M. OKA
KODAI MATH. J.
16 (1993), 181—195

FINITENESS OF FUNDAMENTAL GROUP OF COMPACT CONVEX INTEGRAL POLYHEDRA

BY MUTSUO OKA

§ 1. Introduction and statement of the result.

Let Δ_i, $i=1, \cdots, k$, be given compact convex integral polyhedra in $\boldsymbol{R}^m$. We consider the following integer "combinatorial connectivity" $\alpha(\Delta_1, \cdots, \Delta_k)$ which is defined in [Ok6] by

$$\alpha(\Delta_1, \cdots, \Delta_k)=\min\left\{\dim\left(\sum_{i\in I}\Delta_i\right)-|I|\,;\, I\subset\{1, \cdots, k\},\ I\neq\emptyset\right\}.$$

We assume that $\alpha(\Delta_1, \cdots, \Delta_k)\geqq 0$. For any integral covector P, we consider the restriction $P|_{\Delta_i}$ to Δ_i of the corresponding linear function associated with P. Let $\Delta(P;\Delta_i)$ be the face where $P|_{\Delta_i}$ takes its minimal value ([Ok5, 6]). We denote the lattice of the integral covectors by N. We define the subgroup $K(\Delta_1, \cdots, \Delta_k)$ of N by

$$K(\Delta_1, \cdots, \Delta_k)=\langle P\in N;\, \alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k))\geqq 0\rangle.$$

Here $\langle P\in N;\, P\in S\rangle$ is the subgroup of N which is generated by the covectors P in S. We also define $\Pi_1(\Delta_1, \cdots, \Delta_k):=N/K(\Delta_1, \cdots, \Delta_k)$. We call $K(\Delta_1, \cdots, \Delta_k)$ (respectively $\Pi_1(\Delta_1, \cdots, \Delta_k)$) *the boundary lattice group* (resp. *the fundamental group*) of the k-ple of polyhedra $\{\Delta_1, \cdots, \Delta_k\}$. The purpose of this paper is to prove:

MAIN THEOREM (1.1). *The boundary lattice group $K(\Delta_1, \cdots, \Delta_k)$ has rank m if and only if $\alpha(\Delta_1, \cdots, \Delta_k)\geqq 1$.*

The geometric interpretation is as follows. Let $h_1(\boldsymbol{u}), \cdots, h_k(\boldsymbol{u})$ be Laurent polynomials such that the respective Newton polygon $\Delta(h_i)$ is equal to Δ_i, for $i=1, \cdots, k$. Let us consider the variety:

$$Z^*=\{\boldsymbol{u}\in\boldsymbol{C}^{*m};\, h_1(\boldsymbol{u})=\cdots=h_k(\boldsymbol{u})=0\}.$$

We can choose the coefficients of $h_1, \cdots, h_k$ so that Z^* is a non-degenerate complete intersection variety in the sense of [Kh1, 2, Ok4, 5]. See § 4 for the existence of such Laurent polynomials $h_1(\boldsymbol{u}), \cdots, h_k(\boldsymbol{u})$. Z^* is non-empty if and

Received January 21, 1993.

only if $\alpha(\Delta_1, \cdots, \Delta_k) \geqq 0$ ([Ok4]). Let Σ^* be a regular simplicial cone subdivision of the dual Newton diagram $\Gamma^*(h_1, \cdots, h_k)=\Gamma^*(\Delta_1, \cdots, \Delta_k)$ and let X be the associated toric compactification of the ambient torus C^{*m}. Let $\tilde{Z}$ be the compactification (=closure) of Z^* in X. Recall that for each vertex P of Σ^*, there exists a corresponding rational divisor $\hat{E}(P)$ of X so that X has the toric stratification

$$X=C^{*m}\coprod_{\mathrm{Cone}(P_1, \cdots, P_s)\in\Sigma^*} \hat{E}(P_1, \cdots, P_s)^*$$

where $\hat{E}(P_1, \cdots, P_s)^*=\bigcap_{i=1}^{s} \hat{E}(P_i)-\bigcup_{P\neq P_1, \cdots, P_s} E(P)$. Let $E(P)=\hat{E}(P)\cap\tilde{Z}$. Note that $E(P)$ is non-empty if and only if $\alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k))\geqq 0$ (Proposition (5.4), [Ok4]). We will see in § 2 that the above subgroup $K(\Delta_1, \cdots, \Delta_k)$ is generated by those $P\in\mathrm{Vertex}(\Sigma^*)$ such that $E(P)\neq\emptyset$ (Assertion (2.4), § 2).

Let G be a finite abelian group. We denote by $\rho(G)$ the minimal number of generators of G. We say that Z^* is *full* if $\dim\Delta_i=m$ for each $i=1, \cdots, k$. By Lemma (4.1) and Theorem (4.2) of [Ok5], we have:

THEOREM (1.2). (1) *Assume that* $\alpha(\Delta_1, \cdots, \Delta_k)\geqq 0$. *Then the fundamental group* $\Pi_1(\Delta_1, \cdots, \Delta_k)$ *is generated by at most* k *elements. That is,* $\rho(\Delta_1, \cdots, \Delta_k) \leqq k$.

(2) *If* $\pi_1(Z^*)\to\pi_1(C^{*m})$ *is isomorphic,* $\Pi_1(\Delta_1, \cdots, \Delta_k)$ *is isomorphic to the fundamental group* $\pi_1(\tilde{Z})$. *In particular, this is the case if* Z^* *is full and* $m-k \geqq 2$.

In [Ok6], we have generalized the second assertion for a non-degenerate complete intersection variety with $\alpha(\Delta_1, \cdots, \Delta_k)\geqq 2$ which satisfies the monotone support condition:

$$(Mn) \qquad \dim\Big(\sum_{i=1}^{j}\Delta_i\Big)=\dim\Delta_j, \qquad j=1, \cdots, k.$$

Note that any full non-degenerate complete intersection variety with $m-k\geqq 2$ satisfies the monotone support condition. As an immediate corollary of Theorem (1.1) and Theorem (1.2), we obtain the following.

THEOREM (1.3). (1) *Assume that* $\alpha(\Delta_1, \cdots, \Delta_k)\geqq 1$. *Then the fundamental group* $\Pi_1(\Delta_1, \cdots, \Delta_k)$ *is a finite abelian group with* $\rho(\Pi_1(\Delta_1, \cdots, \Delta_k))\leqq k$.

