Z. YOSIMURA
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KO-HOMOLOGIES OF A FEW CELLS COMPLEXES

BY ZEN-ICHI YOSIMURA

0. Introduction.

Let KO and KU be the real and the complex K-spectrum respectively. For any CW-spectra X and Y we say that X is *quasi KO_*-equivalent* to Y if there exists a map $h: Y \to KO \wedge X$ such that the composite map $(\mu\wedge 1)(1\wedge h)$: $KO\wedge Y \to KO\wedge X$ is an equivalence where μ: $KO\wedge KO \to KO$ denotes the multiplication of KO (see [4] or [3]). Such a map h is called to be a quasi KO_*-equivalence. If X is quasi KO_*-equivalent to Y, then KO_*X is isomorphic to KO_*Y as a KO_*-module and in addition KU_*X is isomorphic to KU_*Y as an abelian group with involution where the conjugation ψ_C^{-1} behaves as an involution. Assume that CW-spectra X and Z have the same quasi KO_*-types as CW-spectra Y and W respectively. For any maps f: $Z \to X$ and g: $W \to Y$ we say that f is *quasi KO_*-equivalent* to g if there exist KO_*-equivalences $h: Y \to KO\wedge X$ and $k: W \to KO\wedge Z$ such that the equality $hg=(1\wedge f)k: W\to KO\wedge X$ holds. In this case their cofibers $C(f)$ and $C(g)$ have the same quasi KO_*-type.

A CW-spectrum X is said to be *stably* quasi KO_*-equivalent to a CW-spectrum Y if X is quasi KO_*-equivalent to the i-fold suspended spectrum $\Sigma^i Y$ for some i. In this note we shall be interested in the stable quasi KO_*-types of complexes with a few cells. Each complex with 2-cells is stably quasi KO_*-equivalent to one of the following spectra $\Sigma^0\vee\Sigma^i$ $(0\leqq i\leqq 7)$, SZ/t $(t\geqq 1)$, $P=C(\eta)$ and $Q=C(\eta^2)$ where SZ/t denotes the Moore spectrum of type Z/t and η: $\Sigma^1\to\Sigma^0$ is the stable Hopf map of order 2. Our purpose of this paper is to determine the stable quasi KO_*-types of any complexes with 3- or 4-cells (Theorems 5.3 and 5.4). In [4] and [5] we introduced some 3-cells spectra X_m and X'_m constructed as the cofibers of certain maps f: $\Sigma^i\to SZ/2^m$ and f': $\Sigma^i SZ/2^m\to\Sigma^0$ and some 4-cells spectra XY_m, $X'Y'_m$ and $Y'X_m$ obtained as the cofibers of their mixed maps. In § 1 and § 4 we study the quasi KO_*-types of their cofibers C(g) for any maps g: $S_i\to\Delta X$ realizing elements of $KO_i X$ when $X=SZ/2^m$, P, Q, X_m or X'_m. In § 2 we introduce some 4-cells spectra $X_{m,n}$ constructed as the cofibers of certain maps f: $\Sigma^i SZ/2^n\to SZ/2^m$, and then study the quasi KO_*-types of their cofibers $C(g)$ for any maps g: $\Sigma^i SZ_n\to\Delta X$ realizing elements of $[\Sigma^i SZ/2^n, KO\wedge X]$ when $X=SZ/2^m$, P or Q. In § 3 we introduce some new small spectra $XV_{m,n}$, $VX_{m,n}$ and $X'X_{n,m}$ needed in § 4. In § 5 we prove Theorems 5.3 and 5.4 by using results obtaind in §§ 1-4.

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1. The cofibers of maps $f\colon \Sigma^t \to SZ/2^m$ and $f'\colon \Sigma^t SZ/2^m \to \Sigma^0$.

1.1. Let $SZ/2^r$ be the Moore spectrum of type $Z/2^r\,(r\geqq 1)$, and $i_r\colon \Sigma^0 \to SZ/2^r$ and $j_r\colon SZ/2^r \to \Sigma^1$ denote the bottom cell inclusion and the top cell projection. For the stable Hopf map $\eta\colon \Sigma^1 \to \Sigma^0$ of order 2 there exists its extension $\bar{\eta}_1\colon \Sigma^1 SZ/2 \to \Sigma^0$ and its coextension $\tilde{\eta}_1\colon \Sigma^2 \to SZ/2$ with $\bar{\eta}_1 i_1 = \eta$ and $j_1 \tilde{\eta}_1 = \eta$. Using the obvious maps $\rho_{r,1}\colon SZ/2^r \to SZ/2$ and $\rho_{1,r}\colon SZ/2 \to SZ/2^r$ we then set $\bar{\eta}_r = \bar{\eta}_1 \rho_{r,1}\colon \Sigma^1 SZ/2^r \to \Sigma^0$ and $\tilde{\eta}_r = \rho_{1,r}\tilde{\eta}_1\colon \Sigma^2 \to SZ/2^r$, which satisfy $\bar{\eta}_r i_r = \eta$ and $j_r \tilde{\eta}_r = \eta$, too. Hereafter we shall often drop as i, j, $\bar{\eta}$ and $\tilde{\eta}$ the subscript "r" in the symbols i_r, j_r, $\bar{\eta}_r$ and $\tilde{\eta}_r$. Choose maps $\varphi\colon \Sigma^1 SZ/2 \to SZ/2\wedge SZ/4$ and $\psi\colon SZ/2\wedge SZ/4 \to SZ/2$ such that $(1\wedge j)\varphi = 1 = \psi(1\wedge i)$ and $(1\wedge i)\psi + \varphi(1\wedge j) = 1$, and then consider the composite maps $\eta_{1,2} = (\bar{\eta}\wedge 1)\varphi\colon \Sigma^2 SZ/2 \to SZ/4$ and $\eta_{2,1} = \psi(\tilde{\eta}\wedge 1)\colon \Sigma^2 SZ/4 \to SZ/2$. It is immediate that $\eta_{1,2} i = \tilde{\eta}$, $j\eta_{1,2} = \bar{\eta}$, $\eta_{2,1} i = \tilde{\eta}$ and $j\eta_{2,1} = \bar{\eta}$ when the maps φ and ψ are replaced by the maps $\varphi + (1\wedge i\eta)$ and $\psi + (1\wedge \eta j)$ if necessary. Set $\eta_{n,m} = \rho_{2,m}\eta_{1,2}\rho_{n,1}\colon \Sigma^2 SZ/2^n \to SZ/2^m$ when $m\geqq 2$, and $\eta'_{n,m} = \rho_{1,m}\eta_{2,1}\rho_{n,2}\colon \Sigma^2 SZ/2^n \to SZ/2^m$ when $n\geqq 2$. Since it is easily shown that $\eta_{n,m} = \eta'_{n,m}$ when $m\geqq 2$ and $n\geqq 2$, we employ the notation $\eta_{n,m}$ instead of $\eta'_{n,m}$ even if $m=1$. Evidently these maps $\eta_{n,m}$ satisfy $\eta_{n,m} i = \tilde{\eta}$ and $j\eta_{n,m} = \bar{\eta}$, too.

Denote by V_m, V'_m, U_m and $U'_m\,(m\geqq 1)$ the small spectra constructed as the cofibers of the maps $i\bar{\eta}\colon \Sigma^1 SZ/2 \to SZ/2^{m-1}$, $\tilde{\eta} j\colon \Sigma^1 SZ/2^{m-1} \to SZ/2$, $\eta_{1,m+1}\colon \Sigma^2 SZ/2 \to SZ/2^{m+1}$ and $\eta_{m+1,1}\colon \Sigma^2 SZ/2^{m+1} \to SZ/2$ respectively. In [4] or [6] these small spectra are written to be V_{2m}, V'_{2m}, U_{2m} and U'_{2m}. We shall denote by $i_V\colon SZ/2^{m-1} \to V_m$, $i'_V\colon SZ/2 \to V'_m$, $i_U\colon SZ/2^{m+1} \to U_m$ and $i'_U\colon SZ/2 \to U'_m$ the canonical inclusions, and by $j_V\colon V_m \to \Sigma^2 SZ/2$, $j'_V\colon V'_m \to \Sigma^2 SZ/2^{m-1}$, $j_U\colon U_m \to \Sigma^3 SZ/2$ and $j'_U\colon U'_m \to \Sigma^3 SZ/2^{m+1}$ the canonical projections. Consider the two cofiber sequences

$$(1.1)\qquad \Sigma^1 SZ/2 \xrightarrow{\bar{\eta}} \Sigma^0 \xrightarrow{\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{j}} \Sigma^2 SZ/2 \quad\text{and}\quad \Sigma^2 \xrightarrow{\tilde{\eta}} SZ/2 \xrightarrow{\tilde{i}} C(\tilde{\eta}) \xrightarrow{\tilde{j}} \Sigma^3$$

in which the cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ have the same quasi KO_*-types as Σ^4 and Σ^{-1} respectively (see [3], [4], [6] or (1.9) below). Then we get the following two cofiber sequences

$$(1.2)\quad \Sigma^0 \xrightarrow{2^m\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{i}_V} V_{m+1} \xrightarrow{\bar{j}_V} \Sigma^1 \quad\text{and}\quad \Sigma^2 \xrightarrow{\tilde{i}'_V} V'_{m+1} \xrightarrow{\tilde{j}'_V} C(\tilde{\eta}) \xrightarrow{2^m\tilde{j}} \Sigma^3.$$

Since $\eta_{1,2} = (\bar{\eta}\wedge 1)\varphi$ and $\eta_{2,1} = \psi(\tilde{\eta}\wedge 1)$ there exist maps $\bar{\eta}_{2,1}\colon C(\bar{\eta})\wedge SZ/4 \to \Sigma^2 SZ/2$ and $\tilde{\eta}_{1,2}\colon \Sigma^1 SZ/2 \to C(\tilde{\eta})\wedge SZ/4$ satisfying $\bar{\eta}_{2,1}(1\wedge i) = \bar{j}$ and $(1\wedge j)\tilde{\eta}_{1,2} = \tilde{i}$, whose cofibers are $\Sigma^1 U_1$ and U'_1 respectively. Hence we can choose maps

$$(1.3)\qquad \bar{\lambda}\colon C(\bar{\eta}) \longrightarrow \Sigma^0 \quad\text{and}\quad \tilde{\lambda}\colon \Sigma^3 \longrightarrow C(\tilde{\eta})$$

satisfying $\bar{i}\bar{\lambda} = 4$ and $\tilde{\lambda}\tilde{j} = 4$ so that their cofibers are U_1 and U'_1 respectively. It is obvious that $\bar{\lambda}\bar{i} = 4 = \tilde{j}\tilde{\lambda}$. So we get the following two cofiber sequences

(1.4) $C(\bar{\eta}) \xrightarrow{2^m\bar{\lambda}} \Sigma^0 \xrightarrow{i_U} U_{m+1} \xrightarrow{j_U} \Sigma^1 C(\bar{\eta})$ and $\Sigma^3 \xrightarrow{2^m\tilde{\lambda}} C(\tilde{\eta}) \xrightarrow{i'_U} U'_{m+1} \xrightarrow{j'_U} \Sigma^4$.

Let P and Q denote the elementary spectra constructed as the cofibers of the stable Hopf map $\eta: \Sigma^1 \to \Sigma^0$ and its square $\eta^2: \Sigma^2 \to \Sigma^0$ respectively. Given such an elementary spectrum X as Σ^i, $SZ/2^m$, P, Q or V_{m+1} each CW-spectrum having the same quasi KO_*-type as X will be represented by ΔX. For simplicity we shall write S_i $(0 \leqq i \leqq 7)$ and SZ_m $(m \geqq 1)$ instead of $\Delta\Sigma^i$ and $\Delta SZ/2^m$.

LEMMA 1.1. *For any map $f: S_i \to S_0$ $(0 \leqq i \leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^0 \vee \Sigma^{i+1}$ or the following small spectrum Y_i: i) $Y_0 = SZ/2^m \vee SZ/q$; ii) $Y_1 = P$; iii) $Y_2 = Q$; iv) $Y_4 = \Sigma^4 V_{m+1} \vee SZ/q$ where $m \geqq 0$ and $q \geqq 1$ is odd.*

Proof. Use the following maps $g_{0,m} = 2^m: \Sigma^0 \to \Sigma^0$, $g_1 = \eta: \Sigma^1 \to \Sigma^0$, $g_2 = \eta^2: \Sigma^2 \to \Sigma^0$ and $g_{4,m} = 2^m \bar{i}: \Sigma^4 \to \Sigma^4 C(\bar{\eta})$, whose cofibers are $SZ/2^m$, P, Q and $\Sigma^4 V_{m+1}$ respectively. Then our result is immediate.

In virtue of Lemma 1.1 we observe that

(1.5) the small spectra $\Sigma^2 V'_m$, $\Sigma^4 U_m$ and $\Sigma^5 U'_m$ $(m \geqq 1)$ have the same quasi KO_*-type as V_m (cf. [6, (1.3) and (1.4)] or [7, (1.9) ii)]).

1.2. Denote by M_m, N_m, P_m, Q_m and $R_m (m \geqq 1)$ the 3-cells spectra constructed as the cofibers of the maps $i\eta: \Sigma^1 \to SZ/2^m$, $i\eta^2: \Sigma^2 \to SZ/2^m$, $\tilde{\eta}: \Sigma^2 \to SZ/2^m$, $\tilde{\eta}\eta: \Sigma^3 \to SZ/2^m$ and $\tilde{\eta}\eta^2: \Sigma^4 \to SZ/2^m$ respectively. Dually we denote by M'_m, N'_m, P'_m, Q'_m and $R'_m (m \geqq 1)$ the 3-cells spectra constructed as the cofibers of the maps $\eta j: SZ/2^m \to \Sigma^0$, $\eta^2 j: \Sigma^1 SZ/2^m \to \Sigma^0$, $\bar{\eta}: \Sigma^1 SZ/2^m \to \Sigma^0$, $\eta\bar{\eta}: \Sigma^2 SZ/2^m \to \Sigma^0$ and $\eta^2\bar{\eta}: \Sigma^3 SZ/2^m \to \Sigma^0$ respectively. When $X = M$, N, P, Q or R we shall denote by $i_X: SZ/2^m \to X_m$ or $i'_X: \Sigma^0 \to X'_m$ the canonical inclusion, and by $j_X: X_m \to \Sigma^d$ or $j'_X: X'_m \to \Sigma^{d'-1} SZ/2^m$ the canonical projection where $d = \dim X_m$ and $d' = \dim X'_m$. In [4, 4.1] these 3-cells spectra X_m and X'_m are written to be X_{2m} and X'_{2m}, and their KU- and KO- homologies have been calculated (see [4, Propositions 4.1 and 4.2]).

LEMMA 1.2. (1) *For any map $f: S_i \to SZ_m (0 \leqq i \leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1} \vee SZ/2^m$ or the following small spectrum Y_i: i) $Y_0 = \Sigma^1 \vee SZ/2^k (0 \leqq k < m)$; ii) $Y_1 = M_m$; iii) $Y_2 = N_m$ or P_m; iv) $Y_3 = Q_m$; v) $Y_4 = R_m$ or $\Sigma^1 \vee \Sigma^4 V_{k+1} (0 \leqq k < m-1)$.*

(2) *For any map $f: \Sigma^{i-1} SZ_m \to S_0 (0 \leqq i \leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^0 \vee \Sigma^i SZ/2^m$ or the following small spectrum Y_i: i) $Y_0 = \Sigma^0 \vee SZ/2^k (0 \leqq k < m)$; ii) $Y_1 = M'_m$; iii) $Y_2 = N'_m$ or P'_m; iv) $Y_3 = Q'_m$; v) $Y_4 = R'_m$ or $\Sigma^4 \vee \Sigma^4 V_{k+1} (0 \leqq k < m-1)$.*

Proof. Consider the following maps $g_{0,k} = 2^k i: \Sigma^0 \to SZ/2^m$, $g_1 = i\eta: \Sigma^1 \to SZ/2^m$, $g_2 = i\eta^2: \Sigma^2 \to SZ/2^m$, $g'_2 = \tilde{\eta}: \Sigma^2 \to SZ/2^m$, $g''_2 = \tilde{\eta} + i\eta^2: \Sigma^2 \to SZ/2^m$, $g_3 =$

$\bar{\eta}\eta: \Sigma^3 \to SZ/2^m$ and $g_{4,k}=2^k i\bar{\lambda}: C(\bar{\eta}) \to SZ/2^m$. The cofibers $C(g_{0,k})$ and $C(g_{4,k})$ are the wedge sums $\Sigma^1 \vee SZ/2^k$ and $\Sigma^1 \vee U_{k+1}$ respectively whenever $0 \leqq k < m-1$, and $C(g_{4,m-1})$ has the same quasi KO_*-type as the 3-cells spectrum R_m since the map $g_{4,m-1}$ is quasi KO_*-equivalent to the map $\bar{\eta}\eta^2: \Sigma^4 \to SZ/2^m$. On the other hand, the cofiber $C(g_2'')$ coincides with the 3-cells spectrum P_m since $\bar{\eta}+i\eta^2=(1+i\eta j)\bar{\eta}$ and $(1+i\eta j)^2=1$. Our result of (1) is now easy, and (2) is dually shown to (1).

For any $m \geqq 1$ we consider the maps $\widetilde{6\bar{\nu}}=\eta_{1,m+1}\bar{\eta}: \Sigma^4 \to SZ/2^{m+1}$ and $\overline{6\bar{\nu}}=\bar{\eta}\eta_{m+1,1}: \Sigma^3 SZ/2^{m+1} \to \Sigma^0$ satisfying $j\widetilde{6\bar{\nu}}=6\nu=\overline{6\bar{\nu}}i$. Then Lemma 1.2 asserts that

(1.6) the cofibers $C(\widetilde{6\bar{\nu}})$ and $C(\overline{6\bar{\nu}})$ have the same quasi KO_*-types as $\Sigma^1 \vee \Sigma^4 V_m$ and $\Sigma^4 \vee \Sigma^4 V_m$ respectively.

