alpha) = 0.\]

This is impossible, so that \(f^\ast(z) = 0\), i.e., \(f(z) = D(z-\gamma)\) where \(D = C - \{0\}\) and \(\gamma \in C - \{\alpha, \beta\}\). Then
\[
g(z) = \frac{D(z-\gamma)}{(z-\alpha)(z-\beta)},
\]
which satisfies the condition (5.7) with \(C = -2D^k(\gamma-\alpha)(\gamma-\beta)\).

Next we shall consider the case where \(\alpha = \beta\). Equation (5.8) is then equal to
\[
g^\ast(z) = \frac{2(m+1)}{z-\alpha} \frac{g'(z) + (m+1)(m+2)}{(z-\alpha)^2} g(z) = 0
\]
Here we make a similar discussion to the above with
\[
g(z) = (z-\alpha)^m f(z),
\]
and obtain a characteristic equation
\[2k(2k-2) - 2k^2m + (m+1)(m+2) = 0,\]
and so \(2k = m+1\) or \(2k = m+2\). Since we now have
\[
W_\alpha(z) = \frac{C}{g(z)^2} = \frac{C}{(z-\alpha)^{2k-m}} = \frac{m+1}{m+2} \frac{2k-m-2}{z-\alpha} \frac{f'(z)}{f(z)} + f''(z),
\]
the latter, \(2k = m+2\), gives immediately a contradiction as \(2k-m\neq 0\). For the former case where \(2k = m+1\) the behavior of two expressions above for \(W_\alpha(z)/g(z)^2\) is compatible. Then \(f(z)\) satisfies the relation
\[
\{(z-\alpha)f'(z) - f(z)\}^2 = -\frac{m+2}{m+1} \cdot C (\neq 0).
\]
Differentiating this we have \(f''(z) = 0\), or \(f(z) = D(z-\gamma)\), \(D = C - \{0\}\), \(\gamma \in C - \{\alpha\}\) again. Hence if \(\alpha = \beta\), \(a = (m+1)/(m+2)\) and
\[
g(z) = D(z-\alpha)^m (z-\gamma),
\]
provided that \(m\) is an integer different from \(0, -1, \) and \(-2\). In order that \(g(z)\) may satisfy (5.7), i.e.,
\[
W_\alpha(z) = C(z-\alpha)^m,
\]
we choose the constant \(C = -aD^k(\gamma-\alpha)^m\).

We have discussed all the possible forms that \(g(z)\) has in Case II:
(1) when \(a = 1/2\), \(g(z) = \alpha z^2 + \beta z + \gamma\), where \(\beta^2 - 4\alpha \gamma \neq 0\);
(2) when \( a = 1 \), \( g(z) = C_1 e^{4z} + C_2 e^{4z} \), where \( \lambda_1 = \lambda_2 \) and \( C_1 C_2 \neq 0 \);

(3) when \( a = 1 \), \( g(z) = (C_2 e^z + C_1 e^{4z}) \), where \( C_2 \neq 0 \);

(4) when \( a \neq 0, 1/2, 1 \), \( g(z) = C_1 (z - \alpha) \), where \( C_1 \neq 0 \);

(5) when \( a = (m - 1)/m \), \( g(z) = C_1 [(z - \alpha)^m - C_1] \), where \( C_1 C_2 \neq 0 \) and \( m \neq 0, 1, 2 \);

(6) when \( a = 2 \), \( g(z) = (C_1 (z - \gamma) -(z - \alpha)(z - \beta)) \), where \( C_1 \neq 0 \), and \( \alpha, \beta, \gamma \) are mutually distinct;

(7) when \( a = (m + 1)/(m + 2) \), \( g(z) = C_1 (z - \alpha)^m [z - (z - \beta)] \), where \( C_1 \neq 0, \alpha \neq \gamma \) and \( m \neq 0, -1, -2 \), provided that \( C_1, C_2, \lambda_1, \lambda_2, \lambda, \alpha, \beta, \gamma \in \mathbb{C} \) and \( m \) is an integer. As their \( W_a(z) \) we obtain also

(1) \( W_{1/2}(z) = -(1/2) (\beta^2 - 4 \alpha \gamma) \);

(2) \( W_1(z) = C_1 C_2 (\lambda_1 - \lambda_2) e^{(2 \lambda_1 + \lambda_2)z} \);

(3) \( W_0(z) = -C_2 e^{2\lambda_2 z} \);

(4) \( W_0(z) = -a \alpha \);

(5) \( W_{(m - 1)/m}(z) = -(m(m - 1)/m) (z - \alpha)^{m-2} \);

(6) \( W_2(z) = (-2C_1 (\gamma - \alpha)(\gamma - \beta))/((z - \alpha)^3 (z - \beta)^3) \);

(7) \( W_{(m + 1)/(m + 2)}(z) = -(m(m + 1)/(m + 2)) C_1 \zeta (\gamma - \alpha)^3 (z - \alpha)^{m+2} \).

Finally we need to examine whether the inequality (2.1) holds or not in each case above. The function \( U_a(r) \) in (2.1) grows at least as rapidly as \( O(\log r) \) for \( r \to \infty \), n.e.. Therefore (2.1) is satisfied by \( g(z) \) given in (4), (5), (6) and (7) as rational functions. As proved in Remark 2° in §2, two possibilities (1) and (2) are the very exceptions. With \( g(z) \) as in (3) it is easily shown that

\[
m(r, g'/g) = O(1), \quad m(r, W_1'/W_1) = O(1),
\]

\[
N(r, 0, W_1) = N(r, g) \equiv 0,
\]

and

\[
T(r, g'/g) = \log r + O(1),
\]
as \( r \to \infty \). Then \( U_a(r) = O(\log r) \) as \( r \) tends to infinity, so Inequality (2.1) also holds. This completes the proof of the theorem.