(2) *Assume that* Z^* *satisfies the monotone support condition and* $\alpha(\Delta_1, \cdots, \Delta_k) \geqq 2$. *Then the fundamental group* $\pi_1(\tilde{Z})$ *is a finite abelian group and* $\rho(\pi_1(\tilde{Z}))\leqq k$.

The finiteness for the case $k=1$ has been proved by [Ok1] and the assertion for the general case has been conjectured in [Ok5, 6]. In § 3, we will construct an algebraic surface whose fundamental group is isomorphic to an arbitrarily given finite abelian group.

§ 2. The proof of Main Theorem (1.1).

We first recall the construction of X ([K-K-M-S], [Kh1], [Dn2], [Eh], [Od1, 2], [Ok4, 5]). Let $A=(a_{i,j})\in GL(m, \boldsymbol{Z})$ with $\det A=\pm 1$. We associate to A a birational morphism

$$\psi_A : \boldsymbol{C}^{*m} \longrightarrow \boldsymbol{C}^{*m}$$

which is defined by $\psi_A(z)=(z_1^{a_{1,1}}\cdots z_m^{a_{1,m}}, \cdots, z_1^{a_{m,1}}\cdots z_m^{a_{m,m}})$. The morphism ψ_A satisfies the property: $\psi_A\circ\psi_B=\psi_{AB}$. In particular, $(\psi_A)^{-1}=\psi_{A^{-1}}$. Assume that $a_{i,j_0}\geqq 0$, $i=1, \cdots, m$ for some j_0. Then ψ_A extends to $\boldsymbol{C}^{*n}\cup\{\boldsymbol{z}\,;\, z_{j_0}=0,\ z_i\neq 0,\ i\neq j_0\}$.

The dual Newton diagram $\Gamma^*(\Delta_1, \cdots, \Delta_k)$ is the polyhedral cone subdivision of the space of covector which is induced by the equivalence relation: $P\sim Q \Leftrightarrow \Delta(P;\Delta_i)=\Delta(Q;\Delta_i)$, $i=1, \cdots, k$. Let Σ^* be a given regular simplicial cone subdivision of the dual Newton diagram $\Gamma^*(\Delta_1, \cdots, \Delta_k)$. Let $\mathcal{M}$ be the set of m-dimensional simplicial cones in Σ^*. For each $\sigma=\mathrm{Cone}(P_1, \cdots, P_m)\in\mathcal{M}$, let $\boldsymbol{C}_\sigma^m$ be the affine space of dimension m with coordinate $\boldsymbol{y}_\sigma=(y_{\sigma,1}, \cdots, y_{\sigma,m})$. Here $P_1, \cdots, P_m$ are primitive integral covectors which generate σ and they are called the vertices of σ. Let $P_j={}^t(p_{1,j}, \cdots, p_{n,j})$ for $j=1, \cdots, m$. We identify σ with the corresponding unimodular matrix $(P_1, \cdots, P_m)=(p_{i,j})$. The original torus $\boldsymbol{C}^{*m}$ is identified with the maximal torus $\boldsymbol{C}_\sigma^{*m}:=\{\boldsymbol{y}_\sigma\in\boldsymbol{C}_\sigma^m\,;\, y_{\sigma,i}\neq 0,\ i=1, \cdots, m\}$ of the coordinate space $\boldsymbol{C}_\sigma^m$ through the isomorphism $\psi_\sigma : \boldsymbol{C}_\sigma^{*m}\to\boldsymbol{C}^{*m}$. X is covered by the affine coordinate charts $\{\boldsymbol{C}_\sigma^m\,;\,\sigma\in\mathcal{M}\}$. Let $\sigma=\mathrm{Cone}(P_1, \cdots, P_m)$, $\tau=\mathrm{Cone}(Q_1, \cdots, Q_m)\in\mathcal{M}$. We recall the gluing of these coordinate spaces, as we use it later. Two points of the different coordinate spaces $\mathrm{u}_\sigma\in\boldsymbol{C}_\sigma^m$ and $\mathrm{u}_\tau\in\boldsymbol{C}_\tau^m$ are identified when and only when the birational map $\psi_{\sigma^{-1}\tau}:\boldsymbol{C}_\tau^m\to\boldsymbol{C}_\sigma^m$ is well-defined on $\boldsymbol{y}_\tau=\boldsymbol{u}_\tau\in\boldsymbol{C}_\tau^m$ and $\boldsymbol{u}_\sigma=\psi_{\sigma^{-1}\tau}(\mathrm{u}_\tau)$. Let $\sigma^{-1}\tau=(\lambda_{i,j})$. This implies that

$$Q_j=\sum_{i=1}^m \lambda_{i,j}P_i \tag{2.1}$$

Thus $\lambda_{i,j}\geqq 0$ for each $i=1, \cdots, m$ if and only if $Q_j\in\sigma$. This is the case if and only if $Q_j=P_l$ for some l. Changing the ordering of the vertices if necessary, we can assume that $\sigma\cap\tau=\mathrm{Cone}(P_1, \cdots, P_s)$ and $Q_i=P_i$, $1\leqq i\leqq s$. Then the matrix $\sigma^{-1}\tau$ can be written as

$$\sigma^{-1}\tau=\begin{pmatrix} I_s & A_{1,2} \\ 0 & A_{2,2}\end{pmatrix}$$

where I_s is the $s\times s$ identity matrix and $\psi_{\sigma^{-1}\tau}$ is well defined precisely on $\{\boldsymbol{y}_\tau\in\boldsymbol{C}_\tau^m\,;\, y_{\tau,i}\neq 0,\ s+1\leqq i\leqq m\}$. Thus applying the same argument for $\tau^{-1}\sigma$, we can see that

$$\psi_{\sigma^{-1}\tau}:\ \{\boldsymbol{y}_\tau\in\boldsymbol{C}_\tau^m\,;\, y_{\tau,i}\neq 0,\ s+1\leqq i\leqq m\}\longrightarrow\{\boldsymbol{y}_\sigma\in\boldsymbol{C}_\sigma^m\,;\, y_{\sigma,i}\neq 0,\ s+1\leqq i\leqq m\}$$

is biholomorphic. In particular,

$$C_\sigma^m - C_\tau^m = C_\sigma^m \cap \Big(\bigcup_{i=s+1}^{m} \hat{E}(P_i) \Big) = \{ \boldsymbol{y}_\sigma \in C_\sigma^m \,;\, y_{\sigma,s+1} \cdots y_{\sigma,m} = 0 \} \tag{2.2}$$