In fact, these cofibers are obtained as those of the composite maps $\tilde{i}j_U: \Sigma^{-1}U_m \to \Sigma^2 C(\bar{\eta})$ and $i'_U\bar{j}: \Sigma^{-1}C(\bar{\eta}) \to \Sigma^1 U'_m$, both of which are KO_*-trivial because $KO_7 V_m=0$. Therefore our assertion (1.6) is certainly valid.

1.3. Recall that $KO_i P \cong Z$ or 0 according as i is even or odd. Using the bottom cell inclusion $i_P: \Sigma^0 \to P$ and the top cell projection $j_P: P \to \Sigma^2$ we get the following two cofiber sequences

$$(1.7) \qquad \Sigma^0 \xrightarrow{2^m i_P} P \xrightarrow{\rho_{P,M}} M_m \xrightarrow{k_M} \Sigma^1 \quad \text{and} \quad \Sigma^1 \xrightarrow{h'_M} M'_m \xrightarrow{\rho_{M',P}} P \xrightarrow{2^m j_P} \Sigma^2.$$

Hence we can immediately show

LEMMA 1.3. (1) *For any map $f: S_i \to \Delta P (0 \leqq i \leqq 1)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1} \vee P$ or the following small spectrum Y_i: $Y_0 = M_m \vee SZ/q$ where $m \geqq 0$ and $q \geqq 1$ is odd.*

(2) *For any map $f: \Sigma^i \Delta P \to S_0 (0 \leqq i \leqq 1)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^0 \vee \Sigma^{i+1} P$ or the following small spectrum Y_i: $Y_0 = \Sigma^{-1} M'_m \vee SZ/q$ where $m \geqq 0$ and $q \geqq 1$ is odd.*

Choose maps $\xi_P: \Sigma^2 \to P$ and $\zeta_P: P \to \Sigma^0$ satisfying $j_P \xi_P = 2 = \zeta_P i_P$, whose cofibers are $C(\eta)=P'_1$ and $C(\bar{\eta})=P_1$ respectively. Then we get the following two cofiber sequences

$$(1.8) \quad \Sigma^2 \xrightarrow{2^m \xi_P} P \xrightarrow{\rho_{P,P'}} P'_{m+1} \xrightarrow{jj'_P} \Sigma^3 \quad \text{and} \quad P \xrightarrow{2^m \zeta_P} \Sigma^0 \xrightarrow{i_P i} P_{m+1} \xrightarrow{\rho_{P,P}} \Sigma^1 P.$$

Lemma 1.3 combined with (1.8) asserts that

(1.9) the 3-cells spectra P'_{m+1} and $P_{m+1} (m \geqq 0)$ have the same quasi KO_*-types as $\Sigma^2 M_m$ and $\Sigma^{-1} M'_m$ respectively, where $M_0 = \Sigma^2$ and $M'_0 = \Sigma^0$ (cf. [4, Corollary 5.4]).

Since $\zeta_P \xi_P = \eta^2: \Sigma^2 \to \Sigma^0$, we obtain maps $\bar{\rho}_Q: C(\bar{\eta}) \to Q$ and $\tilde{\rho}_Q: Q \to C(\bar{\eta})$

satisfying $j_Q\bar{\rho}_Q=j\bar{j}$, $\bar{\rho}_Q\bar{i}=2i_Q$, $\tilde{\rho}_Q i_Q=\tilde{i}$ and $\tilde{j}\tilde{\rho}_Q=2j_Q$ where $i_Q: \Sigma^0\to Q$ and $j_Q: Q\to\Sigma^3$ denote the bottom cell inclusion and the top cell projection. Evidently there exists the following cofiber sequence

(1.10) $$C(\bar{\eta})\xrightarrow{\bar{\rho}_Q} Q\xrightarrow{\tilde{\rho}_Q} C(\bar{\eta})\xrightarrow{\delta}\Sigma^1 C(\bar{\eta}),$$

where δ is the composition of the maps $\rho_{P,P}$ and $\rho_{P,P'}$ in (1.8). We moreover obtain maps $\bar{\lambda}_P: \Sigma^2 C(\bar{\eta})\to P$ and $\tilde{\lambda}_P: \Sigma^3 P\to C(\bar{\eta})$ satisfying $j_P\bar{\lambda}_P=\bar{\lambda}$, $\bar{\lambda}_P\bar{i}=2\xi_P$, $\tilde{\lambda}_P i_P=\tilde{\lambda}$ and $\tilde{j}\tilde{\lambda}_P=2\zeta_P$ because $j_{P*}: [\Sigma^2 C(\bar{\eta}), P]\to[C(\bar{\eta}), \Sigma^0]$ and $i_P^*: [\Sigma^3 P, C(\bar{\eta})]\to[\Sigma^3, C(\bar{\eta})]$ are isomorphisms. Since the elementary spectra P and Q are related by the following cofiber sequences

$$\Sigma^1 P\xrightarrow{\lambda_{P,Q}} Q\xrightarrow{\rho_{Q,P}} P\xrightarrow{i_P j_P}\Sigma^2 P,$$

we here set

(1.11) $$\begin{aligned}
&\xi_Q=\lambda_{P,Q}\xi_P: \Sigma^3\longrightarrow Q, &&\zeta_Q=\zeta_P\rho_{Q,P}: Q\longrightarrow\Sigma^0,\\
&\bar{\rho}_P=\rho_{Q,P}\bar{\rho}_Q: C(\bar{\eta})\longrightarrow P, &&\tilde{\rho}_P=\tilde{\rho}_Q\lambda_{P,Q}: \Sigma^1 P\longrightarrow C(\bar{\eta}),\\
&\bar{\lambda}_Q=\lambda_{P,Q}\bar{\lambda}_P: \Sigma^3 C(\bar{\eta})\longrightarrow Q, &&\tilde{\lambda}_Q=\tilde{\lambda}_P\rho_{Q,P}: \Sigma^3 Q\longrightarrow C(\bar{\eta}).
\end{aligned}$$

Recall that $KO_iQ\cong Z, Z/2, 0, Z$ according as $i\equiv 0, 1, 2, 3 \bmod 4$. As is easily seen, there exist the following cofiber sequences

(1.12) $$\begin{aligned}
&\Sigma^0\xrightarrow{2^m i_Q} Q\xrightarrow{\rho_{Q,N}} N_m\xrightarrow{k_N}\Sigma^1, &&\Sigma^2\xrightarrow{h'_N} N'_m\xrightarrow{\rho_{N',Q}} Q\xrightarrow{2^m j_Q}\Sigma^3,\\
&\Sigma^1\xrightarrow{i_Q\eta} Q\xrightarrow{(j_Q,\rho_{Q,P})}\Sigma^3\vee P\xrightarrow{\eta\vee j_P}\Sigma^2, &&\Sigma^1\xrightarrow{(\eta, i_P)}\Sigma^0\vee\Sigma^1 P\xrightarrow{i_Q\vee\lambda_{P,Q}} Q\xrightarrow{\eta j_Q}\Sigma^2,\\
&\Sigma^3\xrightarrow{2^m\xi_Q} Q\xrightarrow{\rho_{Q,Q'}} Q'_{m+1}\xrightarrow{jj'_Q}\Sigma^4, &&\Sigma^0\xrightarrow{i_Q i} Q_{m+1}\xrightarrow{\rho_{Q,Q}}\Sigma^1 Q\xrightarrow{2^m\zeta_Q}\Sigma^1.
\end{aligned}$$

Hence we can immediately show

LEMMA 1.4. (1) *For any map* $f: S_i\to\Delta Q$ $(0\leqq i\leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^{i+1}\vee Q$ *or the following small spectrum* Y_i: i) $Y_0=N_m\vee SZ/q$; ii) $Y_1=\Sigma^3\vee P$; iii) $Y_3=Q'_{m+1}\vee\Sigma^3 SZ/q$ *where* $m\geqq 0$ *and* $q\geqq 1$ *is odd.*

(2) *For any map* $f: \Sigma^{i+1}\Delta Q\to S_0$ $(0\leqq i\leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^0\vee\Sigma^{i+2}Q$ *or the following small spectrum* Y_i: i) $Y_0=\Sigma^{-2}N'_m\vee SZ/q$; ii) $Y_1=\Sigma^{-1}\vee P$; iii) $Y_3=Q_{m+1}\vee SZ/q$ *where* $m\geqq 0$ *and* $q\geqq 1$ *is odd.*

1.4. Recall that $KO_iV_{m+1}\cong Z/2^m$, 0, $Z/2$, $Z/2$, $Z/2^{m+2}$, $Z/2$, $Z/2$, 0 according as $i=0, 1, \cdots, 7$.

LEMMA 1.5. (1) *For any map* $f: S_i\to\Delta V_{m+1}$ $(0\leqq i\leqq 7)$ *its cofiber* $C(f)$ *is* *quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1}\vee V_{m+1}$ or the following small spectrum Y_i*: i) $Y_0=\Sigma^1\vee V_{k+1}$ $(0\leqq k<m)$; ii) $Y_2=\Sigma^4 P_{m+1}$; iii) $Y_3=\Sigma^4 Q_{m+1}$; iv) $Y_4=\Sigma^4 R_{m+1}$ or $\Sigma^1\vee\Sigma^4 SZ/2^k$ $(0\leqq k\leqq m)$; v) $Y_5=M_{m+1}$; vi) $Y_6=N_{m+1}$.

(2) *For any map f: $\Sigma^{i-1}\Delta V_{m+1}\to S_0$ $(0\leqq i\leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^0\vee\Sigma^i V_{m+1}$ or the following small spectrum Y_i*: i) $Y_0=\Sigma^4 R'_{m+1}$ *or* $\Sigma^4\vee SZ/2^k$ $(0\leqq k\leqq m)$; ii) $Y_1=\Sigma^4 M'_{m+1}$; iii) $Y_2=\Sigma^4 N'_{m+1}$; iv) $Y_4=\Sigma^0\vee\Sigma^4 V_{k+1}$ $(0\leqq k<m)$; v) $Y_6=\Sigma^4 P'_{m+1}$; vi) $Y_7=\Sigma^4 Q'_{m+1}$.

Proof. Consider the following maps $g_{0,k}=2^k i_V i: \Sigma^0\to V_{m+1}$, $g_2=i_V\bar{\eta}: \Sigma^2\to V_{m+1}$, $g_3=i_V\bar{\eta}\eta: \Sigma^3\to V_{m+1}$, $g_{4,k}=2^k\bar{i}_V: C(\bar{\eta})\to V_{m+1}$, $g_5=\bar{i}_V(\eta\wedge 1): \Sigma^1 C(\bar{\eta})\to V_{m+1}$, $g_6=\bar{i}_V(\eta^2\wedge 1): \Sigma^2 C(\bar{\eta})\to V_{m+1}$. The cofibers $C(g_{0,k})$ and $C(g_{4,k})$ are respectively the wedge sums $\Sigma^1\vee V_{k+1}$ and $\Sigma^1\vee(C(\bar{\eta})\wedge SZ/2^k)$ whenever $0\leqq k\leqq m$, and $C(g_{4,m+1})$ coincides with the cofiber of the map $2^m(\bar{i}\wedge i)$: $\Sigma^0\to C(\bar{\eta})\wedge SZ/2^{m+1}$ which is quasi KO_*-equivalent to $\Sigma^4 R_{m+1}$ according to Lemma 1.2. On the other hand, the cofibers $C(g_2)$ and $C(g_3)$ coincide with those of the maps $i_P i\bar{\eta}$: $\Sigma^1 SZ/2\to P_m$ and $i_Q i\bar{\eta}$: $\Sigma^1 SZ/2\to Q_m$, and hence they are obtained as those of the maps $2^{m-1}\bar{i}\zeta_P$: $P\to C(\bar{\eta})$ and $2^{m-1}\bar{i}\zeta_Q$: $Q\to C(\bar{\eta})$. Further the cofibers $C(g_5)$ and $C(g_6)$ coincide with those of the maps $2^m(\bar{i}\wedge i_P)$: $\Sigma^0\to C(\bar{\eta})\wedge P$ and $2^m(\bar{i}\wedge i_Q)$: $\Sigma^0\to C(\bar{\eta})\wedge Q$. Therefore Lemmas 1.3 and 1.4 show that these four cofibers have the same quasi KO_*-types as $\Sigma^4 P_{m+1}$, $\Sigma^4 Q_{m+1}$, M_{m+1} and N_{m+1} respectively. Now our result of (1) is immediate, and (2) is dually shown to (1).

Denote by W_{m+1} and W'_{m+1} $(m\geqq 1)$ the 4-cells spectra constructed as the cofibers of the maps $i\bar{\eta}+\bar{\eta}j$: $\Sigma^1 SZ/2\to SZ/2^m$ and $i\bar{\eta}+\bar{\eta}j$: $\Sigma^1 SZ/2^m\to SZ/2$ respectively. Note that $\Sigma^4 W_{m+1}$ and $\Sigma^2 W'_{m+1}$ have the same quasi KO_*-type as W_{m+1} (see [4, Corollary 5.4] or (4.12) below). Recall that $KO_i W_{m+1}\cong Z/2^m$, 0, $Z/2$, 0 according as $i\equiv 0, 1, 2, 3 \bmod 4$.

LEMMA 1.6. (1) *For any map f: $S_i\to\Delta W_{m+1}$ $(0\leqq i\leqq 3)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1}\vee W_{m+1}$ or the following small spectrum Y_i*: i) $Y_0=\Sigma^5 Q'_{k+1}$ $(0\leqq k<m)$; ii) $Y_2=\Sigma^4 P_{m+1}$.

(2) *For any map f: $\Sigma^{i-1}\Delta W_{m+1}\to S_0$ $(0\leqq i\leqq 3)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^0\vee\Sigma^i W_{m+1}$ or the following small spectrum Y_i*: i) $Y_0=Q_{k+1}$ $(0\leqq k<m)$; ii) $Y_2=\Sigma^4 P'_{m+1}$.

Proof. Consider the following maps $g_{0,k}=2^k i_W i$: $\Sigma^0\to W_{m+1}$ and $g_2=i_W\bar{\eta}$: $\Sigma^2\to W_{m+1}$. The cofiber $C(g_{0,k})$ coincides with that of the map $(\eta j, i\bar{\eta})$: $\Sigma^1 SZ/2\to\Sigma^1\vee SZ/2^k$ whenever $0\leqq k<m$. Therefore it is the cofiber of the composite map $\eta j j_V$: $\Sigma^{-1}V_{k+1}\to\Sigma^1$, which is quasi KO_*-equivalent to $\Sigma^5 Q'_{k+1}$ according to Lemma 1.5. On the other hand, the cofiber $C(g_2)$ coincides with that of the map $i_P i\bar{\eta}$: $\Sigma^1 SZ/2\to P_m$, which is quasi KO_*-equivalent to $\Sigma^4 P_{m+1}$ as shown in the proof of Lemma 1.5.

2. The cofibers $X_{m,n}$ of maps $f: \Sigma^i SZ/2^n \to SZ/2^m$.

2.1. For any $m, n \geqq 1$ we here introduce 4-cells spectra $M_{m,n}$, $N_{m,n}$, $P_{m,n}$, $P'_{m,n}$, $P''_{m,n}$, $Q_{m,n}$, $Q'_{m,n}$, $Q''_{m,n}$, $R_{m,n}$, $R'_{m,n}$ and $R''_{m,n}$ constructed as the cofibers of the following maps respectively:

$$(2.1)\quad \begin{aligned} & i\eta j: SZ/2^n \longrightarrow SZ/2^m, \qquad i\eta^2 j: \Sigma^1 SZ/2^n \longrightarrow SZ/2^m, \\ & \tilde{\eta} j,\ i\bar{\eta} \ \text{ and } \ i\bar{\eta}+\tilde{\eta} j: \Sigma^1 SZ/2^n \longrightarrow SZ/2^m, \\ & \tilde{\eta}\eta j,\ i\eta\bar{\eta} \ \text{ and } \ i\eta\bar{\eta}+\tilde{\eta}\eta j: \Sigma^2 SZ/2^n \longrightarrow SZ/2^m, \ \text{ and} \\ & \tilde{\eta}\eta^2 j,\ i\eta^2\bar{\eta} \ \text{ and } \ i\eta^2\bar{\eta}+\tilde{\eta}\eta^2 j: \Sigma^3 SZ/2^n \longrightarrow SZ/2^m. \end{aligned}$$

Of course $M_{1,1}=SZ/2\wedge SZ/2$, $N_{1,1}=SZ/2\vee\Sigma^2 SZ/2$, $P_{1,n}=V'_{n+1}$, $P'_{m,1}=V_{m+1}$, $P''_{1,n}=W'_{n+1}$, $P''_{m,1}=W_{m+1}$, $P''_{m,m}=P\wedge SZ/2^m$ and $Q''_{m,m}=Q\wedge SZ/2^m$. Moreover we note that $\Sigma^2 P''_{m,n}$ are quasi KO_*-equivalent to $P''_{n,m}$ (see (4.12)). In [4, 4.2] the 4-cells spectra $M_{m,n}$, $N_{m,n}$, $P_{m,n}$, $P'_{m,n}$ and $P''_{m,n}$ are written to be $S_{2m,2n}$, $T_{2m,2n}$, $V'_{2m,2n}$, $V_{2m,2n}$ and $W_{2m,2n}$ respectively. As is easily checked, the maps $(\iota\wedge 1)\tilde{\eta}\eta^2 j: \Sigma^3 SZ/2^k \to KO\wedge SZ/2^l$ and $(\iota\wedge 1)i\eta^2\bar{\eta}: \Sigma^3 SZ/2^l \to KO\wedge SZ/2^k$ are trivial whenever $k<l$, and the map $(\iota\wedge 1)\ (i\eta^2\bar{\eta}+\tilde{\eta}\eta^2 j): \Sigma^3 SZ/2^k \to KO\wedge SZ/2^k$ is also trivial where $\iota: \Sigma^0\to KO$ denotes the unit of KO. So we notice that

(2.2) i) when $k<l$, $R_{l,k}$ and $R'_{k,l}$ have the same quasi KO_*-types as the wedge sums $SZ/2^l\vee\Sigma^4 SZ/2^k$ and $SZ/2^k\vee\Sigma^4 SZ/2^l$ respectively, and

ii) $R_{k,k}$ and $R'_{k,k}$ have the same quasi KO_*-type.