**Remark.** Mention needs to be made of rational functions. Reconsidering the above proof in Case I as a rational function \( g(z) \), we see that all of \( m(r, G_a), m(r, H_a), m(r, \phi_a) \) and \( m(r, \phi_a) \) grow possibly in the degree of \( o(1) \) with the aid of (1.2) in Lemma. Inequality (2.1) can be therefore sharpened by

\[
T(r, g'/g) \leq 5 \{|N(r, 0, W_a) + N(r, g)| + O(1)\}.
\]

Then there exist such the rational functions \( g(z) \) as never satisfy (5.9) only in (3) with \( \lambda = 0 \), (4), (5) with \( m > 5 \) or \( m < -4 \), as well as (1) of Case II. In fact, since Inequality (5.9) equals
\[ N(r, 0, g) \leq 5N(r, 0, W_a) + 4N(r, g) + O(1) \]

in virtue of the equation

\[ T(r, \frac{g'}{g}) = m(r, \frac{g'}{g}) + N(r, \frac{g'}{g}) = N(r, 0, g) + N(r, g) + o(1), \]

this fact can be shown by studying these counting functions in each occasion. In (5) for example, if \( m > 2 \),

\[ N(r, 0, g) = m \log r, \quad N(r, 0, W_a) = \log r, \quad N(r, g) = 0 \]

and if \( m < 0 \),

\[ N(r, 0, g) = -m \log r, \quad N(r, 0, W_a) = 0, \quad N(r, g) = \log r \]

for sufficiently large \( r \). The equality of (5.9) occurs if \( m = 5 \) or \( m = -4 \).

6. Proof of Corollary

In order to prove this result we shall return to the proof of Theorem. At first we are concerned about Case I in Section 4. Assume that \( \phi_a(x) \equiv 0 \). The present assumption (2.2) reduces the equations (4.3) and (4.4) to

\[ m(r, G_a) = m(r, W_a) = S(r, W_a) = S(r, g) \]

and

\[ m(r, H_a) = S(r, g) + S(r, W_a) = S(r, g), \]

respectively. Similarly (4.5) and (4.6) become

\[ N(r, G_a) \leq N(r, 0, W_a) = m(r, W_a) + O(1) = S(r, g) \]

and

\[ N(r, H_a) \leq 2N(r, 0, W_a) = S(r, g). \]

All of them hold independently of the value of \( a \in C \). Therefore it follows also for both \( \phi_a \) and \( \phi_a \) that

\[ T(r, \phi_a) = S(r, g) \quad \text{and} \quad T(r, \phi_a) = S(r, g), \]

so that

\[ T(r, \frac{g'}{g}) = S(r, g). \]

(6.1)

Using a relation

\[ g^2 = \frac{W_a}{(\frac{g'}{g})^2 - (a - 1)(\frac{g'}{g})^2}, \]
we obtain

\[ 2T(r, g) \leq T(r, W_z) + T\left(r, \left(\frac{g'}{g}\right)^\prime - (a-1)\left(\frac{g'}{g}\right)^\ast\right) + O(1) \]

\[ \leq m(r, W_z) + 4T\left(r, \frac{g'}{g}\right) + S\left(r, \frac{g'}{g}\right) + O(1). \]

Then from (2.2) and (6.1) we conclude

\[ T(r, g) = S(r, g), \]

which is impossible. Hence it must be hold \( \phi_a(z) = 0 \).

Concerning the possibilities in Case II we have made a list in Section 5. We shall pick out those what give entire functions \( g(z) \) with the property (2.2). Evidently (5), (6) and (7) are beside our object. If \( g(z) \) is a polynomial, \( W_z(z) \) must be a constant. Possibilities (1) and (4) come under this heading. If \( \lambda_1 + \lambda_2 \neq 0 \) in (2), then \( W_z(z) = C_1 C_2 (\lambda_1 - \lambda_2)^2 e^{(\lambda_1 + \lambda_2)z} \) is an entire function of order one and

\[ m(r, W_z) = \frac{|\lambda_1 + \lambda_2|}{\pi} r + O(1) \quad \text{as} \quad r \to \infty. \]

(See Hayman [1], p. 7.) A similar observation shows

\[ m(r, g) \leq (|\lambda_1| + |\lambda_2|) \frac{r}{\pi} + O(1) \quad \text{as} \quad r \to \infty. \]

Therefore (2.2) fails to hold since

\[ \lim_{r \to \infty} \frac{m(r, W_z)}{m(r, g)} \geq \frac{|\lambda_1 + \lambda_2|}{|\lambda_1| + |\lambda_2|} > 0. \]

When \( \lambda_1 + \lambda_2 = 0 \), \( W_z(z) \) is a constant and \( g(z) \) is such a transcendental entire function that satisfies all the assumptions in Corollary. Functions in the last remaining (3) can satisfy Condition (2.2) only if \( \lambda = 0 \). We have thus proved the corollary.

**References**


Department of Mathematics
Science University of Tokyo
Noda, Chiba, Japan