Recall that in the coordinate space C_σ^m, $\hat{E}(P_i)$ and $E(P_i) := \tilde{Z} \cap \hat{E}(P_i)$ are defined by

$$\hat{E}(P_i) \cap C_\sigma^m = \{ \boldsymbol{y}_\sigma \in C_\sigma^m \,;\, y_{\sigma,i} = 0 \}$$

$$E(P_i) \cap C_\sigma^m = \{ \boldsymbol{y}_\sigma \in C_\sigma^m \,;\, y_{\sigma,i} = h_{1,P_i,\sigma}(\boldsymbol{y}_\sigma) = \cdots = h_{k,P_i,\sigma}(\boldsymbol{y}_\sigma) = 0 \}$$

where $h_{\alpha,P_i,\sigma}(\boldsymbol{y}_\sigma)$ is defined by the equality $h_{\alpha,P_i}(\phi_\sigma(\boldsymbol{y}_\sigma)) = h_{\alpha,P_i,\sigma}(\boldsymbol{y}_\sigma) \cdot \prod_{j=1}^m y_{\sigma j}^{d(P_j;\Delta_\alpha)}$. Here $d(P_j\,;\,\Delta_\alpha)$ is the minimal value of $P_j|_{\Delta_\alpha}$. Note that $\Delta(h_{\alpha,P_i}) = \Delta(P;\Delta_i)$ and $E(P)$ is a non-empty divisor if and only if $\alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k)) \geqq 0$ by Proposition (5.4) of [Ok4].

Now we prove Main Theorem (1.1). Assume first that $\alpha(\Delta_1, \cdots, \Delta_k) = 0$. There exists a non-empty subset $I \subset \{1, \cdots, k\}$ so that $\dim(\sum_{i\in I} \Delta_i) - |I| = 0$. Take any integral covector P such that $\alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k)) \geqq 0$. Then we must have $\Delta(P;\Delta_i) = \Delta_i$ for any $i \in I$ (Proposition (4.1), §4). This implies that K is orthogonal to the affine subspace generated by $\sum_{i\in I} \Delta_i$. Thus $\mathrm{rank}(K(\Delta_1, \cdots, \Delta_k)) \leqq m - |I|$. Now we assume that

$$\alpha(\Delta_1, \cdots, \Delta_k) \geqq 1\,. \tag{2.3}$$

We have to show that $\mathrm{rank}(K(\Delta_1, \cdots, \Delta_k)) = m$. Let $\mathcal{V}$ be the set of the vertices $P \in \mathrm{Vertex}(\Sigma^*)$ such that $\alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k)) \geqq 0$. It is obvious that $\langle P;\, P \in \mathcal{V} \rangle \subset K(\Delta_1, \cdots, \Delta_k)$.

ASSERTION (2.4). *The boundary lattice group* $K(\Delta_1, \cdots, \Delta_k)$ *is equal to* $\langle P;\, P \in \mathcal{V} \rangle$.

Proof. Assume that P is an integral covector such that $\alpha(\Delta(P;\Delta_1), \cdots, \Delta(P;\Delta_k)) \geqq 0$. P is not necessarily a vertex of Σ^*. Let $[P]$ be the closure of the equivalence class of P in $\Gamma^*(\Delta_1, \cdots, \Delta_k)$. It is easy to see that $\dim[P] = m - \dim(\sum_{i=1}^k \Delta(P;\Delta_i))$. Let $r = \dim[P]$. As Σ^* is a regular simplicial subdivision of $\Gamma^*(\Delta_1, \cdots, \Delta_k)$, there exists a simplicial cone $\sigma = \mathrm{Cone}(P_1, \cdots, P_r)$ in Σ^* such that $P_1, \cdots, P_r \in [\bar{P}]$ (=the closure of $[P]$). Note that $P_i \in \mathcal{V}$ for $i=1, \cdots, r$ as $\Delta(P_i\,;\,\Delta_j) \supset \Delta(P;\Delta_j)$, $j=1, \cdots, k$. It is obvious that we can write $P = \sum_{i=1}^r a_i P_i$ for some rational numbers $a_1, \cdots, a_r$. We assert that $a_i \in \boldsymbol{Z}$ for $i = 1, \cdots, r$. Consider a_r for instance. Then the assertion follows from the equality:

$$\begin{aligned}\boldsymbol{Z} \ni \det(P_1, \cdots, P_{r-1}, P) &= \det(P_1, \cdots, P_{r-1}, \sum_{i=1}^r a_i P_i) \\ &= a_r \det(P_1, \cdots, P_r) = a_r\end{aligned}$$

Here $\det(P_1, \cdots, P_r)$ is the greatest common divisor of the $r \times r$-minors of $n \times r$-

matrix $(P_1, \cdots, P_r)$ as in § 3 of [Ok1]. Q. E. D.

Let $r=\mathrm{rank}(K(\Delta_1, \cdots, \Delta_k))$ and assume that $r \leqq m-1$. We will show that this gives a contradiction. Let $K_{\boldsymbol{R}}=K(\Delta_1, \cdots, \Delta_k)\otimes \boldsymbol{R}$ be the linear subspace of the real vector space of covectors $N_{\boldsymbol{R}}=N\otimes \boldsymbol{R}$. Taking a regular subdivision if necessary, we may assume that the restriction of Σ^* to $K_{\boldsymbol{R}}$ is also a regular simplicial cone subdivision of $K_{\boldsymbol{R}}$ (§ 3, [Ok1]). We consider the subset $\mathcal{M}'$ of coordinate charts $\mathcal{M}$ which is defined by:

$$\tau=\mathrm{Cone}(Q_1, \cdots, Q_m)\in \mathcal{M}' \Longleftrightarrow Q_i \in K(\Delta_1, \cdots, \Delta_k), \quad 1 \leqq i \leqq r .$$

ASSERTION (2.5). *The subfamily* $\{\boldsymbol{C}_\sigma^m ; \sigma \in \mathcal{M}'\}$ *is a covering of* $\tilde{Z}$.