In addition, $R''_{m,n}$ has the same quasi KO_*-type as $R_{m,n}$, $SZ/2^m\vee\Sigma^4 SZ/2^n$ or $R'_{m,n}$ according as $m<n$, $m=n$ or $m>n$.

For any $m, n\geqq 1$ we moreover introduce 4-cells spectra $H_{m,n}$ ($(m,n)\neq(1,1)$), $K_{m,n}$ and $L_{m,n}$ constructed as the cofibers of the following maps respectively:

$$(2.3)\quad \begin{aligned} & \eta_{n,m}: \Sigma^2 SZ/2^n \longrightarrow SZ/2^m, \qquad \tilde{\eta}\bar{\eta}: \Sigma^3 SZ/2^n \longrightarrow SZ/2^m \ \text{ and} \\ & \tilde{\eta}\eta\bar{\eta}: \Sigma^4 SZ/2^n \longrightarrow SZ/2^m. \end{aligned}$$

Of course, $H_{m+1,1}=U_m$ and $H_{1,n+1}=U'_n$. Since the map $\bar{i}\bar{j}: \Sigma^{-1}C(\bar{\eta})\to\Sigma^2 C(\bar{\eta})$ is quasi KO_*-equivalent to the multiplication by 4 on Σ^6, the 4-cells spectrum $K_{1,1}$ has the same quasi KO_*-type as $\Sigma^6 SZ/4$. We can easily calculate the KU- and KO-homologies of these 4-cells spectra $X=X_{m,n}$ ($m, n\geqq 1$) as follows (cf. [4, Propositions 4.4 and 4.5]).

PROPOSITION 2.1. *The KU-homologies KU_0X, KU_1X and the conjugation ψ_C^{-1} on $KU_0X\oplus KU_1X$ are given as follows*:

$X=$	$M_{m,n}$	$N_{m,n}$	$P_{m,n}$		$P'_{m,n}$
			$m\geqq n+1$	$m\leqq n+1$	$m+1\leqq n$
$KU_0X\cong$	$Z/2^m$	$Z/2^m\oplus Z/2^n$	$Z/2^m\oplus Z/2^n$	$Z/2^{n+1}\oplus Z/2^{m-1}$	$Z/2^n\oplus Z/2^m$
$KU_1X\cong$	$Z/2^n$	0	0	0	0
$\psi_C^{-1}=$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&2^{m-n}\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&-2^{n-m+2}\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\-1&1\end{pmatrix}$

$X=$	$P'_{m,n}$	$P''_{m,n}$		
	$m+1\geqq n$	$m<n$	$m=n$	$m>n$
$KU_0X\cong$	$Z/2^{m+1}\oplus Z/2^{n-1}$	$Z/2^{n+1}\oplus Z/2^{m-1}$	$Z/2^n\oplus Z/2^m$	$Z/2^{m+1}\oplus Z/2^{n-1}$
$KU_1X\cong$	0	0	0	0
$\psi_C^{-1}=$	$\begin{pmatrix}1&0\\1&-1\end{pmatrix}$	$-A_{n-m}$	$\begin{pmatrix}-1&0\\-1&1\end{pmatrix}$	A_{m-n}

$X=$	$Q_{m,n}$, $Q'_{m,n}$, $Q''_{m,n}$	$R_{m,n}$, $R'_{m,n}$	$H_{m,n}$	$K_{m,n}$	$L_{m,n}$
			$(m,n)\neq(1,1)$	$(m,n)\neq(1,1)$	
$KU_0X\cong$	$Z/2^m$	$Z/2^m\oplus Z/2^n$	$Z/2^{m-1}$	$Z/2^m\oplus Z/2^n$	$Z/2^m$
$KU_1X\cong$	$Z/2^n$	0	$Z/2^{n-1}$	0	$Z/2^n$
$\psi_C^{-1}=$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&2^{m-1}\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Here $A_k=\begin{pmatrix}1-2^{k+1}&2^{k+2}(1-2^k)\\1&-1+2^{k+1}\end{pmatrix}$ *and this matrix operates on* $Z/2^{k+l+2}\oplus Z/2^l$ *as left action.*

PROPOSITION 2.2. *The KO-homologies* KO_iX $(0\leqq i\leqq 7)$ *are tabled as follows*:

$X\backslash i=$	0	1	2	3	4	5	6	7
$M_{m,n}$	$Z/2^m$	$Z/2^{n+1}$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$	$Z/2^{m+1}$	$Z/2^n$	0	0
$N_{m,n}$	$Z/2^m$	$Z/2$	$Z/2^{n+1}\oplus Z/2$	$Z/2\oplus Z/2$	$Z/2^{m+1}\oplus Z/2$	$Z/2$	$Z/2^n$	0
$P_{m,n}$	$Z/2^m$	$Z/2$	$(*)_{n,m}$	$Z/2$	$Z/2^{m-1}\oplus Z/2$	0	$Z/2^n$	0
$P'_{m,n}$	$Z/2^m$	0	$Z/2^{n-1}\oplus Z/2$	$Z/2$	$(*)_{m,n}$	$Z/2$	$Z/2^n$	0
$P''_{m,n}$	$Z/2^m$	0	$Z/2^n$	0	$Z/2^m$	0	$Z/2^n$	0
$Q_{m,n}$	$Z/2^m$	$Z/2$	$(*)_m$	$Z/2^{n+1}$	$Z/2^{m-1}\oplus Z/2$	$Z/2$	$Z/2$	$Z/2^n$
$Q'_{m,n}$	$Z/2^m$	$Z/2$	$Z/2$	$Z/2^{n-1}\oplus Z/2$	$Z/2^{m+1}$	$(*)_n$	$Z/2$	$Z/2^n$
$Q''_{m,n}$	$Z/2^m$	$Z/2$	$Z/2$	$Z/2^n$	$Z/2^m$	$Z/2$	$Z/2$	$Z/2^n$

$R_{m,n}$ $(m\leqq n)$	$Z/2^m\oplus Z/2^n$	$Z/2$	$(*)_m$	$Z/2$	$Z/2^{m-1}\oplus Z/2^{n+1}$	$Z/2$	$(*)_n$	$Z/2$
$R'_{m,n}$ $(m\geqq n)$	$Z/2^m\oplus Z/2^n$	$Z/2$	$(*)_m$	$Z/2$	$Z/2^{m+1}\oplus Z/2^{n-1}$	$Z/2$	$(*)_n$	$Z/2$
$H_{m,n}$ $(m,\ n\geqq 2)$	$Z/2^m$	$Z/2$	$Z/2$	$Z/2^{n-1}$	$Z/2^{m-1}$	$Z/2$	$Z/2$	$Z/2^n$
$K_{m,n}$	$Z/2^m\oplus Z/2^n$	$Z/2$	$(*)_m$	0	$Z/2^{m-1}\oplus Z/2^{n-1}$	0	$(*)_n$	$Z/2$
$L_{m,n}$	$Z/2^m\oplus Z/2$	$Z/2^n\oplus Z/2$	$(*)_m$	$Z/2$	$Z/2^{m-1}$	$Z/2^{n-1}$	$Z/2$	$(*)_n$

Here $(*)_{m,1}\cong Z/2^{m+2}$ *and* $(*)_{m,n}\cong Z/2^{m+1}\oplus Z/2$ *if* $n\geqq 2$, *and* $(*)_{0,n}$ *is abbreviated to be* $(*)_n$.

For the 4-cells spectra $R_{m,n}$ and $R'_{n,m}$ $(2\leqq m\leqq n)$ their KU-, KO- and KT-homologies are all equal, but their induced homomorphisms by $\tau:\ \Sigma^1 KT\rightarrow KO$ (see [1] or [3]) are not equal when $m<n$. In fact, the induced homomorphisms $\tau_*:\ KT_{2i}X\rightarrow KO_{2i+1}X$ are represented by the following rows T_{2i+1} for $X=R_{m,n}$ $(m\leqq n)$ and $R'_{m,n}$ $(m\geqq n)$:

$$\begin{aligned} &T_1=(1\ 1):\ Z/2^m\oplus Z/2^n\longrightarrow Z/2, \qquad T_3=(1\ 0):\ Z/2\oplus Z/2\longrightarrow Z/2,\\ &T_5=(0\ 1):\ Z/2^m\oplus Z/2^n\longrightarrow Z/2, \qquad T_7=(0\ 1):\ Z/2\oplus Z/2\longrightarrow Z/2. \end{aligned} \tag{2.4}$$

2.2. We here show

LEMMA 2.3. *For any map* $f:\ \Sigma^{i-1}SZ_n\rightarrow SZ_m$ $(0\leqq i\leqq 7)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $SZ/2^m\vee\Sigma^i SZ/2^n$ *or the following small spectrum* Y_i: i) $Y_0=SZ/2^k\vee SZ/2^{m+n-k}$ $(0\leqq k<\mathrm{Min}\{m,\ n\})$; ii) $Y_1=M_{m,n}$, $SZ/2^k\vee\Sigma^1 SZ/2^{n-m+k}$, $M_{k,n-m+k}$, $SZ/2^{m-n+l}\vee\Sigma^1 SZ/2^l$ *or* $M_{m-n+l,l}$ $(0\leqq k<m\leqq n$ *and* $0\leqq l<n\leqq m)$; iii) $Y_2=N_{m,n}$, $P_{m,n}$, $P'_{m,n}$ *or* $P''_{m,n}$; iv) $Y_3=Q_{m,n}$, $Q'_{m,n}$, $Q''_{m,n}$ *or* $H_{m,n}$; v) $Y_4=R_{m,n}\,(m\leqq n)$, $R'_{m,n}\,(m\geqq n)$, $K_{m,n}$, $\Sigma^4 V_{k+1}\vee V_{m+n-k-1}$ *or* $\Sigma^4 V_{k+1}\vee W_{m+n-k-1}$ $(0\leqq k<\mathrm{Min}\{m-1,\ n-1\})$; vi) $Y_5=L_{m,n}$, $\Sigma^4 V_{k+1}\vee\Sigma^5 V_{n-m+k+1}$ *or* $\Sigma^4 V_{m-n+l+1}\vee\Sigma^5 V_{l+1}$ $(0\leqq k<m-1<n$ *and* $0\leqq l<n-1<m)$.

Proof. Consider the following maps: i) $g_{0,k}=2^k ij:\ \Sigma^{-1}SZ/2^n\rightarrow SZ/2^m$, ii) $g_1=i\eta j:\ SZ/2^n\rightarrow SZ/2^m$, $g_{1,k}=2^k\rho_{n,m}:\ SZ/2^n\rightarrow SZ/2^m$, $g'_{1,k}=2^k\rho_{n,m}+i\eta j:$ $SZ/2^n\rightarrow SZ/2^m$, iii) $g_2=i\eta^2 j$, $\tilde{\eta}j$, $i\bar{\eta}$, $i\bar{\eta}+\tilde{\eta}j:\ \Sigma^1 SZ/2^n\rightarrow SZ/2^m$, iv) $g_3=\tilde{\eta}\eta j$, $i\eta\bar{\eta}$, $i\eta\bar{\eta}+\tilde{\eta}\eta j$, $\eta_{n,m}:\ \Sigma^2 SZ/2^n\rightarrow SZ/2^m$, v) $g_4=\tilde{\eta}\eta^2 j$, $i\eta^2\bar{\eta}$, $\tilde{\eta}\bar{\eta}:\ \Sigma^3 SZ/2^n\rightarrow SZ/2^m$, $g_{4,k}=2^k i(\bar{\lambda}\wedge j):\ \Sigma^{-1}C(\bar{\eta})\wedge SZ/2^n\rightarrow SZ/2^m$, $g'_{4,k}=2^k i(\bar{\lambda}\wedge j)+\tilde{\eta}j\bar{\eta}_{n,1}:$ $\Sigma^{-1}C(\bar{\eta})\wedge SZ/2^n\rightarrow SZ/2^m$ and vi) $g_5=\tilde{\eta}\eta\bar{\eta}:\ \Sigma^4 SZ/2^n\rightarrow SZ/2^m$, $g_{5,k}=2^k(\bar{\lambda}\wedge\rho_{n,m}):$ $C(\bar{\eta})\wedge SZ/2^n\rightarrow SZ/2^m$ where $\rho_{n,m}:\ SZ/2^n\rightarrow SZ/2^m$ is the obvious map and $\bar{\eta}_{n,1}=\bar{\eta}_{2,1}(1\wedge\rho_{n,2}):\ C(\bar{\eta})\wedge SZ/2^n\rightarrow\Sigma^2 SZ/2$ for the map $\bar{\eta}_{2,1}$ given in 1.1. For any k with $0\leqq k<\mathrm{Min}\{m,\ n\}$ the cofiber $C(g_{0,k})$ is the wedge sum $SZ/2^k\vee SZ/2^{m+n-k}$, and $C(g_{1,k})$ is the wedge sum $SZ/2^k\vee\Sigma^1 SZ/2^{n-m+k}$ or $SZ/2^{m-n+k}\vee$

$\Sigma^1 SZ/2^k$ according as $m\leqq n$ or $m\geqq n$. The cofiber $C(g'_{1,k})$ is obtained as that of the map $(2^{n-m+k}, \imath\eta)\colon \Sigma^1\to\Sigma^1\vee SZ/2^k$ when $m\leqq n$, and as that of the map $2^{m-n+k}\vee\eta\jmath\colon \Sigma^0\vee SZ/2^k\to\Sigma^0$ when $m\geqq n$. Therefore it is the 4-cells spectrum $M_{k,n-m+k}$ or $M_{m-n+k,k}$ according as $m\leqq n$ or $m\geqq n$. Assume that $0\leqq k<\mathrm{Min}\{m-1, n-1\}$. For the cofiber sequence

$$\Sigma^1 SZ/2 \longrightarrow U_{n-1} \xrightarrow{\pi_U} C(\bar\eta)\wedge SZ/2^n \xrightarrow{\bar\eta_{n,1}} \Sigma^2 SZ/2$$

we note that $(1\wedge\jmath)\pi_U=\bar\jmath_U\colon U_{n-1}\to\Sigma^1 C(\bar\eta)$ and the cofiber of the map $2^k\imath\bar\lambda\bar\jmath_U\colon \Sigma^{-1}U_{n-1}\to SZ/2^m$ is the wedge sum $SZ/2^{m+n-k-2}\vee U_{k+1}$. As is easily checked, the cofibers $C(g_{4,k})$ and $C(g'_{4,k})$ coincide with those of the maps $(\imath\bar\eta, 0)$ and $(\imath\bar\eta+\tilde\eta\jmath,\ a\imath_U\eta^2\jmath)\colon \Sigma^1 SZ/2\to SZ/2^{m+n-k-2}\vee U_{k+1}$ for some $a\in Z/2$. So they are respectively the wedge sums $V_{m+n-k-1}\vee U_{k+1}$ and $W_{m+n-k-1}\vee U_{k+1}$ because $\imath_U\eta^2\jmath=\imath_U\eta\jmath(\imath\bar\eta+\tilde\eta\jmath)$. Of course, $C(g_{4,k})$ may be determined more easily since it is obtained as the cofiber of the map $2^{m+n-k-2}\tilde\imath\vee 0\colon \Sigma^0\vee\Sigma^{-1}U_{k+1}\to C(\bar\eta)$. On the other hand, the cofiber $C(g_{5,k})$ is obtained as that of the map $(2^{n-m+k}\bar\lambda, 0)\colon \Sigma^1 C(\bar\eta)\to\Sigma^1\vee U_{k+1}$ when $m\leqq n$, and as that of the map $(2^k\bar\lambda, 0)\colon \Sigma^1 C(\bar\eta)\to \Sigma^1\vee U_{m-n+k+1}$ when $m\geqq n$. Therefore it is the wedge sum $\Sigma^1 U_{n-m+k+1}\vee U_{k+1}$ or $\Sigma^1 U_{k+1}\vee U_{m-n+k+1}$ according as $m\leqq n$ or $m\geqq n$. Since $\tilde\eta\jmath+\imath\eta^2\jmath=(1+\imath\eta\jmath)\tilde\eta\jmath$, $\eta_{n,m}+\tilde\eta\eta\jmath=\eta_{n,m}(1+\imath\eta\jmath)$, $\tilde\eta\bar\eta+\tilde\eta\eta^2\jmath=\tilde\eta\bar\eta(1+\imath\eta\jmath)$ and so on, our result is now established.

For any $m,\ n\geqq 2$ we here consider the map $\nu_{n,m}=\eta_{1,m}\eta_{n,1}\colon \Sigma^4 SZ/2^n\to SZ/2^m$ satisfying $\nu_{n,m}\imath=\tilde6\tilde\nu$ and $\jmath\nu_{n,m}=\bar6\bar\nu$. Then lemma 2.3 asserts that

(2.5) the cofibers of the maps $\tilde6\tilde\nu\jmath$ and $\tilde6\tilde\nu\jmath+\tilde\eta\bar\eta\colon \Sigma^3 SZ/2^n\to SZ/2^m\ (2\leqq m\leqq n)$, $\imath\bar6\bar\nu$ and $\imath\bar6\bar\nu+\tilde\eta\bar\eta\colon \Sigma^3 SZ/2^n\to SZ/2^m\ (2\leqq n\leqq m)$ and $\nu_{n,m}\colon \Sigma^4 SZ/2^n\to SZ/2^m (m, n\geqq 2)$ have the same quasi KO_*-types as the wedge sums $\Sigma^4 V_{m-1}\vee V_{n+1}$, $\Sigma^4 V_{m-1}\vee W_{n+1}$, $\Sigma^4 V_{n-1}\vee V_{m+1}$, $\Sigma^4 V_{n-1}\vee W_{m+1}$ and $\Sigma^4 V_{m-1}\vee\Sigma^5 V_{n-1}$ respectively.

In fact, these cofibers are obtained as those of the composite maps $\imath'_V\jmath_U\colon \Sigma^{-1}U_{m-1}\to\Sigma^2 V'_{n+1}$, $\imath'_W\jmath_U\colon \Sigma^{-1}U_{m-1}\to\Sigma^2 W'_{n+1}$, $\imath'_U\jmath_V\colon \Sigma^{-1}V_{m+1}\to\Sigma^1 U'_{n-1}$, $\imath'_U\jmath_W\colon \Sigma^{-1}W_{m+1}\to\Sigma^1 U'_{n-1}$ and $\imath'_U\jmath_U\colon \Sigma^{-1}U_{m-1}\to\Sigma^2 U'_{n-1}$. Since $\jmath_U=\bar\jmath\bar\jmath_U\colon \Sigma^{-1}U_{m-1}\to \Sigma^2 SZ/2$ and $\imath'_U=\tilde\imath'_U\tilde\imath\colon SZ/2\to U'_{n-1}$, the first two maps are KO_*-trivial when $2\leqq m\leqq n$, the next two maps are KO_*-trivial when $2\leqq n\leqq m$, and the last one is always KO_*-trivial. Hence our assertion (2.5) is certainly valid.

2.3. The cofibers of the maps $2^k\imath_P\jmath\colon \Sigma^{-1}SZ/2^m\to P$ and $2^k\imath\jmath_P\colon P\to\Sigma^2 SZ/2^m$ are the wedge sums $\Sigma^0\vee M_k$ and $\Sigma^3\vee\Sigma^1 M'_k$ respectively whenever $0\leqq k<m$. So we obtain

LEMMA 2.4. (1) *For any map* $f\colon \Sigma^{i-1}SZ_m\to\Delta P\ (0\leqq i\leqq 1)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^i SZ/2^m\vee P$ *or the following small spectrum* Y_i: $Y_0=\Sigma^0\vee M_k\ (0\leqq k<m)$.

(2) *For any map* $f\colon \Sigma^i\Delta P\to SZ_m\ (0\leqq i\leqq 1)$ *its cofiber* $C(f)$ *is quasi* KO_*-

equivalent to the wedge sum $SZ/2^m \vee \Sigma^{i+1}P$ *or the following small spectrum* Y_i: $Y_0 = \Sigma^1 \vee \Sigma^{-1} M'_k$ $(0 \leqq k < m)$.

The cofibers of the maps $2^k i_Q j$: $\Sigma^{-1} SZ/2^m \to Q$, $i_Q \eta j$: $SZ/2^m \to Q$, $i_Q \bar{\eta}$: $\Sigma^1 SZ/2^m \to Q$ and $2^k \xi_Q j$: $\Sigma^2 SZ/2^m \to Q$ are the wedge sums $\Sigma^0 \vee N_k$, $\Sigma^3 \vee M'_m$, $\Sigma^3 \vee P'_m$ and $\Sigma^3 \vee Q'_{k+1}$ respectively whenever $0 \leqq k < m$. From this fact and its dual we obtain.

LEMMA 2.5. (1) *For any map* f: $\Sigma^{i-1} SZ_m \to \Delta Q$ $(0 \leqq i \leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^i SZ/2^m \vee Q$ *or the following small spectrum* Y_i: i) $Y_0 = \Sigma^0 \vee N_k$ $(0 \leqq k < m)$; ii) $Y_1 = \Sigma^3 \vee M'_m$; iii) $Y_2 = \Sigma^3 \vee P'_m$; iv) $Y_3 = \Sigma^3 \vee Q'_{k+1}$ $(0 \leqq k < m)$..

(2) *For any map* f: $\Sigma^{i+1} \Delta Q \to SZ_m$ $(0 \leqq i \leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $SZ/2^m \vee \Sigma^{i+2} Q$ *or the following small spectrum* Y_i: i) $Y_0 = \Sigma^1 \vee \Sigma^{-2} N'_k$ $(0 \leqq k < m)$; ii) $Y_1 = \Sigma^{-1} \vee M_m$; iii) $Y_2 = \Sigma^0 \vee P_m$; iv) $Y_3 = \Sigma^1 \vee Q_{k+1}$ $(0 \leqq k < m)$.

The cofibers of the maps $2^k i_P \bar{j}_V$: $\Sigma^{-1} V_{m+1} \to P$ and $2^k \bar{i}_U j_P$: $P \to \Sigma^2 U_{m+1}$ are the wedge sums $C(\bar{\eta}) \vee M_k$ and $\Sigma^3 C(\bar{\eta}) \vee \Sigma^1 M'_k$ respectively whenever $0 \leqq k \leqq m$. So we obtain

LEMMA 2.6. (1) *For any map* f: $\Sigma^{i-1} \Delta V_{m+1} \to \Delta P (0 \leqq i \leqq 1)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^i V_{m+1} \vee P$ *or the following small spectrum* Y_i: $Y_0 = \Sigma^4 \vee M_k$ $(0 \leqq k \leqq m)$.

(2) *For any map* f: $\Sigma^i \Delta P \to \Delta V_{m+1}$ $(0 \leqq i \leqq 1)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $V_{m+1} \vee \Sigma^{i+1} P$ *or the following small spectrum* Y_i: $Y_0 = \Sigma^1 \vee \Sigma^3 M'_k$ $(0 \leqq k \leqq m)$.

LEMMA 2.7. (1) *For any* map f: $\Sigma^{i-1} \Delta V_{m+1} \to \Delta Q$ $(0 \leqq i \leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^i V_{m+1} \vee Q$ *or the following small spectrum* Y_i: i) $Y_0 = \Sigma^4 \vee N_k$ $(0 \leqq k \leqq m)$; ii) $Y_1 = \Sigma^3 \vee \Sigma^4 M'_{m+1}$; iii) $Y_2 = \Sigma^7 \vee P'_{m+1}$; iv) $Y_3 = \Sigma^7 \vee Q'_{k+1}$ $(0 \leqq k \leqq m)$.

(2) *For any map* f: $\Sigma^{i+1} \Delta Q \to \Delta V_{m+1}$ $(0 \leqq i \leqq 3)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $V_{m+1} \vee \Sigma^{i+2} Q$ *or the following small spectrum* Y_i: i) $Y_0 = \Sigma^1 \vee \Sigma^2 N'_k$ $(0 \leqq k \leqq m)$; ii) $Y_1 = \Sigma^3 \vee M_{m+1}$; iii) $Y_2 = \Sigma^0 \vee \Sigma^4 P_{m+1}$; iv) $Y_3 = \Sigma^1 \vee \Sigma^4 Q_{k+1}$ $(0 \leqq k \leqq m)$.

Proof. Consider the following maps $g_{0,k} = 2^k i_Q \bar{j}_V$: $\Sigma^{-1} V_{m+1} \to Q$, $g_1 = i_Q \eta \bar{j}_V$: $V_{m+1} \to Q$, $g_2 = i_Q j j_V$: $\Sigma^1 V_{m+1} \to \Sigma^4 Q$ and $g_{3,k} = 2^k \xi_Q \bar{j}_V$: $\Sigma^2 V_{m+1} \to Q$. The cofiber $C(g_{0,k})$ is the wedge sum $C(\bar{\eta}) \vee N_k$ whenever $0 \leqq k \leqq m$, and $C(g_1)$ and $C(g_2)$ are the wedge sums $\Sigma^3 \vee C(\eta \bar{j}_V)$ and $\Sigma^7 \vee \Sigma^2 M_m$ respectively. Here the cofiber $C(\eta \bar{j}_V)$ has the same quasi KO_*-type as $\Sigma^4 M'_{m+1}$ in virtue of Lemma 1.5. On the other hand, the cofiber $C(g_{3,k})$ coincides with that of the map $2^m \bar{i} j j'_Q$: $\Sigma^{-1} Q'_{k+1} \to \Sigma^3 C(\bar{\eta})$. When $0 \leqq k < m$ it is just the wedge sum $\Sigma^3 C(\bar{\eta}) \vee Q'_{k+1}$, and when $k = m$ it has the same quasi KO_*-type as $\Sigma^3 C(\bar{\eta}) \vee Q'_{m+1}$ because the map

$2^m\bar{i}j: \Sigma^3SZ/2^{m+1}\to\Sigma^4C(\bar{\eta})$ is quasi KO_*-equivalent to the map $\eta^2\bar{\eta}: \Sigma^3SZ/2^{m+1}\to\Sigma^0$. Our result of (1) is now immediate, and (2) is dually shown to (1).

3. Some small spectra $XV_{m,n}$, $VX_{m,n}$ and $X'X_{n,m}$.

3.1. For any maps $f: \Sigma^i\to SZ/2^m$ and $g: \Sigma^j\to SZ/2^m$ $(i\leqq j)$ we denote by XY_m the cofiber of the map $f\vee g: \Sigma^i\vee\Sigma^j\to SZ/2^m$ when the cofibers of the maps f and g are denoted by X_m and Y_m respectively. Dually we denote by $X'Y'_m$ the cofiber of the map $(f', g'): \Sigma^jSZ/2^m\to\Sigma^{j-i}\vee\Sigma^0$ when the cofibers of any maps $f': \Sigma^iSZ/2^m\to\Sigma^0$ and $g': \Sigma^jSZ/2^m\to\Sigma^0$ $(i\leqq j)$ are denoted by X'_m and Y'_m respectively. In [5] these 4-cells spectra XY_m and $X'Y'_m$ are written to be XY_{2m} and XY'_{2m}, and their KU- and KO-homologies have been calculated in [5, Propositions 1.2 and 1.3] when $X=M$ or N, and $Y=P$, Q or R. Let X_m and Y'_m denote the cofibers of any maps $f: \Sigma^i\to SZ/2^m$ and $g': \Sigma^jSZ/2^m\to\Sigma^0$. If the composite map $g'f: \Sigma^{i+j}\to\Sigma^0$ is trivial, then the maps f and g' admit a coextension $h: \Sigma^{i+j+1}\to Y'_m$ and an extension $k: \Sigma^jX_m\to\Sigma^0$ so that their cofibers $C(h)$ and $C(k)$ coincide. Its coincident cofiber is denoted by $Y'X_m$ when a suitable pair (h, k) is chosen as in [5, (2.1) and (2.2)]. In [5] these 4-cells spectra $Y'X_m$ are written to be $Y'X_{2m}$, and their KU- and KO-homologies have been calculated in [5, Propositions 2.3 and 2.4].

For any map $f: \Sigma^iSZ/2\to SZ/2^m$ we denote by $XV_{m,n}$ $(m, n\geqq 1)$ the cofiber of the map $(f, i\bar{\eta}): \Sigma^iSZ/2\to SZ/2^m\vee\Sigma^{i-1}SZ/2^{n-1}$ when the cofiber of the map f is denoted by $X_{m,1}$. We are interested in $XV_{m,n}$ only when $X=M$, N, P and Q because the other cases are of little importance. Note that $XV_{m,1}=X_{m,1}$ and $NV_{m,n}=SZ/2^m\vee V_n$ whenever $m\leqq n$. In [7, (2.2)] the small spectrum $PV_{m,n}$ is written to be $U_{n-1,m,1}$. Moreover we introduce new small spectra $NV^k_{m,n}$, $PV^k_{m,n}$ and $QV^0_{m,n}$ $(m, n\geqq 1$ and $k\geqq 0)$ constructed as the cofibers of the following maps respectively:

$$
\begin{aligned}
&g^k_N=2^ki\bar{j}_V+i\eta^2jj_V: \Sigma^{-1}V_n \longrightarrow SZ/2^m,\\
(3.1)\qquad &g^k_P=2^ki\bar{j}_V+\bar{\eta}jj_V: \Sigma^{-1}V_n \longrightarrow SZ/2^m \quad\text{and}\\
&g^0_Q=i\eta\bar{j}_V+\bar{\eta}\eta jj_V: V_n \longrightarrow SZ/2^m.
\end{aligned}
$$

Since $2^{n-1}\bar{j}_V=\bar{\eta}j_V: V_n\to\Sigma^1$, it is immediate that $g^n_N=0$, $g^n_P=(1+i\eta j)\bar{\eta}jj_V$, $g^k_N=i\eta^2jj_V$, $g^k_P=\bar{\eta}jj_V$ and $g^l_N=2^l(1+2^{n-l})i\bar{j}_V$ when $k\geqq\mathrm{Min}\{m, n+1\}$ and $l<n$. Hence it is easily shown that

$$
(3.2)\qquad
\begin{aligned}
NV^k_{m,n}&=\begin{cases} SZ/2^m\vee V_n & \text{when } k=n\\ NV_{m,n} & \text{when } k\geqq\mathrm{Min}\{m, n+1\}\\ SZ/2^k\vee V_{m+n-k} & \text{when } k<\mathrm{Min}\{m, n\}\end{cases}\\
PV^k_{m,n}&=\begin{cases} PV_{m,n} & \text{when } k\geqq\mathrm{Min}\{m, n\}\\ SZ/2^k\vee W_{m+n-k} & \text{when } k<\mathrm{Min}\{m, n\}.\end{cases}
\end{aligned}
$$

For any map $f: \Sigma^i SZ/2^n \to SZ/2$ there exists a map $h: \Sigma^{i+2}SZ/2^n \to V_m$ satisfying $j_V h = f$ if the composite map $i\bar{\eta} f: \Sigma^{i+1}SZ/2^n \to SZ/2^{m-1}$ is trivial. By choosing such a map h suitably we introduce a new small spectrum $VX_{m,n}$ $(m, n \geqq 1)$ constructed as the cofiber of its map h when the cofiber of the map f is denoted by $X_{1,n}$. Evidently $VX_{1,n} = \Sigma^2 X_{1,n}$. Choose a map $\bar{\xi}_V: \Sigma^5 SZ/2^n \to V_m$ satisfying $j_V \bar{\xi}_V = \bar{\eta}\bar{\eta}$, and then set $\xi_V = \bar{\xi}_V i: \Sigma^5 \to V_m$. Such a map ξ_V with $j_V \xi_V = \bar{\eta}\eta$ is uniquely determined, although $\bar{\xi}_V$ is unique only up to quasi KO_*-equivalences. We are only interested in the following new spectra $VQ_{m,n}$, $VR_{m,n}$, $VK_{m,n}$ and $VL_{m,n}$ $(m, n \geqq 1)$ constructed as the cofibers of the maps $\xi_V j: \Sigma^4 SZ/2^n \to V_m$, $\xi_V \eta j: \Sigma^5 SZ/2^n \to V_m$, $\bar{\xi}_V: \Sigma^5 SZ/2^n \to V_m$ and $\bar{\xi}_V(\eta \wedge 1): \Sigma^6 SZ/2^n \to V_m$ respectively. According to Lemma 1.5 the cofibers $C(\xi_V)$ and $C(\xi_V \eta)$ have the same quasi KO_*-types as the elementary spectra M_m and N_m respectively. The cofibers $C(\xi_V j)$, $C(\bar{\xi}_V)$ and $C(\bar{\xi}_V(\eta \wedge 1))$ are given as those of certain maps $g_Q: C(\xi_V) \to \Sigma^6$, $g_K: \Sigma^6 \to C(\xi_V)$ and $g_L: \Sigma^7 \to C(\xi_V \eta)$, which induce $g_Q^*(1) = 2^n \in KO^6 C(\xi_V) \cong Z$, $g_{K*}(1) = 2^{n-1} \in KO_6 C(\xi_V) \cong Z$ and $g_{L*}(1) = 2^{n-1} \in KO_7 C(\xi_V \eta) \cong Z$. Applying Propositions 4.1 and 4.2 and the dual of Proposition 4.5 established below we can observe that

(3.3) the small spectra $VQ_{m,n}$, $VK_{m,n}$ and $VL_{m,n}$ are quasi KO_*-equivalent to $\Sigma^5 H_{n+1,m+1}$, $\Sigma^6 P'_{n-1,m+1}$ and $MV_{m,n}$ respectively. In particular, $Q_{1,n}$, $K_{1,n}$ and $L_{1,n}$ are quasi KO_*-equivalent to $\Sigma^3 H_{n+1,2}$, $\Sigma^4 P'_{n-1,2}$ and $\Sigma^6 MV_{1,n}$ respectively.

3.2. We can easily compute the KU- and KO-homologies of the new small spectra $Y = XV_{m,n}$, $QV^0_{m,n}$ and $VR_{m,n}$ $(m, n \geqq 1)$ for $X = M, N, P$ and Q, where $XV_{m,1} = X_{m,1}$, $QV^0_{m,1} = Q''_{m,1}$ and $VR_{1,n} = \Sigma^2 R_{1,n}$.

PROPOSITION 3.1. i) *The KU-homologies KU_0Y, KU_1Y and the conjugation ψ_C^{-1} on $KU_0Y \oplus KU_1Y$ are given as follows:*

$Y=$	$MV_{m,n}$	$NV_{m,m}$	$PV_{m,n}$	$QV_{m,n}$ $QV^0_{m,n}$	$VR_{m,n}$
$KU_0Y \cong$	$Z/2^m$	$Z/2^m \oplus Z/2^n$	$Z/2^m \oplus Z/2^n$	$Z/2^m$	$Z/2^m \oplus Z/2^n$
$KU_1Y \cong$	$Z/2^n$	0	0	$Z/2^n$	0
$\psi_C^{-1} =$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2^{m-1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

ii) *The KO-homologies KO_iY $(0 \leqq i \leqq 7)$ are tabled as follows:*

$Y \backslash i =$	0	1	2	3	4	5	6	7
$MV_{m,n}$	$Z/2^m$	$(*)_n$	$Z/2 \oplus Z/2$	$Z/2^n \oplus Z/2$	$Z/2^{m+1}$	$Z/2$	0	$Z/2^{n-1}$
$NV_{m,n}$ $(m>n)$	$Z/2^m \oplus Z/2^{n-1}$	$Z/2$	$(*)_n \oplus Z/2$	$Z/2 \oplus Z/2$	$Z/2^{m+1} \oplus Z/2^n$	$Z/2$	$Z/2$	0

$PV_{m,n}$ $(n\geqq 2)$	$Z/2^m\oplus Z/2^{n-1}$	$Z/2$	$(*)_m\oplus Z/2$	$Z/2$	$Z/2^{m-1}\oplus Z/2^n$	0	$Z/2$	0
$QV_{m,n}$	$Z/2^m$	$Z/2^{n-1}\oplus Z/2$	$(*)_m$	$(*)_n$	$Z/2^{m-1}\oplus Z/2$	$Z/2^n$	$Z/2$	$Z/2$
$QV^0_{m,n}$	$Z/2^m$	$Z/2^n$	$Z/2$	$Z/2$	$Z/2^m$	$Z/2^n$	$Z/2$	$Z/2$
$VR_{m,n}$ $(m\geqq 2)$	$Z/2^m\oplus Z/2$	$Z/2$	$Z/2^n\oplus Z/2$	$Z/2$	$Z/2^{m+1}$	$Z/2$	$Z/2^{n+1}$	$Z/2$

in which $(*)_1\cong Z/4$ and $(*)_l\cong Z/2\oplus Z/2$ if $l\geqq 2$.