Proof. Take an arbitrary point $p \in \tilde{Z} \cap \boldsymbol{C}_\sigma^m$ where $\sigma=\mathrm{Cone}(P_1, \cdots, P_m)$. Changing the ordering if necessary, we may assume that p corresponds to $(0, \cdots, 0, \alpha_{t+1}, \cdots, \alpha_m)$ with $\alpha_i \neq 0$, $t+1 \leqq i \leqq m$, in this coordinate chart. This implies that $P_j \in \mathcal{V}$ for $j \leqq t$. In particular $t \leqq r$. If $t=r$, $\sigma \in \mathcal{M}'$. Assume that $t<r$. We can find a simplicial cone $\tau=\mathrm{Cone}(Q_1, \cdots, Q_m)$ in $\mathcal{M}'$ such that $Q_j=P_j$ for $j=1, \cdots, t$. Then we see easily that $p \in \tilde{Z} \cap \boldsymbol{C}_\tau^m$ by (2.2). Q. E. D.

Let $\sigma=\mathrm{Cone}(P_1, \cdots, P_m)$ be a fixed simplicial cone in $\mathcal{M}'$. We consider the canonical extension of the coordinate function $y_{\sigma, j}$ for $r+1 \leqq j \leqq m$. They are rational functions on X. We assert:

LEMMA (2.6). *For any* j, $r+1 \leqq j \leqq m$, *the restriction of the rational function* $y_{\sigma, j}$ *to* $\tilde{Z}$ *is holomorphic. In particular, it is constant on each connected component of* $\tilde{Z}$.

Proof. Take a coordinate chart $\boldsymbol{C}_\tau^m$, $\tau=\mathrm{Cone}(Q_1, \cdots, Q_m)\in \mathcal{M}'$, and let $\sigma^{-1}\tau=(\lambda_{i,j})$. Recall that the rational function $y_{\sigma, j}$ is written in the coordinate chart $\boldsymbol{C}_\tau^m$ as $y_{\sigma, j}=y_{\tau, 1}^{\lambda_{j,1}} \cdots y_{\tau, m}^{\lambda_{j,m}}$. By the assumption, both of $\{P_1, \cdots, P_r\}$ and $\{Q_1, \cdots, Q_r\}$ are the basis of $K(\Delta_1, \cdots, \Delta_k)$. Therefore the matrix $\sigma^{-1}\tau=(\lambda_{i,j})$ takes the following form:

$$\sigma^{-1}\tau=\begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix}$$

Namely $\lambda_{i,j}=0$ for $r+1 \leqq i \leqq m$, $1 \leqq j \leqq r$. Therefore we have $y_{\sigma, j}=y_{\tau, r+1}^{\lambda_{j, r+1}} \cdots y_{\tau, m}^{\lambda_{j,m}}$, for $r+1 \leqq j \leqq m$. As $\tilde{Z} \cap \boldsymbol{C}_\tau^m \subset \{\mathrm{y}_\tau ; y_{\tau, i} \neq 0, i=r+1, \cdots, m\}$, the above expression implies that $y_{\sigma, j}$ is a holomorphic function on $\tilde{Z} \cap \boldsymbol{C}_\tau^m$. As $\tilde{Z}$ is a compact complex manifold, the second assertion follows immediately. Q. E. D.

Now we are ready to finish the proof of Theorem (1.1). We assume that $r<m$. (Recall that $r=\mathrm{rank}(K(\Delta_1, \cdots, \Delta_k))$.) By Assertion (2.6), the restriction $y_{\sigma, m}|_{\tilde{Z}}$ is constant on each connected component of $\tilde{Z}$. Let $\{\delta_1, \cdots, \delta_l\}$ be the values of $y_{\sigma, m}|_{\tilde{Z}}$. Let $h_{\sigma, k+1}(\boldsymbol{y}_\sigma):=y_{\sigma, m}-\delta$ for $\delta \in \boldsymbol{C}$. We can choose δ so that

$\delta\neq\delta_1, \cdots, \delta_l$ and the subvariety of Z^*

$$V^*:=\{\boldsymbol{y}_\sigma\in\boldsymbol{C}_\sigma^{*m}\,;\, h_{1,\sigma}(\boldsymbol{y}_\sigma)=\cdots=h_{k,\sigma}(\boldsymbol{y}_\sigma)=h_{k+1,\sigma}(\boldsymbol{y}_\sigma)=0\}$$

is a non-degenerate complete intersection variety. See the Appendix in §4 for the existence of such a δ. By the assumption $\delta\neq\delta_1, \cdots, \delta_l$, V^* is empty. Let $\Delta'_i=\Delta(h_{i,\sigma})$ for $i=1, \cdots, k+1$. The assumption (2.3) implies that $\alpha(\Delta'_1, \cdots, \Delta'_k)\geqq 1$. We assert that $\alpha(\Delta'_1, \cdots, \Delta'_{k+1})\geqq 0$. In fact, for any subset $I\subset\{1, \cdots, k+1\}$, we have

$$\dim\Big(\sum_{i\in I}\Delta'_i\Big)-|I|\begin{cases}\geqq 1 & \text{if } k+1\notin I\\ \geqq 0 & \text{if } k+1\in I,\ |I|\geqq 2\\ =0 & \text{if } I=\{k+1\}.\end{cases}$$

Thus again by Proposition (5.4) in [Ok4], V^* is non-empty. This is a contradiction to the emptiness $V^*=\emptyset$. This completes the proof of Theorem (1.1).

§3. Construction of an algebraic surface with a given fundamental group.

In this section, we will construct an algebraic surface which has an arbitrary given fundamental group. We first give several basic properties of the boundary lattice group $K(\Delta_1, \cdots, \Delta_k)$ and the fuudamental group $\Pi_1(\Delta_1, \cdots, \Delta_k)$.

(3.1) Let $\Delta_i, \Delta'_i, i=1, \cdots, k$, be compact convex integral polyhedra. We say that $\{\Delta_1, \cdots, \Delta_k\}$ and $\{\Delta'_1, \cdots, \Delta'_k\}$ are similar if there exist integral vectors $A_1, \cdots, A_k$ and positive rational numbers $r_1, \cdots, r_k$ so that $\Delta'_i=r_i\Delta_i+A_i$, $i=1, \cdots, k$, and we write $\{\Delta_1, \cdots, \Delta_k\}\overset{s}{\sim}\{\Delta'_1, \cdots, \Delta'_k\}$. Assume that $\{\Delta_1, \cdots, \Delta_k\}\overset{s}{\sim}\{\Delta'_1, \cdots, \Delta'_k\}$. Then it is immediate from the definition that

(3.1.1) $\qquad K(\Delta_1, \cdots, \Delta_k)=K(\Delta'_1, \cdots, \Delta'_k),\quad \Pi_1(\Delta_1, \cdots, \Delta_k)=\Pi_1(\Delta'_1, \cdots, \Delta'_k)$

(3.2) There is a canonical action of the unimodular matrices $SL(m\,;\boldsymbol{Z})$ to the set of compact convex integral polyhedra. Let ξ be a unimodular matrix and let Δ be a compact convex integral polyhedron. We denote the image of Δ by the action of ξ by Δ^ξ. Then we have canonical isomorphisms which are induced by the equality $\Delta(\xi P;\Delta)=\Delta(P;\Delta^\xi)$

(3.2.1) $\qquad K(\Delta_1, \cdots, \Delta_k)\cong K(\Delta_1^\xi, \cdots, \Delta_k^\xi),\quad \Pi_1(\Delta_1, \cdots, \Delta_k)\cong\Pi_1(\Delta_1^\xi, \cdots, \Delta_k^\xi),$

$\xi\in SL(m\,;\boldsymbol{Z})$

(3.3) Let $I=\{i_1, \cdots, i_s\}$ be a subset of $\{1, \cdots, k\}$. Then we have the canonical inclusion: $K(\Delta_1, \cdots, \Delta_k)\subset K(\Delta_{i_1}, \cdots, \Delta_{i_s})$. This gives the canonical surjective homomorphism:

(3.3.1) $$\Pi_1(\Delta_1, \cdots, \Delta_k) \longrightarrow \Pi_1(\Delta_{i_1}, \cdots, \Delta_{i_s}) \longrightarrow 0$$

(3.4) Let us consider the case: $\Delta_1 = \cdots = \Delta_k = \Delta$. The corresponding variety is called a strictly similar complete intersection variety ([Ok5, 6]). By the definition, $K(\Delta, \cdots, \Delta)$ is generated by the $(m-k)$-skeleton of the dual Newton diagram $\Gamma^*(\Delta)$. Thus the calculation of $K(\Delta, \cdots, \Delta)$ and $\Pi_1(\Delta, \cdots, \Delta)$ is easy.

Let G be an arbitrary finite abelian group. Now we construct an algebraic surface M such that $\pi_1(M) \cong G$.

Example (3.5). We first consider the case $\rho(G)=1$. Then we can write $G \cong \boldsymbol{Z}/n\boldsymbol{Z}$. We consider the algebraic surface $\tilde{M}_n$ which is the compactification of

$$M_n^* = \{(x, y, z) \in \boldsymbol{C}^{*3};\ h(x, y, z) = x^{6n}z^3 + y^{2n}z^2 + z + 1 = 0\}$$

and let $\Delta_n := \Delta(h)$. The dual Newton diagram is generated by four vertices:

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -1 \\ -2 \\ 2n \end{pmatrix}, \quad P_4 = \begin{pmatrix} 2 \\ 3 \\ -6n \end{pmatrix}.$$

Thus $K(\Delta_n)$ is generated by integral covectors in $\mathrm{Cone}(P_i, P_j)$, $1 \leq i < j \leq 4$. Let $\langle \mathrm{Cone}(P_i, P_j) \rangle_{\boldsymbol{Z}}$ be the subgroup which is generated by the integral covectors in $\mathrm{Cone}(P_i, P_j)$. Note that $\langle \mathrm{Cone}(P_i, P_j) \rangle_{\boldsymbol{Z}}$ is generated by P_i and P_j if and only if $\det(P_i, P_j)=1$. Otherwise $\langle \mathrm{Cone}(P_i, P_j) \rangle_{\boldsymbol{Z}} = \langle P_i, T \rangle$ where T is an integral covector $T \in \mathrm{Cone}(P_i, P_j)$ such that $\det(P_i, T)=1$. See the proof of Assertion (2.4). In our case, $\langle \mathrm{Cone}(P_i, P_j) \rangle_{\boldsymbol{Z}} = \langle P_i, P_j \rangle$ for $(i, j)=(1, 2), (2, 3)$. As $\det(P_1, P_3)=2$, $\det(P_1, P_4)=3$ and $\det(P_2, P_4)=2$, $\langle \mathrm{Cone}(P_1, P_3) \rangle_{\boldsymbol{Z}} = \langle P_1, T \rangle$, $\langle \mathrm{Cone}(P_1, P_4) \rangle_{\boldsymbol{Z}} = \langle P_1, S \rangle$ and $\langle \mathrm{Cone}(P_2, P_4) \rangle_{\boldsymbol{Z}} = \langle P_2 R \rangle$ where

$$T := (P_1+P_3)/2 = \begin{pmatrix} 0 \\ -1 \\ n \end{pmatrix}, \quad S := (P_4+P_1)/3 = \begin{pmatrix} 1 \\ 1 \\ -2n \end{pmatrix}, \quad R := (P_4+P_2)/2 = \begin{pmatrix} 1 \\ 2 \\ -3n \end{pmatrix}.$$

Thus $K(\Delta_n)$ is generated by covectors $P_1, \cdots, P_4, T, S, R$ and we can easily see that

$$K(\Delta_n) = \left\{ \begin{pmatrix} a \\ b \\ cn \end{pmatrix};\ a, b, c \in \boldsymbol{Z} \right\}, \qquad \Pi_1(\Delta_n) = \pi_1(\tilde{M}_n) = \boldsymbol{Z}/n\boldsymbol{Z}$$

Remark (3.6). To construct an explicit algebraic surface with fundamental group $\boldsymbol{Z}/n\boldsymbol{Z}$ whose topological Euler characteristic or geometric genus is as small as possible, the above example is not the best for n relatively coprime to 6. Let

$$N_n^* = \{(x, y, z) \in \boldsymbol{C}^{*3};\ x^n z^3 + y^n z^2 + z + 1\}.$$

Then we have $\pi_1(\tilde{N}_n) = z/n'\boldsymbol{Z}$ where $n' = n/\gcd\langle n, 6)$. This series contains many interesting surfaces. For example, $\tilde{N}_4$ is called an Enriques surface and

$\pi_1(\tilde{N}_4)=\boldsymbol{Z}/2\boldsymbol{Z}$. $\tilde{N}_5$ has the fundamental group $\boldsymbol{Z}/5\boldsymbol{Z}$ and it is called a Godeaux surface. We have studied these cases in [Ok3, Ok2].