For the small spectra $QV^0_{m,n}$ and $\Sigma^1 Q''_{n,m}$ their KU- and KO-homologies are equal, but their KT-homologies are not equal. In fact,

(3.4) i) $KT_i QV^0_{m,n}\cong Z/2^m\oplus Z/2^n$, $Z/2^{n+1}$, $Z/2\oplus Z/2$, $Z/2^{m+1}$ according as $i=0, 1, 2, 3$ when $n\geqq 2$;

ii) $KT_0 Q''_{m,n}\cong Z/2^m\oplus Z/2$, $KT_1 Q''_{m,n}\cong Z/4$, $Z/4$ or $Z/2\oplus Z/2$ when $m>n=1$, $n>m=1$ or otherwise, $KT_2 Q''_{m,n}\cong Z/2^n\oplus Z/2$ and $KT_3 Q''_{m,n}\cong Z/2^{m+1}\oplus Z/2^{n-1}$, $Z/2^m\oplus Z/2^n$ or $Z/2^{m-1}\oplus Z/2^{n+1}$ when $m>n$, $m=n$ or $m<n$.

3.3. Consider the maps

$$\phi_n=2^{n-1}i'_N\bar{\lambda}:\ C(\bar{\eta})\longrightarrow N'_m \quad\text{and}$$
(3.5)
$$\phi_{n,0}=2^{n-1}i'_N\bar{\lambda}+h'_N\eta j\bar{j}:\ C(\bar{\eta})\longrightarrow N'_m$$

where the map $h'_N:\ \Sigma^2\to N'_m$ given in (1.12) satisfies $j'_N h'_N=\iota$ and $2^m h'_N=i'_N\eta^2$. Since it coincides with the cofiber of the map $i_U\eta^2 j:\ \Sigma^1 SZ/2^m\to U_n$, the cofiber $C(\phi_n)$ is quasi KO_*-equivalent to the small spectrum $\Sigma^4 VR_{n,m}$ constructed as the cofiber of the map $\xi_V\eta j:\ \Sigma^9 SZ/2^m\to\Sigma^4 V_n$. On the other hand, the cofiber $C(\phi_{n,0})$, denoted by $N'N_{n,m}$ $(m, n\geqq 1)$, has the following KU- and KO-homologies:

PROPOSITION 3.2. i) *$KU_0 N'N_{n,m}\cong Z/2^n\oplus Z/2^m$ on which* $\psi_C^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ *and* $KU_1 N'N_{n,m}=0$.

ii) $KO_i N'N_{n,m}\cong Z/2^{n+1}$, $Z/2$, $Z/2^{m+1}$, $Z/2$ *according as* $i\equiv 0, 1, 2, 3 \bmod 4$ *unless* $(m, n)=(1, 1)$, *and* $KO_i N'N_{1,1}\cong Z/4$, $Z/2$, $Z/4$, $Z/2$, $Z/2\oplus Z/2$, $Z/2$, $Z/4$, $Z/2$ *according as* $i=0, 1, \cdots, 7$.

Denote by $\bar{R}'_m$ $(m\geqq 1)$ the cofiber of the map $2^{m-1}(\bar{\lambda}\wedge j):\ \Sigma^{-1}C(\bar{\eta})\wedge SZ/2^m\to\Sigma^0$, which has the same quasi KO_*-type as the elementary spectrum R'_m. Then there exists a cofiber sequence

$$\text{(3.6)}\qquad C(\bar{\eta})\xrightarrow{(2^{m-1}\bar{\lambda},\ 2^m)}\Sigma^0\vee C(\bar{\eta})\xrightarrow{\bar{\rho}'_R}\bar{R}'_m\xrightarrow{(1\wedge j)\bar{j}'_R}\Sigma^1 C(\bar{\eta})$$

where $\bar{j}'_R:\ \bar{R}'_m\to C(\bar{\eta})\wedge SZ/2^m$ is the canonical projection. Using the map $f_{n,k}=$

$(2^k+2^n, 2^{k-1}\bar{i})$: $\Sigma^0\to\Sigma^0\vee C(\bar{\eta})$ we here introduce a new small spectrum $R'_{n,k,m}$ constructed as the cofiber of the composite map $\bar{\rho}'_R f_{n,k}$: $\Sigma^0\to\bar{R}'_m$. Assume that $1\leqq k<m$. Then the small spectrum $R'_{n,k,m}$ coincides with the cofiber of the map $h_{n,k,m}=2^{m+n-k-1}\bar{\lambda}\vee(2^n+2^k)\bar{j}_V$: $C(\bar{\eta})\vee\Sigma^{-1}V_k\to\Sigma^0$ because $\bar{j}'_R\bar{\rho}'_R f_{n,k}=2^{k-1}(\bar{i}\wedge i)$: $\Sigma^0\to C(\bar{\eta})\wedge SZ/2^m$. Note that $h_{n,k,m}=2^s\bar{\lambda}\vee 2^n\bar{j}_V$, $2^s\bar{\lambda}\vee 0$ or $2^s\bar{\lambda}\vee 2^k\bar{j}_V$ according as $k>n$, $k=n$ or $k<n$ where $s=m+n-k-1$. When $k=n$ the cofiber $C(h_{n,k,m})$ is evidently the wedge sum $U_m\vee V_n$, and when $k>n$ it coincides with the cofiber of the map $(2^m, 0)$: $C(\bar{\eta})\to C(2^n\bar{j}_V)=C(\bar{\eta})\vee SZ/2^n$. When $k<n$, it is given as the cofiber of a certain map $l_{n,k,m}$: $C(\bar{\eta})\to C(2^k\bar{j}_V)$ which is quasi KO_*-equivalent to the map $2^{s+1}i'_R$: $\Sigma^4\to\Sigma^4R'_k$. Consequently we observe that

(3.7) whenever $1\leqq k<m$ the small spectrum $R'_{n,k,m}$ has the same quasi KO_*-type as $\Sigma^4SZ/2^m\vee SZ/2^n$, $\Sigma^4V_m\vee V_n$ or $\Sigma^4R'_{m+n-k,k}$ according as $k>n$, $k=n$ or $k<n$.

When $k>m$ the map $f_{n,k}=(2^k+2^n, 2^{k-1}\bar{i})$ is replaced by the simpler map $f_n=(2^n, 0)$. Thus the small spectrum $R'_{n,k,m}$ is constructed as the cofiber of the composite map $\bar{\rho}'_R f_n$: $\Sigma^0\to\bar{R}'_m$. Therefore it coincides with the cofiber of the map $(2^{m-1}i\bar{\lambda}, 2^m)$: $C(\bar{\eta})\to SZ/2^n\vee C(\bar{\eta})$ when $k>\mathrm{Min}\{m, n\}$. Since it is the cofiber of the map $2^{m-1}i(\bar{\lambda}\wedge j)$: $\Sigma^{-1}C(\bar{\eta})\wedge SZ/2^m\to SZ/2^n$, we see that

(3.8) the small spectrum $R'_{n,k,m}$ has the same quasi KO_*-type as $R'_{n,m}$ whenever $k>\mathrm{Min}\{m, n\}$.

We here rewrite the small spectrum $R'_{n,m,m}$ to be $R'R_{n,m}$. Since it is obtained as the cofiber of the map $2^m\bar{i}_V\bar{j}_U$: $\Sigma^{-1}U_m\to V_m$, the small spectrum $R'R_{m,m}$ is quasi KO_*-equivalent to the small spectrum constructed as the cofiber of the map $i_V\tilde{\eta}\eta^2\bar{j}_V$: $\Sigma^3V_m\to V_m$ or $i\eta^2\bar{\eta}$: $\Sigma^5SZ/2\to\Sigma^2SZ/2$ according as $m\geqq 2$ or $m=1$. In particular, $R'R_{1,1}$ has the same quasi KO_*-type as $\Sigma^2R'_{1,1}$. By (2.2) and (3.8) we note that the small spectrum $R'R_{n,m}$ has the same quasi KO_*-type as $SZ/2^n\vee\Sigma^4SZ/2^m$ when $n<m$.

PROPOSITION 3.3. i) *$KU_0R'R_{n,m}\cong Z/2^n\oplus Z/2^m$ on which $\psi_C^{-1}=1$ and $KU_1R'R_{n,m}=0$.*

ii) *$KO_iR'R_{n,m}\cong Z/2^{n+1}\oplus Z/2^{m-1}$, $Z/2$, $(*)_m$, $Z/2$ according as $i\equiv 0, 1, 2, 3$ mod 4 when $m<n$ or $m=n\geqq 2$. Here $(*)_1\cong Z/4$ and $(*)_m\cong Z/2\oplus Z/2$ if $m\geqq 2$.*

For the small spectra $R'R_{m,m}$ and $V_m\vee\Sigma^4V_m$ $(m\geqq 2)$ their KU-, KO- and KT-homologies are all equal, but their induced homomorphisms by τ: $\Sigma^1KT\to KO$ are not equal. In fact, the induced homomorphisms τ_*: $KT_{2i}R'R_{m,m}\to KO_{2i+1}R'R_{m,m}$ $(m\geqq 1)$ are represented by the following rows T_{2i+1}:

$$\begin{aligned}(3.9)\quad &T_1=(0\ 1): Z/2^m\oplus Z/2^m\longrightarrow Z/2, \qquad T_3=(1\ 1): Z/2\oplus Z/2\longrightarrow Z/2,\\ &T_5=(1\ 0): Z/2^m\oplus Z/2^m\longrightarrow Z/2, \qquad T_7=(0\ 1): Z/2\oplus Z/2\longrightarrow Z/2.\end{aligned}$$

4. The cofibers of maps $f: \Sigma^i \to X_m$ and $f': \Sigma^i \to X'_m$.

4.1. Using the maps $\rho_{P,M}: P \to M_m$ and $\rho_{Q,N}: Q \to N_m$ given in (1.7) and (1.12) we set

$$\begin{aligned}
&\xi_M=\rho_{P,M}\xi_P: \Sigma^2 \longrightarrow M_m, \qquad \xi_N=\rho_{Q,N}\xi_Q: \Sigma^3 \longrightarrow N_m, \\
(4.1)\qquad &\bar{\rho}_M=\rho_{P,M}\bar{\rho}_P: C(\bar{\eta}) \longrightarrow M_m, \qquad \bar{\rho}_N=\rho_{Q,N}\bar{\rho}_Q: C(\bar{\eta}) \longrightarrow N_m, \\
&\bar{\lambda}_M=\rho_{P,M}\bar{\lambda}_P: \Sigma^2 C(\bar{\eta}) \longrightarrow M_m, \qquad \bar{\lambda}_N=\rho_{Q,N}\bar{\lambda}_Q: \Sigma^3 C(\bar{\eta}) \longrightarrow N_m .
\end{aligned}$$

These maps satisfy $j_M\xi_M=2=j_N\xi_N$, $j_M\bar{\rho}_M=\eta j\bar{j}$, $j_N\bar{\rho}_N=j\bar{j}$ and $j_M\bar{\lambda}_M=\bar{\lambda}=j_N\bar{\lambda}_N$. Recall that $KO_iM_m \cong Z/2^m$, 0, $Z\oplus Z/2$, $Z/2$, $Z/2^{m+1}$, 0, Z, 0 according as $i=0, 1, \cdots, 7$.

PROPOSITION 4.1. *For any map $f: S_i \to \Delta M_m$ ($0 \leqq i \leqq 7$) its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1} \vee M_m$ or the following small spectrum Y_i:* i) $Y_0=\Sigma^1 \vee M_k$ $(0 \leqq k < m)$; ii) $Y_2=MP_m$, $P'_{m,n+1} \vee \Sigma^2 SZ/q$ *or* $P''_{m,n+1} \vee \Sigma^2 SZ/q$ $(n \geqq 0)$; iii) $Y_3=MQ_m$; iv) $Y_4=MR_m$ *or* $\Sigma^1 \vee \Sigma^4 M_k$ $(0 \leqq k<m)$; v) $Y_6=\Sigma^6 P'_{n,m+1} \vee \Sigma^2 SZ/q$ $(n \geqq 0)$ *where $q \geqq 1$ is odd.*

Proof. Consider the following maps $g_{0,k}=2^k i_M \iota: \Sigma^0 \to M_m$, $g_2=i_M\bar{\eta}: \Sigma^2 \to M_m$, $g_{2,n}=2^n\xi_M: \Sigma^2 \to M_m$, $g'_{2,n}=2^n\xi_M+i_M\bar{\eta}: \Sigma^2 \to M_m$, $g_3=i_M\bar{\eta}\eta: \Sigma^3 \to M_m$, $g_{4,k}=2^k\bar{\rho}_M: C(\bar{\eta}) \to M_m$ and $g_{6,n}=2^n\bar{\lambda}_M: \Sigma^2 C(\bar{\eta}) \to M_m$. The cofibers $C(g_{4,k})$ and $C(g_{6,n})$ are given as those of certain maps $h_{4,k}: \Sigma^0 \to C(2^k\bar{\rho}_P)$ and $h_{6,n}: \Sigma^0 \to C(2^n\bar{\lambda}_P)$. Here the map $h_{4,k}$ is KO_*-trivial whenever $0 \leqq k<m$, and $h_{6,n}$ is quasi KO_*-equivalent to the map $2^m\xi_M: \Sigma^0 \to \Sigma^{-2}M_n$. Hence they have the same quasi KO_*-types as $\Sigma^1 \vee \Sigma^4 M_k$ and $\Sigma^{-2}P'_{n,m+1}$ respectively when $0 \leqq k<m$ and $n \geqq 0$. Moreover the cofiber $C(g_{4,m})$ has the same quasi KO_*-type as MR_m because the map $g_{4,m}$ is quasi KO_*-equivalent to the map $i_M\bar{\eta}\eta^2: \Sigma^4 \to M_m$. Since the remaining cofibers are easily observed, our result is shown.

Recall that $KO_iN_m \cong Z/2^m$, $Z/2$, $Z/2$, $Z\oplus Z/2$, $Z/2^{m+1}$, $Z/2$, 0, Z according as $i=0, 1, \cdots, 7$.

PROPOSITION 4.2. *For any map $f: S_i \to \Delta N_m$ ($0 \leqq i \leqq 7$) its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1} \vee N_m$ or the following small spectrum Y_i:* i) $Y_0=\Sigma^1 \vee N_k$ $(0 \leqq k<m)$; ii) $Y_1=\Sigma^3 \vee M_m$; iii) $Y_2=NP_m$; iv) $Y_3=NQ_m$, $Q'_{m,n+1} \vee \Sigma^3 SZ/q$ *or* $Q''_{m,n+1} \vee \Sigma^3 SZ/q$ $(n \geqq 0)$; v) $Y_4=NR_m$ *or* $\Sigma^1 \vee \Sigma^4 N_k$ $(0 \leqq k<m)$; vi) $Y_5=\Sigma^7 \vee M_m$; vii) $Y_7=MV_{m,n+1} \vee \Sigma^3 SZ/q$ $(n \geqq 0)$ *where $q \geqq 1$ is odd.*

Proof. Use the following maps $g_{0,k}=2^k i_N i: \Sigma^0 \to N_m$, $g_1=i_N i\eta: \Sigma^1 \to N_m$, $g_2=i_N\bar{\eta}: \Sigma^2 \to N_m$, $g_3=i_N\bar{\eta}\eta: \Sigma^3 \to N_m$, $g_{3,n}=2^n\xi_N: \Sigma^3 \to N_m$, $g'_{3,n}=2^n\xi_N+i_N\bar{\eta}\eta: \Sigma^3 \to N_m$, $g_{4,k}=2^k\bar{\rho}_N: C(\bar{\eta}) \to N_m$, $g_5=\bar{\rho}_N(\eta \wedge 1): \Sigma^1 C(\bar{\eta}) \to N_m$ and $g_{7,n}=2^n\bar{\lambda}_N: \Sigma^3 C(\bar{\eta}) \to N_m$. By a similar argument to the proof of Proposition 4.1 we can easily show our result.