Example (3.7). Let l, n be a given positive integer. We consider the case that $G\cong\boldsymbol{Z}/n\boldsymbol{Z}\oplus\boldsymbol{Z}/nl\boldsymbol{Z}$. We consider a strictly similar non-degenerate complete intersection variety $M_{n,l}^{*}=\{\boldsymbol{u}\in\boldsymbol{C}^{*4};\ h_1(\boldsymbol{u})=h_2(\boldsymbol{u})=0\}$ whose dual Newton diagram is generated by five vertices:

$$P_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad P_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad P_3=\begin{pmatrix}-1\\-1\\n\\0\end{pmatrix},\quad P_4=\begin{pmatrix}-1\\-2\\-2n\\2ln\end{pmatrix},\quad P_5=\begin{pmatrix}1\\3\\3n\\-6ln\end{pmatrix}.$$

For example, we can take

$$h_i(\boldsymbol{u})=a_{i,1}u_1^{6ln}u_3^{6l}u_4^{9}+a_{i,2}u_2^{5ln}u_3^{5l}u_4^{10}+a_{i,3}u_3^{10l}u_4^{10}+a_{i,4}u_4^{5}+1,\qquad i=1,2.$$

Let $\Delta_{n,l}=\Delta(h_i(\boldsymbol{u}))$. As $\det(P_1,P_4)=2$, $\det(P_1,P_5)=3$ and $\det(P_3,P_5)=2$, we have $\langle\mathrm{Cone}(P_1,P_4)\rangle_{\boldsymbol{Z}}=\langle P_1,T\rangle$, $\langle\mathrm{Cone}(P_1,P_5)\rangle_{\boldsymbol{Z}}=\langle P_1,S_1\rangle$ and $\langle\mathrm{Cone}(P_3,P_5)\rangle_{\boldsymbol{Z}}=\langle P_3,R\rangle$ where $T=(P_1+P_4)/2={}^t(0,-1,-n,ln)$, $S=(P_5+2P_1)/3={}^t(1,1,n,-2ln)$ and $R=(P_5+P_3)/2={}^t(0,1,2n,-3ln)$. Thus $K(\Delta_{n,l},\Delta_{n,l})$ is generated by those vertices and we have

$$K(\Delta_{n,l},\Delta_{n,l})=\left\{\begin{pmatrix}a\\b\\cn\\dln\end{pmatrix};\ a,b,c,d\in\boldsymbol{Z}\right\},$$

$$\Pi_1(\Delta_{n,l},\Delta_{n,l})=\pi_1(\tilde{M}_{n,l})=\boldsymbol{Z}/n\boldsymbol{Z}\oplus\boldsymbol{Z}/ln\boldsymbol{Z}.$$

Now we consider $K(\Delta_{n,l})$. As $K(\Delta_{n,l})$ is generated by 3-skeleton of $\Gamma^*(\Delta_{n,l})$, we have to add $\langle\mathrm{Cone}(P_i,P_j,P_k)\rangle_{\boldsymbol{Z}}$ to $K(\Delta_{n,l},\Delta_{n,l})$. First we have $E_3:=(P_1+P_2+P_3)/n={}^t(0,0,1,0)\in\mathrm{Cone}(P_1,P_2,P_3)$. Secondly $-E_4:=(P_1+3P_4+2P_5)/6ln={}^t(0,0,0,-1)$. Thus we have $K(\Delta_{n,l})=N$ and $\Pi_1(\Delta_{n,l})=0$.

Example (3.8). Let n, m, l be given positive integers and assume that $G\cong\boldsymbol{Z}/n\boldsymbol{Z}\oplus\boldsymbol{Z}/nm\boldsymbol{Z}\oplus\boldsymbol{Z}/nml\boldsymbol{Z}$. We consider an algebraic surface $\tilde{M}_{n,m,l}$ which is the compactification of the non-degenerate complete intersection variety

$$M_{n,m,l}^{*}=\{\boldsymbol{u}\in\boldsymbol{C}^{*5};\ h_1(\boldsymbol{u})=h_2(\boldsymbol{u})=h_3(\boldsymbol{u})=0\}$$

whose dual Newton diagram is generated by six vertices

$$P_1=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\quad P_2=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\quad P_3=\begin{pmatrix}-1\\-1\\n\\0\\0\end{pmatrix},\quad P_4=\begin{pmatrix}-1\\-2\\-2n\\2nm\\0\end{pmatrix},$$

$$P_5=\begin{pmatrix}-1\\-3\\-3n\\-6nm\\6nml\end{pmatrix},\quad P_6=\begin{pmatrix}1\\4\\8n\\12nm\\-24nml\end{pmatrix}.$$

For example, we can take

$$h_i(\boldsymbol{u})=a_{i,1}(u_1^n u_3)^{288ml}u_4^{432l}u_5^{624}+a_{i,2}(u_2^n u_3)^{200ml}u_4^{400l}u_5^{600}$$
$$+a_{i,3}(u_3^m u_4)^{450l}u_5^{675}+a_{i,4}(u_4^l u_5)^{600}+a_{i,5}u_5^{300}+1,\quad i=1, 2, 3.$$

Let $\Delta=\Delta(h_i)$. As $\det(P_1, P_4)=2$, $\det(P_1, P_5)=3$, $\det(P_1, P_6)=4$, $\det(P_3, P_5)=2$, $\det(P_3, P_6)=3$ and $\det(P_4, P_6)=2$, we have

$$\langle\mathrm{Cone}(P_1, P_4)\rangle_{\boldsymbol{Z}}=\langle P_1, P_{1,4}\rangle,\qquad \langle\mathrm{Cone}(P_1, P_5)\rangle_{\boldsymbol{Z}}=\langle P_1, P_{1,5}\rangle,$$
$$\langle\mathrm{Cone}(P_1, P_6)\rangle_{\boldsymbol{Z}}=\langle P_1, P_{1,6}\rangle,\qquad \langle\mathrm{Cone}(P_3, P_5)\rangle_{\boldsymbol{Z}}=\langle P_3, P_{3,5}\rangle,$$
$$\langle\mathrm{Cone}(P_3, P_6)\rangle_{\boldsymbol{Z}}=\langle P_3, P_{3,6}\rangle,\qquad \langle\mathrm{Cone}(P_4, P_6)\rangle_{\boldsymbol{Z}}=\langle P_4, P_{4,6}\rangle$$

and $\langle\mathrm{Cone}(P_i, P_j)\rangle_{\boldsymbol{Z}}=\langle P_i, P_j\rangle$ for other (i, j) as $\det(P_i, P_j)=1$. Here $P_{1,4}, \cdots, P_{4,6}$ are defined by

$$P_{1,4}=(P_1+P_4)/2={}^t(0, -1, -n, nm, 0)$$
$$P_{1,5}=(P_1+P_5)/3={}^t(0, -1, -n, -2nm, 2nml)$$
$$P_{1,6}=(3P_1+P_6)/4={}^t(1, 1, 2n, 3nm, -6nml)$$
$$P_{3,5}=(P_3+P_5)/2={}^t(-1, -2, -n, -3nm, 3nml)$$
$$P_{3,6}=(P_3+P_6)/3={}^t(0, 1, 3n, 4nm, -8nml)$$
$$P_{4,6}=(P_4+P_6)/2={}^t(0, 1, 3n, 7nm, -12nml).$$

Thus we can easily conclude that

$$K(\Delta, \Delta, \Delta)=\left\{\begin{pmatrix}a\\b\\cn\\dnm\\enml\end{pmatrix};\ a, b, c, d, e\in\boldsymbol{Z}\right\}\quad\text{and}$$

$$\Pi_1(\Delta, \Delta, \Delta)=\pi_1(\tilde{M}_{n,m,l})=\boldsymbol{Z}/n\boldsymbol{Z}\oplus\boldsymbol{Z}/nm\boldsymbol{Z}\oplus\boldsymbol{Z}/nml\boldsymbol{Z}\,.$$

Now we consider $K(\Delta, \Delta)$ and $K(\Delta)$. As generators of $K(\Delta, \Delta)$ (respectively of $K(\Delta)$) we need to add $\langle\mathrm{Cone}(P_i, P_j, P_k)\rangle_{\boldsymbol{Z}}$ (resp. $\langle\mathrm{Cone}(P_i, P_j, P_k, P_l)\rangle_{\boldsymbol{Z}}$). For brevity, we assume that $m\not\equiv 0$ modulo 2, 3 and $l\not\equiv 0$ modulo 3. In addition to the generators of $K(\Delta, \Delta, \Delta)$, we have the following in $K(\Delta, \Delta)$:

$$P_{1,2,3} := {}^t(0, 0, 1, 0, 0) = (P_1+P_2+P_3)/n$$

$$P_{1,3,4} := {}^t(0, -1, -2+n, m, 0) = (P_4+(2n-2)P_3+(2n-1)P_1)/2n$$

$$P_{3,4,6} := {}^t(-1, -1, n, 3, -4l) = (P_6+3P_4+(6nm-2)P_3)/6nm$$

$$P_{1,3,5} := {}^t(0, -1, -1+n, -m, ml) = (P_5+(6n-3)P_3+(6n-2)P_1)/6n$$

$$P_{1,4,5} := {}^t(0, -1, n, -3+nm, 2l) = (P_5+(3nm-3)P_{1,4}+P_1)/3nm$$

and the following is also contained in $K(\Delta)$:

$$\begin{aligned} {}^t(0, -1, n, 0, -1) = (P_6&+(12nml-12n+4)P_3 \\ &+(12nml-12n+3)P_1+12nP_{1,3,5})/12nml. \end{aligned}$$

Thus $\Pi_1(\Delta, \Delta)=\boldsymbol{Z}/l\boldsymbol{Z}$ and $\Pi_1(\Delta)=0$. We leave the details for the calculation of this assertion to the reader.

Example (3.9). A polynomial $h(\boldsymbol{u})$ is called *strongly full* if for any subset $I\subset\{1, \cdots, k\}$, the restriction $h^I := h|_{C^I}$ is not constantly zero and $\dim(\Delta(h^I)) = |I|$ ([Ok6]). We also call $\Delta(h)$ a strongly full polyhedron. Assume that $\Delta_1, \cdots, \Delta_k$ are strongly full and $m-k\geqq 2$. Then it is easy to see that $E_i := {}^t(0,\cdots,\overset{i}{\check{1}},\cdots,0)$ is in $K(\Delta_1, \cdots, \Delta_k)$ for any $i=1, \cdots, m$. Therefore we have that $K(\Delta_1, \cdots, \Delta_k)=N$ and $\Pi_1(\Delta_1, \cdots, \Delta_k)=0$. In particular, any non-degenerate strongly full complete intersection variety of dimension $m-k\geqq 2$ is always simply-connected. The simply connectedness of a smooth complete intersection variety, with dimension greater than 1, in the projective space $\boldsymbol{P}^n$ can be reduced to this criterion.