4.2. Consider the following cofiber sequences

$$\Sigma^2 P \xrightarrow{\lambda_{P,Q}} Q_m \xrightarrow{\rho_{Q,P}} P_m \xrightarrow{i_P j_P} \Sigma^3 P \quad \text{and} \quad \Sigma^2 Q \xrightarrow{\lambda_{Q,R}} R_m \xrightarrow{\rho_{R,P}} P_m \xrightarrow{i_Q j_P} \Sigma^3 Q$$

and then set

$$\begin{aligned} &\xi_Q=\lambda_{P,Q}\xi_P : \Sigma^4 \longrightarrow Q_m, && \xi_R=\lambda_{Q,R}\xi_Q : \Sigma^5 \longrightarrow R_m, \\ &\bar\rho_Q=\lambda_{P,Q}\bar\rho_P : \Sigma^2 C(\bar\eta) \longrightarrow Q_m, && \bar\rho_R=\lambda_{Q,R}\bar\rho_Q : \Sigma^2 C(\bar\eta) \longrightarrow R_m, \\ &\bar\lambda_Q=\lambda_{P,Q}\bar\lambda_P : \Sigma^4 C(\bar\eta) \longrightarrow Q_m, && \bar\lambda_R=\lambda_{Q,R}\bar\lambda_Q : \Sigma^5 C(\bar\eta) \longrightarrow R_m. \end{aligned} \tag{4.2}$$

These maps satisfy $j_Q\xi_Q=2=j_R\xi_R$, $j_Q\bar\rho_Q=\eta j\bar j$, $j_R\bar\rho_R=j\bar j$ and $j_Q\bar\lambda_Q=\bar\lambda=j_R\bar\lambda_R$. Denote by $\bar Q_m$ and $\bar R_m$ $(m\geqq 1)$ the cofibers of the maps $\tilde\eta j\bar j : \Sigma^{-1}C(\bar\eta)\to SZ/2^m$ and $\tilde\eta\eta j\bar j : C(\bar\eta)\to SZ/2^m$, which have the same quasi KO_*-types as the elementary spectra Q_m and R_m respectively. Choose maps $\bar h_Q : \Sigma^0\to\bar Q_m$ and $\bar h_R : \Sigma^1\to\bar R_m$ satisfying $\bar j_Q\bar h_Q=\bar i=\bar j_R\bar h_R$, $\bar h_Q\bar\eta=\bar i_Q\tilde\eta j$ and $\bar h_R\bar\eta=i_R\tilde\eta\eta j$ where $\bar i_Q : SZ/2^m\to\bar Q_m$ and $\bar i_R : SZ/2^m\to\bar R_m$ are the canonical inclusions, and $\bar j_Q : \bar Q_m\to C(\bar\eta)$ and $\bar j_R : \bar R_m\to\Sigma^1 C(\bar\eta)$ are the canonical projections. We moreover choose a map $\bar{\bar\xi}_Q : C(\bar\eta)\to\bar Q_m$ satisfying $\bar j_Q\bar{\bar\xi}_Q=2$ and $\bar{\bar\xi}_Q(1\wedge\bar j)=\bar i_Q\rho_{1,m}(\bar j\wedge\bar\eta_1)$. Recall that $KO_iQ_m\cong Z\oplus Z/2^m, Z/2, (*)_m, 0, Z\oplus Z/2^{m-1}, 0, Z/2, 0$ according as $i=0, 1, \cdots, 7$ where $(*)_1\cong Z/4$ and $(*)_m\cong Z/2\oplus Z/2$ if $m\geqq 2$.

PROPOSITION 4.3. *For any map $f : S_i\to\Delta Q_m$ $(0\leqq i\leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1}\vee Q_m$ or the following small spectrum Y_i*: i) $Y_0=\Sigma^1 Q\vee SZ/2^k$ $(0\leqq k<m)$, $PV_{m,n+1}\vee SZ/q$ $(n\geqq 0)$ *or* $SZ/2^k\vee W_{m+n+1-k}\vee SZ/q$ $(0\leqq k<\mathrm{Min}\{m, n+1\})$; ii) $Y_1=MQ_m$; iii) $Y_2=NQ_m$ *or* $\Sigma^4\vee P_m$; iv) $Y_4=\Sigma^1 Q\vee\Sigma^4 V_{k+1}$ $(0\leqq k<m-1)$, $K_{m,n+1}\vee SZ/q$ $(n\geqq 0)$ *or* $\Sigma^4 V_{k+1}\vee W_{m+n-k}\vee SZ/q$ $(0\leqq k<\mathrm{Min}\{m-1, n\})$; v) $Y_6=\Sigma^0\vee P_m$ *where $q\geqq 1$ is odd.*

Proof. Consider the following maps $g_{0,k}=2^k i_Q i : \Sigma^0\to Q_m$, $g'_{0,n}=2^n\bar h_Q : \Sigma^0\to\bar Q_m$, $g'_{0,n,k}=2^n\bar h_Q+2^k\bar i_Q i : \Sigma^0\to\bar Q_m$, $g_1=i_Q i\eta : \Sigma^1\to Q_m$, $g_2=i_Q i\eta^2 : \Sigma^2\to Q_m$, $g'_2=i_Q\tilde\eta : \Sigma^2\to Q_m$, $g''_2=i_Q(\tilde\eta+i\eta^2) : \Sigma^2\to Q_m$, $g_{4,k}=2^k i_Q i\bar\lambda : C(\bar\eta)\to Q_m$, $g'_{4,n}=2^n\bar{\bar\xi}_Q : C(\bar\eta)\to\bar Q_m$, $g'_{4,n,k}=2^n\bar{\bar\xi}_Q+2^k\bar i_Q i\bar\lambda : C(\bar\eta)\to\bar Q_m$ and $g_6=\bar\rho_Q : \Sigma^2 C(\bar\eta)\to Q_m$. The cofibers $C(g'_{0,n})$, $C(g'_{0,n,k})$, $C(g'_{4,n})$ and $C(g'_{4,n,k})$ coincide with those of the maps $\tilde\eta j\bar j_V : \Sigma^{-1}V_{n+1}\to SZ/2^m$, $\tilde\eta j\bar j_V+2^k i\bar j_V : \Sigma^{-1}V_{n+1}\to SZ/2^m$, $\rho_{1,m}(\bar j\wedge\bar\eta) : \Sigma^{-1}C(\bar\eta)\wedge SZ/2^{n+1}$ $SZ/2^m$ and $\rho_{1,m}(\bar j\wedge\bar\eta)+2^k i(\bar\lambda\wedge j) : \Sigma^{-1}C(\bar\eta)\wedge SZ/2^{n+1}\to SZ/2^m$ respectively. When $0\leqq n<k$, both of the first two cofibers are the small spectrum $PV_{m,n+1}$ since $\tilde\eta j\bar j_V+2^{n+1}i\bar j_V=(1+i\eta j)\tilde\eta j\bar j_V$. Moreover the second cofiber is the wedge sum $SZ/2^k\vee W_{m+n-k+1}$ whenever $0\leqq k\leqq n$, because it is obtained as the cofiber of the map $(0, i\bar\eta+\tilde\eta j) : \Sigma^1 SZ/2\to SZ/2^k\vee SZ/2^{m+n-k}$. Since the maps $\rho_{1,m}(\bar j\wedge\bar\eta)$ and $\rho_{1,m}(\bar j\wedge\bar\eta)+2^n i(\bar\lambda\wedge j)$ are quasi KO_*-equivalent to the maps $\tilde\eta\bar\eta$ and $\tilde\eta\bar\eta+i\eta^2\bar\eta=(1+i\eta j)\tilde\eta\bar\eta : \Sigma^3 SZ/2^{n+1}\to SZ/2^m$, both of the last two cofibers have the same quasi KO_*-type as the small spectrum $K_{m,n+1}$ when $0\leqq n\leqq k$. Moreover, according to Lemma 2.3 the last cofiber has the same quasi KO_*-type as the

wedge sum $\Sigma^4 V_{k+1} \vee W_{m+n-k}$ whenever $0 \leqq k < \mathrm{Min}\{m-1, n\}$. Since the remaining cofibers are more easily observed, our result is established.

Recall that $KO_i R_m \cong Z/2^m$, $Z \oplus Z/2$, $(*)_m$, $Z/2$, $Z/2^{m-1}$, Z, $Z/2$, $Z/2$ according as $i=0, 1, \cdots, 7$ where $(*)_1 \cong Z/4$ and $(*)_m \cong Z/2 \oplus Z/2$ if $m \geqq 2$.

PROPOSITION 4.4. *For any map* $f: S_i \to \Delta R_m$ $(0 \leqq i \leqq 7)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^{i+1} \vee R_m$ *or the following small spectrum* Y_i: i) $Y_0 = \Sigma^1 \vee \Sigma^5 \vee SZ/2^k$ $(0 \leqq k < m)$; ii) $Y_1 = MR_m$, $QV_{m,n+1} \vee \Sigma^1 SZ/q$ *or* $QV^0_{m,n+1} \vee \Sigma^1 SZ/q$ $(n \geqq 0)$; iii) $Y_2 = NR_m$ *or* $\Sigma^5 \vee P_m$; iv) $Y_3 = \Sigma^5 \vee Q_m$; v) $Y_4 = \Sigma^1 \vee \Sigma^5 \vee \Sigma^4 V_{k+1}$ $(0 \leqq k < m-1)$; vi) $Y_5 = L_{m,n+1} \vee \Sigma^1 SZ/q$ $(n \geqq 0)$; vii) $Y_6 = \Sigma^1 \vee P_m$; viii) $Y_7 = \Sigma^1 \vee Q_m$ *where* $q \geqq 1$ *is odd.*

Proof. Use the following maps $g_{0,k} = 2^k i_R i: \Sigma^0 \to R_m$, $g_1 = i_R i \eta: \Sigma^1 \to R_m$, $g_{1,n} = 2^n \bar{h}_R: \Sigma^1 \to \bar{R}_m$, $g'_{1,n} = 2^n \bar{h}_R + \bar{i}_R i \eta: \Sigma^1 \to \bar{R}_m$, $g_2 = i_R i \eta^2: \Sigma^2 \to R_m$, $g'_2 = i_R \tilde{\eta}: \Sigma^2 \to R_m$, $g''_2 = i_R(\tilde{\eta} + i\eta^2): \Sigma^2 \to R_m$, $g_3 = i_R \tilde{\eta} \eta: \Sigma^3 \to R_m$, $g_{4,k} = 2^k i_R i \bar{\lambda}: C(\bar{\eta}) \to R_m$, $g_{5,n} = 2^n \xi_R: \Sigma^5 \to R_m$, $g_6 = \bar{\rho}_R: \Sigma^2 C(\bar{\eta}) \to R_m$ and $g_7 = \bar{\rho}_R(\eta \wedge 1): \Sigma^3 C(\bar{\eta}) \to R_m$. Then we can easily show our result by a similar argument to the proof of Proposition 4.1.

4.3. Note that the elementary spectrum M'_m is quasi KO_*-equivalent to $\Sigma^1 P_{m+1}$. We can choose a map $\xi_P: \Sigma^3 \to P_{m+1}$ $(m \geqq 1)$ satisfying $j_P \xi_P = 2$ whose cofiber is the small spectrum $H_{m+1,1}$. In other words, there exists a map f_P: $\Sigma^1 H_{m+1,1} \to \Sigma^3$ whose cofiber is P_{m+1}. Since the map $f_P: \Sigma^{-1} H_{2,1} \to \Sigma^3$ is paticularly quasi KO_*-equivalent to the map $\eta \bar{\eta}: \Sigma^5 SZ/2 \to \Sigma^3$ we notice that

(4.3) the elementary spectra M'_1 and M_1 are quasi KO_*-equivalent to $\Sigma^4 Q'_1$ and $\Sigma^2 Q_1$ respectively.

Recall that $KO_i M'_m \cong Z$, $Z/2^{m+1}$, $Z/2$, $Z/2$, Z, $Z/2^m$, 0, 0 according as $i=0, 1, \cdots, 7$.

PROPOSITION 4.5. *For any map* $f: S_i \to \Delta M'_m$ $(0 \leqq i \leqq 7)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^{i+1} \vee M'_m$ *or the following small spectrum* Y_i: i) $Y_0 = M_{n,m} \vee SZ/q$ $(n \geqq 0)$; ii) $Y_1 = P \vee \Sigma^1 SZ/2^k$ $(0 \leqq k \leqq m)$; iii) $Y_2 = M'M_m$; iv) $Y_3 = M'N_m$; v) $Y_4 = \Sigma^1 H_{m+1,n+1} \vee SZ/q$ $(n \geqq 0)$; vi) $Y_5 = P \vee \Sigma^5 V_{k+1}$ $(0 \leqq k < m)$ *where* $q \geqq 1$ *is odd.*

Proof. Use the following maps $g_{0,n} = 2^n i'_M: \Sigma^0 \to M'_m$, $g_{1,k} = 2^k h'_M: \Sigma^1 \to M'_m$, $g_2 = h'_M \eta: \Sigma^2 \to M'_m$, $g_3 = h'_M \eta^2: \Sigma^3 \to M'_m$, $g_{4,n} = 2^n \xi_P: \Sigma^4 \to \Sigma^1 P_{m+1}$ and $g_{5,k} = 2^k h'_M \bar{\lambda}: \Sigma^1 C(\bar{\eta}) \to M'_m$. Then our result is easily shown.

We can choose a map $\bar{\rho}'_N: C(\bar{\eta}) \to N'_m$ satisfying $\rho_{N',Q} \bar{\rho}'_N = \bar{\rho}_Q$ so that its cofiber is the elementary spectrum V'_m obtained as that of the map $2^{m-1}\tilde{j}$: $\Sigma^{-1} C(\bar{\eta}) \to \Sigma^2$, where the map $\rho_{N',Q}: N'_m \to Q$ is given in (1.12). In other words, there exists a map $f'_N: \Sigma^{-1} V'_m \to C(\bar{\eta})$ whose cofiber is N'_m. Since the map $f'_N: \Sigma^{-1} V'_1 \to C(\bar{\eta})$ is quasi KO_*-equivalent to the map $\eta^2 \bar{\eta}: \Sigma^7 SZ/2 \to \Sigma^4$, we notice that

(4.4) the elementary spectra N_1' and N_1 are quasi KO_*-equivalent to $\Sigma^4 R_1'$ and $\Sigma^2 R_1$ respectively.

Recall that $KO_i N_m' \cong Z, Z/2, Z/2^{m+1}, Z/2, Z\oplus Z/2, Z/2, Z/2^m, 0$ according as $i=0, 1, \cdots, 7$.

PROPOSITION 4.6. *For any map* $f: S_i \to \Delta N_m'$ $(0\leqq i\leqq 7)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^{i+1}\vee N_m'$ *or the following small spectrum* Y_i: i) $Y_0=N_{n,m}'\vee SZ/q$ $(n\geqq 0)$; ii) $Y_1=P\vee\Sigma^2 SZ/2^m$; iii) $Y_2=Q\vee\Sigma^2 SZ/2^k$ $(0\leqq k\leqq m)$; iv) $Y_3=N'M_m$; v) $Y_4=N'N_m$, $\Sigma^6 V_m\vee SZ/q$, $\Sigma^4 VR_{n+1,m}\vee SZ/q$ *or* $N'N_{n+1,m}\vee SZ/q$ $(n\geqq 0)$; vi) $Y_5=P\vee\Sigma^6 V_m$; vii) $Y_6=Q\vee\Sigma^6 V_{k+1}$ $(0\leqq k<m)$ *where* $q\geqq 1$ *is odd.*

Proof. Consider the following maps $g_{0,n}=2^n i_N'$: $\Sigma^0\to N_m'$, $g_1=i_N'\eta$: $\Sigma^1\to N_m'$, $g_{2,k}=2^k h_N'$: $\Sigma^2\to N_m'$, $g_3=h_N'\eta$: $\Sigma^3\to N_m'$, $g_4=h_N'\eta^2$: $\Sigma^4\to N_m'$, $g_{4,n}=2^n\bar{\rho}_N'$: $C(\bar{\eta})\to N_m'$, $g_{4,n}'=2^n\bar{\rho}_N'+h_N'\eta j\bar{j}$: $C(\bar{\eta})\to N_m'$, $g_5=\bar{\rho}_N'(\eta\wedge 1)$: $\Sigma^1 C(\bar{\eta})\to N_m'$ and $g_{6,k}=2^k h_N'\bar{\lambda}$: $\Sigma^2 C(\bar{\eta})\to N_m'$. The cofibers $C(g_{4,0})$ and $C(g_{4,0}')$ are given as those of certain maps $h_{4,0}$ and $h_{4,0}'$: $\Sigma^{-1}C(\tilde{\eta})\to\Sigma^2$, both of which are quasi KO_*-equivalent to the map $2^{m-1}\tilde{j}$: $\Sigma^{-1}C(\tilde{\eta})\to\Sigma^2$. Hence they have the same quasi KO_*-type as V_m'. When $n\geqq 1$ the maps $g_{4,n}$ and $g_{4,n}'$ may be replaced by the maps $\phi_n=2^{n-1}i_N'\bar{\lambda}$ and $\phi_{n,0}=2^{n-1}i_N'\bar{\lambda}+h_N'\tilde{\eta}\eta\bar{j}$ given in (3.5). In fact, these maps ϕ_n and $\phi_{n,0}$ are respectively quasi KO_*-equivalent to the maps $g_{4,n}$ and $g_{4,n}'$ when $n\geqq 2$, and ϕ_1 and $\phi_{1,0}$ are respectively quasi KO_*-equivalent to the maps $g_{4,1}'$ and $g_{4,1}$. Since the remaining cofibers are easily observed, our result is shown.

4.4. Using the map $\rho_{Q,Q'}$: $Q\to Q_m'$ given in (1.12) we set

$$(4.5)\qquad \begin{aligned} &\xi_Q'=\rho_{Q,Q'}\xi_Q: \Sigma^3 \longrightarrow Q_m', \qquad \bar{\rho}_Q'=\rho_{Q,Q'}\bar{\rho}_Q: C(\bar{\eta}) \longrightarrow Q_m' \quad\text{and}\\ &\bar{\lambda}_Q'=\rho_{Q,Q'}\bar{\lambda}_Q: \Sigma^3 C(\bar{\eta}) \longrightarrow Q_m'. \end{aligned}$$

These maps satisfy $j_Q'\xi_Q'=2i$, $j_Q'\bar{\rho}_Q'=ij\bar{j}$ and $j_Q'\bar{\lambda}_Q'=i\bar{\lambda}$. Moreover we choose maps h_Q': $\Sigma^5\to Q_m'$ and $\tilde{h}_Q$: $\Sigma^5\to Q_m'$ satisfying $j_Q'h_Q'=i\eta^2$ and $j_Q'\tilde{h}_Q=\tilde{\eta}$ as in [5, (2.1) and (2.2)]. Recall that $KO_i Q_m'\cong Z, Z/2, 0, Z/2^{m-1}, Z, (*)_m, Z/2, Z/2^m$ according as $i=0, 1, \cdots, 7$ where $(*)_1\cong Z/4$ and $(*)_m\cong Z/2\oplus Z/2$ if $m\geqq 2$.

PROPOSITION 4.7. *For any map* $f: S_i\to\Delta Q_m'$ $(0\leqq i\leqq 7)$ *its cofiber* $C(f)$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^{i+1}\vee Q_m'$ *or the following small spectrum* Y_i: i) $Y_0=Q_{n,m}'\vee SZ/q$ $(n\geqq 0)$; ii) $Y_1=P\vee\Sigma^3 SZ/2^m$; iii) $Y_3=\Sigma^4\vee Q_{k+1}'$ $(0\leqq k<m-1)$; iv) $Y_4=\Sigma^4 MV_{n,m}\vee SZ/q$ $(n\geqq 0)$; v) $Y_5=P\vee\Sigma^3 V_m$ *or* $Q'P_m$; vi) $Y_6=Q'Q_m$; vii) $Y_7=Q'R_m$ *or* $\Sigma^4\vee\Sigma^4 Q_{k+1}'$ $(0\leqq k<m-1)$ *where* $q\geqq 1$ *is odd.*

Proof. Use the following maps $g_{0,n}=2^n i_Q'$: $\Sigma^0\to Q_m'$, $g_1=i_Q'\eta$: $\Sigma^1\to Q_m'$, $g_{3,k}=2^k\xi_Q'$: $\Sigma^3\to Q_m'$, $g_{4,n}=2^n\bar{\rho}_Q'$: $C(\bar{\eta})\to Q_m'$, $g_5=\tilde{h}_Q$: $\Sigma^5\to Q_m'$, $g_5'=\eta\bar{\rho}_Q$: $\Sigma^1 C(\bar{\eta})\to$

Q'_m, $g''_5=\tilde{h}_Q+h'_Q: \Sigma^5\to Q'_m$, $g_6=\tilde{h}_Q\eta: \Sigma^6\to Q'_m$ and $g_{7,k}=2^k\bar{\lambda}'_Q: \Sigma^3C(\bar{\eta})\to Q'_m$. Then we can easily show our result by a similar argument to the proof of Proposition 4.1.

4.5. Consider the following cofiber sequences

$$\Sigma^1 Q\xrightarrow{\lambda_{Q,R}} R\xrightarrow{\rho_{R,P}} P\xrightarrow{i_Q j_P}\Sigma^2 Q \quad\text{and}\quad \Sigma^2 P\xrightarrow{\lambda_{P,R}} R\xrightarrow{\rho_{R,Q}} Q\xrightarrow{i_P j_Q}\Sigma^3 P,$$

and then set $\xi_R=\lambda_{P,R}\xi_P: \Sigma^4\to R$, $\bar{\rho}_R=\lambda_{Q,R}\bar{\rho}_Q: \Sigma^1C(\bar{\eta})\to R$ and $\bar{\lambda}_R=\lambda_{P,R}\bar{\lambda}_P: \Sigma^4C(\bar{\eta})\to R$ where R denotes the cofiber of the map $\eta^3: \Sigma^3\to\Sigma^0$. Since the elementary spectrum R'_m is related to R by the following cofiber sequence

$$\Sigma^4\xrightarrow{2^{m-1}\xi_R} R\xrightarrow{\rho_{R,R'}} R'_m\xrightarrow{jj'_R}\Sigma^5,$$

we get maps

$$(4.6)\qquad \begin{aligned}&\xi'_R=\rho_{R,R'}\xi_R: \Sigma^4\longrightarrow R'_m, \qquad \bar{\rho}'_R=\rho_{R,R'}\bar{\rho}_R: \Sigma^1C(\bar{\eta})\longrightarrow R'_m \quad\text{and}\\ &\bar{\lambda}'_R=\rho_{R,R'}\bar{\lambda}_R: \Sigma^4C(\bar{\eta})\longrightarrow R'_m,\end{aligned}$$

which satisfy $j'_R\xi'_R=2i$, $j'_R\bar{\rho}'_R=ij\bar{j}$ and $j'_R\bar{\lambda}'_R=i\bar{\lambda}$. Moreover we choose maps $h'_R: \Sigma^5\to R'_m$ and $\tilde{h}_R: \Sigma^6\to R'_m$ satisfying $j'_Rh'_R=i\eta$ and $j'_R\tilde{h}_R=\tilde{\eta}$ as in [5, (2.1) and (2.2)]. Using the map $\bar{\rho}'_R: \Sigma^0\vee C(\bar{\eta})\to\bar{R}'_m$ given in (3.6) we here set

$$(4.7)\qquad \begin{aligned}&\lambda'_R=\bar{\rho}'_R(2, \bar{i}): \Sigma^0\longrightarrow\bar{R}'_m, \qquad \xi'_R=\bar{\rho}'_R(\bar{\lambda}, 2): C(\bar{\eta})\longrightarrow\bar{R}'_m \quad\text{and}\\ &\bar{\kappa}'_R=\bar{\rho}'_R(0, 1): C(\bar{\eta})\longrightarrow\bar{R}'_m.\end{aligned}$$

These maps satisfy $\bar{j}'_R\lambda'_R=\bar{i}\wedge\imath$, $\bar{j}'_R\xi'_R=2(1\wedge i)$ and $\bar{j}'_R\bar{\kappa}'_R=1\wedge i$. Recall that $KO_iR'_m\cong Z\oplus Z/2^m$, $Z/2$, $Z/2$, 0, $Z\oplus Z/2^{m-1}$, $Z/2$, $(*)_m$, $Z/2$ according as $i=0, 1, \cdots, 7$ where $(*)_1\cong Z/4$ and $(*)_m\cong Z/2\oplus Z/2$ if $m\geqq 2$.

PROPOSITION 4.8. *For any map $f: S_i\to\Delta R'_m$ $(0\leqq i\leqq 7)$ its cofiber $C(f)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{i+1}\vee R'_m$ or the following small spectrum Y_i:* i) $Y_0=R'R_m$, $\Sigma^5\vee\Sigma^4R'_{k+1}$ $(0\leqq k<m-1)$, $R'_{n,m}\vee SZ/q$ $(n\geqq m)$, $\Sigma^4SZ/2^m\vee SZ/2^n\vee SZ/q$ $(0\leqq n\leqq m-1)$, $\Sigma^4V_m\vee V_n\vee SZ/q$ $(1\leqq n\leqq m-1)$, $\Sigma^4R'_{m+n-k-1,k+1}\vee SZ/q$ $(0\leqq k<\mathrm{Min}\{m-1, n-1\})$ *or* $R'R_{n,m}\vee SZ/q$ $(n\geqq m)$; ii) $Y_1=P\vee\Sigma^4SZ/2^m$; iii) $Y_2=Q\vee\Sigma^4SZ/2^m$: iv) $Y_4=\Sigma^5\vee R'_{k+1}$ $(0\leqq k<m-1)$, $\Sigma^4V_m\vee\Sigma^4SZ/2^n\vee SZ/q$ $(0\leqq n\leqq m)$, $\Sigma^4SZ/2^m\vee\Sigma^4V_n\vee SZ/q$ $(1\leqq n\leqq m-1)$ *or* $\Sigma^4NV_{m+n-k-1,k+1}\vee SZ/q$ $(0\leqq k<\mathrm{Min}\{m, n-1\})$; v) $Y_5=P\vee\Sigma^4V_m$; vi) $Y_6=Q\vee\Sigma^4V_m$ *or* $R'P_m$; vii) $Y_7=R'Q_m$ *where $q\geqq 1$ is odd.*

Proof. Consider the following maps $g_{0,n}=2^ni'_R: \Sigma^0\to R'_m$, $g'_{0,k}=2^k\lambda'_R: \Sigma^0\to\bar{R}'_m$, $g_{0,n,k}=2^n\bar{i}'_R+2^k\lambda'_R: \Sigma^0\to\bar{R}'_m$, $g_1=i'_R\eta: \Sigma^1\to R'_m$, $g_2=i'_R\eta^2: \Sigma^2\to R'_m$, $g_{4,n}=2^n\bar{\kappa}'_R: C(\bar{\eta})\to\bar{R}'_m$, $g'_{4,k}=2^k\xi'_R: \Sigma^4\to R'_m$, $g_{4,n,k}=2^n\bar{\kappa}'_R+2^k\bar{\xi}'_R: C(\bar{\eta})\to\bar{R}'_m$, $g_5=\bar{\kappa}'_R(\eta\wedge 1): \Sigma^1C(\bar{\eta})\to R'_m$, $g_6=\bar{\kappa}'_R(\eta^2\wedge 1): \Sigma^2C(\bar{\eta})\to R'_m$, $g'_6=\tilde{h}_R: \Sigma^6\to R'_m$, $g''_6=\tilde{h}_R+h'_R\eta$:

$\Sigma^6 \to R'_m$ and $g_7 = \tilde{h}_R \eta : \Sigma^7 \to R'_m$. The cofiber $C(g_{4,n})$ coincides with that of the map $h_{4,n} = (2^{m-1}\bar{\lambda}, 2^m(1\wedge i)) : C(\bar{\eta}) \to \Sigma^0 \vee (C(\bar{\eta}) \wedge SZ/2^n)$. When $n \leqq m$ it is the wedge sum $U_m \vee (C(\bar{\eta}) \wedge SZ/2^n)$, and when $n > m$ it is obtained as the cofiber of the map $2^{n-1}\bar{\lambda} \vee 2^{m-1}\bar{\lambda}(1\wedge j) : C(\bar{\eta}) \vee (\Sigma^{-1}C(\bar{\eta}) \wedge SZ/2^m) \to \Sigma^0$ which is quasi KO_*-equivalent to the map $k_{4,n} = 2^{n-1}\bar{\lambda} \vee \eta^2\bar{\eta} : C(\bar{\eta}) \vee \Sigma^3 SZ/2^m \to \Sigma^0$. The cofiber $C(k_{4,n})$ is given as that of a certain map $l_{4,n} : \Sigma^3 SZ/2^m \to U_n$ which is quasi KO_*-equivalent to the map $q_{4,n} = 2^{m-1} i_V i j : \Sigma^3 SZ/2^m \to \Sigma^4 V_n$. As is easily seen, the cofiber $C(q_{4,n})$ is the small spectrum $\Sigma^4 NV_{n,m}$. Since $g_{0,n,k} = \bar{\rho}'_R(2^{k+1}+2^n, 2^k\bar{i})$, its cofiber $C(g_{0,n,k})$ is exactly the small spectrum $R'_{n,k+1,m}$. From (3.7) and (3.8) we recall that it has the same quasi KO_*-type as $\Sigma^4 SZ/2^m \vee SZ/2^n$, $\Sigma^4 V_m \vee V_n$ or $\Sigma^4 R'_{m+n-k-1,k+1}$ according as $n < k+1 \leqq m$, $n = k+1 < m$ or $k+1 < \mathrm{Min}\{m, n\}$. And $R'_{n,m,m}$ is written to be $R'R_{n,m}$ when $n \geqq m$. Assume that $0 \leqq k < m-1$. Since $g_{4,n,k} = \bar{\rho}'_R(2^k\bar{\lambda}, 2^{k+1}+2^n)$, its cofiber $C(g_{4,n,k})$ is given as the cofiber of a certain map $h_{4,n,k} : C(\bar{\eta}) \to C(\varphi_{n,k})$ where $\varphi_{n,k} = (2^k\bar{\lambda}, 2^{k+1}+2^n) : C(\bar{\eta}) \to \Sigma^0 \vee C(\bar{\eta})$. Note that $C(\varphi_{n,k})$ is $\Sigma^0 \vee (C(\bar{\eta}) \wedge SZ/2^n)$ or $U_{k+1} \vee C(\bar{\eta})$ according as $k \geqq n$ or $k = n-1$, and it has the same quasi KO_*-type as R'_{k+1} when $k \leqq n-2$. Then the map $h_{4,n,k}$ is expressed as $(2^{m-1}\bar{\lambda}, 2^m(1\wedge i)) : C(\bar{\eta}) \to \Sigma^0 \vee (C(\bar{\eta}) \wedge SZ/2^n)$ when $k \geqq n$, and as $(0, 2^m) : C(\bar{\eta}) \to U_n \vee C(\bar{\eta})$ when $k = n-1$. Therefore the cofiber $C(h_{4,n,k})$ is the wedge sum $U_m \vee (C(\bar{\eta}) \wedge SZ/2^n)$ or $U_n \vee (C(\bar{\eta}) \wedge SZ/2^m)$ according as $k \geqq n$ or $k = n-1$. When $k \leqq n-2$ the map $h_{4,n,k}$ is expressed as $-i_{n,k}(0, 2^{m+n-k-1}) : C(\bar{\eta}) \to C(\varphi_{n,k})$ where $i_{n,k} : \Sigma^0 \vee C(\bar{\eta}) \to C(\varphi_{n,k})$ is the canonical inclusion. So its cofiber coincides with that of the map $l_{4,n,k} = (2^k\bar{\lambda}, (2^{k+1}+2^m)(1\wedge i)) : C(\bar{\eta}) \to \Sigma^0 \vee (C(\bar{\eta}) \wedge SZ/2^{m+n-k-1})$ which is quasi KO_*-equivalent to the map $q_{4,n,k} = (2^k\bar{i}, 2^{k+1}i) : \Sigma^4 \to \Sigma^4 C(\bar{\eta}) \vee \Sigma^4 SZ/2^{m+n-k-1}$. Since it is obtained as the cofiber of the map $\bar{i}(2^{m+n-k-2}, 2^k j) : \Sigma^4 \vee \Sigma^3 SZ/2^{k+1} \to \Sigma^4 C(\bar{\eta})$, the cofiber $C(q_{4,n,k})$ is the small spectrum $\Sigma^4 NV_{m+n-k-1,k+1}$. Thus $C(h_{4,n,k})$ has the same quasi KO_*-type as $\Sigma^4 NV_{m+n-k-1,k+1}$ when $0 \leqq k \leqq \mathrm{Min}\{m-2, n-2\}$. Since the remaining cofibers are easily observed, our result is established.

4.6. We first consider the maps $\tilde{h}_M : \Sigma^5 \to M'_m$, $\tilde{h}_N : \Sigma^5 \to N'_m$, $h'_Q : \Sigma^5 \to Q'_m$ and $h'_R : \Sigma^5 \to R'_m$ satisfying $j'_M \tilde{h}_M = \bar{\eta}\eta^2$, $j'_N \tilde{h}_N = \bar{\eta}\eta$, $j'_Q h'_Q = i\eta^2$ and $j'_R h'_R = i\eta$ as in [5, (2.1) and (2.2)]. The cofibers of the maps $\tilde{h}_M$, $\tilde{h}_N$, $\tilde{h}_N\eta$, h'_Q, h'_R and $h'_R\eta$ are denoted by $M'R_m$, $N'Q_m$, $N'R_m$, $Q'N_m$, $R'M_m$ and $R'N_m$ respectively. According to Propositions 4.5, 4.6, 4.7 and 4.8 we observe that

(4.8) the 4-cells spectra $M'R_m$, $N'Q_m$, $N'R_m$, $Q'N_m$, $R'M_m$ and $R'N_m$ are quasi KO_*-equivalent to the wedge sums $P \vee \Sigma^5 V_m$, $P \vee \Sigma^6 V_m$, $Q \vee \Sigma^6 V_m$, $P \vee \Sigma^3 V_m$, $P \vee \Sigma^4 V_m$ and $Q \vee \Sigma^4 V_m$ respectively (cf. [5, Corollary 4.5]).

On the other hand, it follows from Proposition 4.3 that

(4.9) the cofiber of the map $i_Q \tilde{\sigma}\tilde{\nu} : \Sigma^4 \to Q_{m+1}$ $(m \geqq 1)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^1 Q \vee \Sigma^4 V_m$.

Consider the maps $i_P i\eta: \Sigma^1 \to P_m$, $i_P i\eta^2: \Sigma^2 \to P_m$, $\tilde{h}_P: \Sigma^5 \to P'_m$ and $\tilde{h}_P\eta: \Sigma^6 \to P'_m$ whose cofibers are respectively denoted by MP_m, NP_m, $P'Q_m$ and $P'R_m$ where the map $\tilde{h}_P$ satisfies $j'_P\tilde{h}_P=\tilde{\eta}\eta$ as in [5, (2.2)]. Since P_{m+1} and P'_{m+1} have the same quasi KO_*-types as $\Sigma^{-1}M'_m$ and Σ^2M_m, Propositions 4.1 and 4.5 imply that

(4.10) i) the small spectra MP_{m+1}, NP_{m+1}, $P'Q_{m+1}$ and $P'R_{m+1}$ ($m\geqq 1$) are quasi KO_*-equivalent to $\Sigma^{-1}M'M_m$, $\Sigma^{-1}M'N_m$, Σ^2MQ_m and Σ^2MR_m respectively, and dually

ii) the small spectra $M'P'_{m+1}$, $N'P'_{m+1}$, $Q'P_{m+1}$ and $R'P_{m+1}$ ($m\geqq 1$) are quasi KO_*-equivalent to $\Sigma^1M'M_m$, $N'M_m$, $M'Q'_m$ and $M'R'_m$ respectively.

Moreover we notice that

(4.11) i) the small spectra $P'Q_1$ and $Q'P_1$ have the same quasi KO_*-type as the elementary spectrum P,

ii) the small spectra $P'R_1$, $R'P_1$, Σ^1MP_1 and $\Sigma^{-1}M'P'_1$ have the same quasi KO_*-type as the elementary spectrum Q, and

iii) the small spectra Σ^1NP_1 and $N'P'_1$ have the same quasi KO_*-type as the wedge sum $\Sigma^0\vee\Sigma^4$.

Choose a map $\rho'_P: \Sigma^2SZ/2\to P'_{m+1}$ ($m\geqq 1$) satisfying $j'_P\rho'_P=\rho_{1,m+1}$ whose cofiber is P'_m, and then consider the map $g'_{4,n}=2^n\bar{\rho}'_P+\rho'_P\bar{j}: C(\bar{\eta})\to P'_{m+1}$ where $\bar{\rho}'_P=\rho_{P,P'}\bar{\rho}_P: C(\bar{\eta})\to P\to P'_{m+1}$ and it satisfies $\bar{\rho}'_P j'_P=i\eta j\bar{j}$. According to Proposition 4.1 the cofiber $C(g'_{4,n})$ has the same quasi KO_*-type as $\Sigma^2P''_{m,n+1}$. On the other hand, it is obtained as the cofiber of a certain map $h'_{4,n}: C(j'_Pg'_{4,n})\to\Sigma^0$ where $C(j'_Pg'_{4,n})$ has the same quasi KO_*-type as M'_m. Applying the dual of Proposition 4.1 we can verify that it has the same quasi KO_*-type as $P''_{n+1,m}$. Consequently it follows that

(4.12) $\Sigma^2P''_{m,n}$ (m, $n\geqq 1$) are quasi KO_*-equivalent to $P''_{n,m}$.

By virtue of (4.3) and (4.4) we can compare Propositions 4.1, 4.2, 4.5 and 4.6 with Propositions 4.3, 4.4, 4.7 and 4.8 to observe that

(4.13) i) the small spectra MQ_1, MR_1 and NR_1 are quasi KO_*-equivalent to Σ^2MQ_1, Σ^2NQ_1 and Σ^2NR_1 respectively,

ii) the small spectra $M'M_1$, $M'N_1$, $N'M_1$ and $N'N_1$ are quasi KO_*-equivalent to $\Sigma^4Q'Q_1$, $\Sigma^4Q'R_1$, $\Sigma^4R'Q_1$ and $\Sigma^4R'R_1$ respectively,

iii) the small spectra $PV_{1,n+1}$, $QV_{1,n+1}$ and $QV^0_{1,n+1}$ ($n\geqq 0$) are quasi KO_*-equivalent to $\Sigma^2P'_{1,n+1}$, $\Sigma^2Q'_{1,n+1}$ and $\Sigma^2Q''_{1,n+1}$ respectively,

iv) the small spectra $H_{2,n+1}$, $K_{1,n+1}$ and $L_{1,n+1}$ ($n\geqq 0$) are quasi KO_*-equivalent to $\Sigma^3Q'_{n,1}$, $\Sigma^4P'_{n,2}$ and $\Sigma^6MV_{1,n+1}$ respectively where $Q'_{0,1}=\Sigma^3SZ/2$ and $P'_{0,2}=\Sigma^2SZ/4$, and

v) the small spectra $VR_{n,1}$ and $N'N_{n,1}$ ($n\geqq 2$) are quasi KO_*-equivalent to $R'_{n,1}$ and $\Sigma^4R'R_{n,1}$ respectively, and $VR_{1,1}$, $R'R_{1,1}$ and $\Sigma^6N'N_{1,1}$ are quasi KO_*-equivalent to $\Sigma^2 R'_{1,1}$.

5. The quasi KO_*-types of a few cells spectra.

5.1. For any finite CW-spectrum X we denote by $\#X$ the number of all the cells in X. Let (X, Y) be a relative CW-spectrum such that X is obtained from Y by attaching one $(j+1)$-cell, thus $X=Y\cup e^{j+1}$. For any map $f: \Sigma^k\to X$ there exists a map $g: \Sigma^{-1}C(\pi f)\to Y$ whose cofiber $C(g)$ coincides with $C(f)$ where $\pi: X\to\Sigma^{j+1}$ denotes the collapsing map. Assume that $\dim Y\leqq j+1\leqq k+1$. If $j<k-1$, then any map $f: \Sigma^k\to X$ is always SQ_*-trivial. If $j=k-1$ or k, then $C(\pi f)=\Sigma^{j+1}\vee\Sigma^{k+1}$ or $\Sigma^{j+1}SZ/t$ for some $t\geqq 1$. Therefore, in order to determine the quasi KO_*-types of any CW-spectra with $(n+1)$-cells it is sufficient to deal with the cofibers of the following maps:

(5.1)
i) any SQ_*-trivial map $f: \Sigma^k\to X$,
ii) any map $g: \Sigma^j SZ/2^m\to Y$ and
iii) any map $g: \Sigma^j\vee\Sigma^k\to Y$ with $k=j$ or $j+1$

where $\#X=n$, $\#Y=n-1$ and $\dim Y\leqq j+1$. For any graded abelian group $G=\{G_i\}$ the wedge sum $\vee\Sigma^i SG_i$ of Moore spectra is simply written to be SG.

LEMMA 5.1. *Let X be a CW-spectrum having the same quasi KO_*-type as $Y=SA\vee(P\wedge SB)\vee(Q\wedge SC)$ with $A=\{A_i\}_{0\leq i\leq 7}$, $B=\{B_j\}_{0\leq j\leq 1}$ and $C=\{C_k\}_{0\leq k\leq 3}$ free. If any map $f: S_0\to X$ is SQ_*-trivial, then its cofiber $C(f)$ is quasi KO_*-equivalent to one of the following spectra $\Sigma^1\vee Y$, $Y_{-7,1,*}$, $Y_{-6,*,2}$ and $Y_{2,1,-3}$ where $Y_{-7,1,*}\vee\Sigma^7=Y\vee\Sigma^1 P$, $Y_{-6,*,2}\vee\Sigma^6=Y\vee\Sigma^2 Q$ and $Y_{2,1,-3}\vee\Sigma^3 Q=Y\vee\Sigma^2\vee\Sigma^1 P$.*

Proof. The cofibers of the maps $i_Q\eta: \Sigma^0\to\Sigma^{-1}Q$ and $(\eta^2, i_Q\eta): \Sigma^0\to\Sigma^{-2}\vee\Sigma^{-1}Q$ are the wedge sums $\Sigma^2\vee\Sigma^{-1}P$ and $\Sigma^{-2}R\vee\Sigma^{-1}P$ respectively where R denotes the cofiber of the map $\eta^3: \Sigma^3\to\Sigma^0$. In these cases they are quasi KO_*-equivalent to the spectrum $Y_{2,1,-3}$. Now our result is easy.

If any map $f=(f_1, f_2): S_k\to S_0\vee Y$ is SQ_*-trivial, then there exists an SQ_*-trivial map $g: \Sigma^{-1}C(f_1)\to Y$ whose cofiber $C(g)$ coincides with $C(f)$. Note that $C(f_1)$ has the same quasi KO_*-type as the elementary spectrum P or Q unless f_1 is KO_*-trivial. By the aid of Lemmas 1.2, 1.5 and 2.4–2.7 it is verified that

(5.2) the quasi KO_*-type of $C(f)$ is completely determined when $Y=\Sigma^i SZ/2^m$ or $\Sigma^i V_m$ and $f=(f_1, f_2): S_k\to S_0\vee Y$ is SQ_*-trivial.

As is easily seen, we obtain

LEMMA 5.2. *For any map $g: \Sigma^j\vee\Sigma^k\to\Sigma^0$ $(0\leqq j\leqq k)$ its cofiber $C(g)$ is quasi*

KO_*-equivalent to the wedge sum $\Sigma^0\vee\Sigma^{j+1}\vee\Sigma^{k+1}$ *or the following spectrum* $Y_{j,k}: Y_{0,i}=\Sigma^{i+1}\vee SZ/2^m\vee SZ/q$, $Y_{0,8r+1}=M_m\vee SZ/q$, $Y_{0,8r+2}=N_m\vee SZ/q$, $Y_{8r+1,i}=Y_{i,8r+1}=\Sigma^{i+1}\vee P$ *or* $Y_{8r+2,i}=Y_{i,8r+2}=\Sigma^{i+1}\vee Q$ $(i, r\geqq 0)$ *where* $m\geqq 0$ *and* $q\geqq 1$ *is odd.*

For any finite CW-spectrum X we denote by $k_0(X)$ the rank of $KU_*X\otimes Q$ and by $k_p(X)$ the rank of $\mathrm{Tor}(KU_*X, Z/p)$ for each prime p where $KU_*X\cong KU_0X\oplus KU_1X$. Set $k(X)=k_0(X)+\underset{p}{\mathrm{Max}}\{2k_p(X)\}$. Then it is immediately checked that

(5.3) $$\#X\geqq k(X) \quad\text{and}\quad \#X\equiv k(X) \bmod 2.$$

In particular, $KU_*X\cong Z\oplus Z\oplus Z$ or $Z\oplus Z/2^m\oplus Z/q$ when $\#X=3$, and $KU_*X\cong Z\oplus Z\oplus Z\oplus Z$, $Z\oplus Z\oplus Z/2^m\oplus Z/q$ or $Z/2^m\oplus Z/2^n\oplus Z/q\oplus Z/r$ when $\#X=4$, where $m, n\geqq 0$ and both of $q, r\geqq 1$ are odd.

Recall that each CW-spectrum with 2-cells is stably quasi KO_*-equivalent to one of the following spectra: $\Sigma^0\vee\Sigma^i$ $(0\leqq i\leqq 7)$, P, Q or $SZ/2^m\vee SZ/q$ where $m\geqq 0$ and $q\geqq 1$ is odd. Using Lemmas 1.2, 1.3, 1.4, 5.1 and 5.2 and (1.6) we can immediately show

THEOREM 5.3. *Let X be a CW-spectrum with 3-cells. Then it is stably quasi KO_*-equivalent to the following spectrum Y:*

i) *The "$KU_*X\cong Z\oplus Z\oplus Z$" case:* $Y=\Sigma^0\vee\Sigma^i\vee\Sigma^j$, $P\vee\Sigma^j$ *or* $Q\vee\Sigma^j$ $(0\leqq i\leqq j\leqq 7)$.

ii) *The "$KU_*X\cong Z\oplus Z/q$ ($q\geqq 1$ odd)" case:* $Y=\Sigma^j\vee SZ/q$ $(0\leqq j\leqq 7)$.

iii) *The "$KU_*X\cong Z\oplus Z/2^m\oplus Z/q$ ($m\geqq 1$, and $q\geqq 1$ odd)" case:* $Y=W\vee SZ/q$ *and* $W=\Sigma^j\vee SZ/2^m$ $(0\leqq j\leqq 7)$, $\Sigma^0\vee V_m$, $\Sigma^5\vee V_m$, M_m, N_m, Q_m, R_m, $\Sigma^{-1}M'_m$, $\Sigma^{-2}N'_m$, $\Sigma^{-3}Q'_m$ *or* $\Sigma^{-4}R'_m$.

5.2. Let X be a CW-spectrum with 3-cells and $f: S_k\to X$ an SQ_*-trivial map. Since the quasi KO_*-type of X is completely observed in Theorem 5.3, we can easily determine the quasi KO_*-type of the cofiber $C(f)$ by means of Propositions 4.1-4.8, Lemma 5.1 and (5.2). We next deal with any map $g=g_1\vee g_2: S_j\vee S_k\to SZ_m$. Evidently there exists an SQ_*-trivial map $h: S_k\to C(g_1)$ whose cofiber $C(h)$ coincides with $C(g)$. Since the quasi KO_*-type of $C(g_1)$ is completely given in Lemma 1.2, we can easily determine the quasi KO_*-type of $C(g)$ by means of Propositions 4.1-4.5 and (5.2), too. Dually we can determine the quasi KO_*-type of $C(g')$ for any map $g'=(g'_1, g'_2): \Sigma^j SZ_m\to S_0\vee S_i$.

Let Y be a CW-spectrum with 2-cells having the same quasi KO_*-type as the elementary spectrum P or Q. For such a CW-spectrum $Y=S^0\cup e^{8r+2}$ or $S^0\cup e^{8r+3}$ it is easily shown that

(5.4) any map $g=g_1\vee g_2: \Sigma^j\vee\Sigma^k\to Y$ is quasi KO_*-equivalent to the map $g_1\vee 0$ or $0\vee g_2$ if $8r+1\leqq j\leqq k\leqq j+1$.

Let Y be a CW-spectrum with 2-cells whose attaching map $\alpha: \Sigma^i\to\Sigma^0$ is KO_*-

trivial, and $g=g_1\vee g_2$: $\Sigma^j\vee\Sigma^k\to Y$ $(-1\leqq i\leqq j\leqq k\leqq j+1)$ be any map. Assume that the map g is never quasi KO_*-equivalent to the map $g_1\vee 0$ or $0\vee g_2$. When $k>i+1$ the cofiber $C(g)$ is obtained as that of a certain SQ_*-trivial map h: $\Sigma^k\to C(g_1)$. In this case it is easy to determine the quasi KO_*-type of $C(g)$ as is stated above. In the $k=i$ or $i+1$ case the cofiber $C(g)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^1\vee\Sigma^l\vee SZ/2^m\vee SZ/q$ $(l=0,\ 1)$ or $\Sigma^1\vee M_m\vee SZ/q$ for some $m\geqq 0$ and some odd $q\geqq 1$ if the composite map πg: $\Sigma^j\vee\Sigma^k\to\Sigma^{i+1}$ is trivial. If not so, there exists a map h: $\Sigma^j\vee\Sigma^i SZ/t\to\Sigma^0$ for some $t\geqq 1$, whose cofiber $C(h)$ coincides with $C(g)$. When such a map h is SQ_*-trivial, the quasi KO_*-type of $C(h)$ is easily determined by a dual argument to (5.2). If not so, then the cofiber $C(h)$ is the wedge sum $SZ/2^m\vee\Sigma^l SZ/2^n\vee SZ/q\vee\Sigma^l SZ/r$ $(l=0,\ 1)$ or $M_{m,n}\vee SZ/q\vee\Sigma^1 SZ/r$ for some $m,\ n\geqq 0$ and some odd $q,\ r\geqq 1$.

In virtue of (5.1) we can now show our main result by the above observations combined with (2.5), (4.8), (4.9) and Lemmas 2.3, 2.4 and 2.5.

THEOREM 5.4. *Let X be a CW-spectrum with 4-cells. Then it is stably quasi KO_*-equivalent to the following spectrum Y:*

i) *The "$KU_*X\cong Z\oplus Z\oplus Z\oplus Z$" case: $Y=\Sigma^0\vee\Sigma^i\vee\Sigma^j\vee\Sigma^k$, $P\vee\Sigma^j\vee\Sigma^k$, $Q\vee\Sigma^j\vee\Sigma^k$, $P\vee\Sigma^j P$, $P\vee\Sigma^j Q$ or $Q\vee\Sigma^j Q$ $(0\leqq i\leqq j\leqq k\leqq 7)$.*

ii) *The "$KU_*X\cong Z\oplus Z\oplus Z/q$ ($q\geqq 1$ odd)" case: $Y=\Sigma^j\vee\Sigma^k\vee SZ/q$, $\Sigma^j P\vee SZ/q$ or $\Sigma^j Q\vee SZ/q$ $(0\leqq j\leqq k\leqq 7)$.*

iii) *The "$KU_*X\cong Z\oplus Z\oplus Z/2^m\oplus Z/q$ ($m\geqq 1$, and $q\geqq 1$ odd)" case: $Y=W\vee SZ/q$ and $W=\Sigma^j\vee\Sigma^k\vee SZ/2^m$, $\Sigma^j P\vee SZ/2^m$, $\Sigma^j Q\vee SZ/2^m$, $\Sigma^0\vee\Sigma^k\vee V_m$, $\Sigma^5\vee\Sigma^k\vee V_m$, $\Sigma^j P\vee V_m$, $\Sigma^l Q\vee V_m$, $\Sigma^k\vee X_m$, $\Sigma^k\vee X'_m$, XY_m, $X'Y'_m$, $Y'X_m$ $(0\leqq j\leqq k\leqq 7$ and $0\leqq l\leqq 2)$ where $X_m=M_m$, N_m, Q_m or R_m; $X'_m=\Sigma^{-1}M'_m$, $\Sigma^{-2}N'_m$, $\Sigma^{-3}Q'_m$ or $\Sigma^{-4}R'_m$; $XY_m=MQ_m$, MR_m, NQ_m or NR_m; $X'Y'_m=\Sigma^{-3}M'Q'_m$, $\Sigma^{-4}M'R'_m$, $\Sigma^{-3}N'Q'_m$ or $\Sigma^{-4}N'R'_m$; and $Y'X_m=\Sigma^{-1}M'M_m$, $\Sigma^{-1}M'N_m$, $\Sigma^{-2}N'M_m$, $\Sigma^{-2}N'N_m$, $\Sigma^{-3}Q'Q_m$, $\Sigma^{-3}Q'R_m$, $\Sigma^{-4}R'Q_m$ or $\Sigma^{-4}R'R_m$.*

iv) *The "$KU_*X\cong Z/2^m\oplus Z/q\oplus Z/r$ ($m\geqq 0$, and $q,\ r\geqq 1$ odd)" case: $Y=SZ/2^m\vee SZ/q\vee\Sigma^j SZ/r$ $(0\leqq j\leqq 3)$, $V_m\vee SZ/q\vee\Sigma^l SZ/r$ $(1\leqq l\leqq 3)$ or $W_m\vee SZ/q\vee\Sigma^2 SZ/r$.*

v) *The "$KU_*X\cong Z/2^m\oplus Z/2^n\oplus Z/q\oplus Z/r$ ($m,\ n\geqq 1$, and $q,\ r\geqq 1$ odd)" case: $Y=U\vee SZ/q\vee\Sigma^j SZ/r$ and $U=SZ/2^m\vee\Sigma^j SZ/2^n$ $(0\leqq j\leqq 7)$, $V_m\vee\Sigma^1 V_n$ $(j=1)$, $V_m\vee\Sigma^4 V_n$ $(|m-n|\geqq 2$ and $j=0)$, $V_m\vee W_n$ $(m+2\leqq n$ and $j=0)$, $W_m\vee V_n$ $(m\geqq n+2$ and $j=0)$ or $X_{m,n}$ $(j=\dim X_{m,n}-1)$ where $X_{m,n}=M_{m,n}$, $N_{m,n}$, $P_{m,n}$ $(m\geqq n+1)$, $P_{m+1,n-1}$ $(m+1\leqq n)$, $P'_{m,n}$ $(m+1\leqq n)$, $P'_{m-1,n+1}$ $(m\geqq n+1)$, $P''_{m+1,n-1}$ $(m+2<n)$, $P''_{m,n}$ $(m=n)$, $P''_{m-1,n+1}$ $(m>n+2)$, $Q_{m,n}$, $Q'_{m,n}$, $Q''_{m,n}$, $R_{m,n}$ $(m\leqq n)$, $R'_{m,n}$ $(m\geqq n)$, $H_{m+1,n+1}$, $K_{m,n}$ $((m,\ n)\neq(1,\ 1))$ or $L_{m,n}$.*

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DEPARTMENT OF MATHEMATICS
OSAKA CITY UNIVERSITY
SUGIMOTO SUMIYOSHI-KU
OSAKA 558, JAPAN