General Case (3.10). Let $n_1, \cdots, n_s$ be given positive integers. We will construct an algebraic surface whose fundamental group is isomorphic to $\boldsymbol{Z}/n_1\boldsymbol{Z} \oplus\cdots\oplus\boldsymbol{Z}/n_s\boldsymbol{Z}$. Probably we can construct such a surface as a strongly similar non-degenerate complete intersection variety as we have constructed in the case of $s\leqq 3$ in Example (3.6), (3.7) and (3.8). However to give a uniform series at a time seems fairly complicated as is already the case in Example (3.8). We propose a slightly different point of view. We start from the product variety of dimension $2s$

$$W^* = M^*_{n_1}\times\cdots\times M^*_{n_s} = \{(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_s)\in \boldsymbol{C}^{*3s};\ h_i(\boldsymbol{u}_i)=0,\ i=1, \cdots, s\}$$

where $\boldsymbol{u}_i=(x_i, y_i, z_i)$ and $h_i(\boldsymbol{u}_i)=x_i^{6n_i}z_i^3+y_i^{2n_i}z_i^2+z_i+1$. The surface $M^*_{n_i} = \{\boldsymbol{u}_i\in \boldsymbol{C}^{*3};\ h_i(\boldsymbol{u}_i)=0\}$ is studied in Example (3.6). The surfaces $\tilde{N}_n$ in Remark (3.6) can be equally used for the following construction. Let $\Delta_i=\Delta(h_i)$. It is easy to see that

$$K(\Delta_1, \cdots, \Delta_s)=K_3(\Delta_1)\times \cdots \times K_3(\Delta_s), \quad \Gamma^*(\Delta_1, \cdots, \Delta_s)=\Gamma_3^*(\Delta_1)\times \cdots \times \Gamma_3^*(\Delta_s)$$

where $K_3(\Delta_i)$ and $\Gamma_3^*(\Delta_i)$ are the boundary lattice group and the dual Newton diagram of Δ_i as a polyhedron in $\boldsymbol{R}^3$. Taking the product compactification $X=X_1\times \cdots \times X_s$ associated with a product regular simplicial cone subdivision $\Sigma^*=\Sigma_1\times \cdots \times \Sigma_s$ of $\Gamma_3^*(\Delta_1)\times \cdots \times \Gamma_3^*(\Delta_s)$, we can see that the compactification $\widetilde{W}$ of W^* is nothing but the product $\widetilde{M}_{n_1}\times \cdots \times \widetilde{M}_{n_s}$. Therefore

$$\pi_1(\widetilde{W})=\Pi_1(\Delta_1, \cdots, \Delta_s)=\boldsymbol{Z}/n_1\boldsymbol{Z}\oplus \cdots \oplus \boldsymbol{Z}/n_s\boldsymbol{Z}\,. \tag{3.10.1}$$

Let $\Xi=\Delta_1+\cdots+\Delta_s$. Note that $\Xi=\Delta_1\times \cdots \times \Delta_s$ if we consider $\Delta_i\subset \boldsymbol{R}^3$ and that $\dim \Xi=3s$. Now we consider the following non-degenerate complete intersection variety of dimension 2 (= a surface) which is given as an iterated admissible hypersurface section of W^* in the sense of [Ok6]:

$$M^*=\{\boldsymbol{u}\in \boldsymbol{C}^{*3s}\,;\ k_j(\boldsymbol{u})=0,\ j=1, \cdots, 3s-2\}$$

where $k_j(\boldsymbol{u})=h_j(\boldsymbol{u}_j)$ for $j=1, \cdots, s$ and $\{k_{s+1}(\boldsymbol{u}), \cdots, k_{3s-2}(\boldsymbol{u})\}$ are generic polynomials with $\Delta(k_j)=\Xi$, for j, $s+1\leqq j\leqq 3s-2$. Let $\widetilde{M}$ be the corresponding compactification. The following lemma and Theorem (1.2) implies that $\pi_1(\widetilde{M})=z/n_1\boldsymbol{Z}\oplus \cdots \oplus \boldsymbol{Z}/n_s\boldsymbol{Z}$. Thus $\widetilde{M}$ is a surface which we are looking for.

LEMMA (3.11). *We have* $K(\Delta_1, \cdots, \Delta_s, \Xi, \cdots, \Xi)=K(\Delta_1, \cdots, \Delta_s)$. (*Here there are* $(2s-2)$-*copies of* Ξ *in the left side.*) *Therefore*

$$\Pi_1(\Delta_1, \cdots, \Delta_s, \Xi, \cdots, \Xi)=\Pi_1(\Delta_1, \cdots, \Delta_s)=\boldsymbol{Z}/n_1\boldsymbol{Z}\oplus \cdots \oplus \boldsymbol{Z}/n_s\boldsymbol{Z}\,.$$

Proof. We have seen that $K(\Delta_1, \cdots, \Delta_s, \Xi, \cdots, \Xi)\subset K(\Delta_1, \cdots, \Delta_s)$ in (3.3). We have to show the opposite inclusion. Let N_i be the lattice of covectors corresponding to the variable $\boldsymbol{u}_i$ and let $p_i: N\to N_i$ be the canonical projection and let $\iota_i: N_i\to N$ be the canonical inclusion. Then $\psi: N\to N_1\oplus \cdots \oplus N_s$ is an isomorphism where $\psi=\sum_{i=1}^s p_i$ and $\psi^{-1}=\sum_{i=1}^s \iota_i$. Let $P\in N$ and let $P_i=p_i(P)$. Then we have that

$$\alpha(\Delta(P; \Delta_1), \cdots, \Delta(P; \Delta_s))\geqq 0 \Longleftrightarrow \dim(\Delta(P_i\,; \Delta_i))\geqq 1\,.$$

Assume that $P\in N$ satisfies $\alpha(\Delta(P; \Delta_1), \cdots, \Delta(P; \Delta_s))\geqq 0$. Let $P_i\in N_i$ be as above and let $P_i'=\iota_i(P_i)\in N$. Note that $p_j(P_i')=0$ for $j\neq i$ and $p_i(P_i')=P_i$. Thus it is easy to see that

$$\Delta(P_i'\,; \Delta_j)=\begin{cases} \Delta_j, & j\neq i \\ \Delta(P_i\,; \Delta_i), & j=i \end{cases} \quad \text{and}$$

$$\Delta(P_i'\,; \Xi)=\Delta(P_i\,; \Delta_i)+\sum_{j\neq i}\Delta_j\,.$$

Thus $\dim \Delta(P_i'\,; \Xi)\geqq 3s-2$ and it is easy to see that

$$\alpha(\Delta(P'_i;\Delta_1),\cdots,\Delta(P'_i;\Delta_s),\Delta(P'_i;\Xi),\cdots,\Delta(P'_i;\Xi))\geqq 0.$$

This implies that $P'_i\in K(\Delta_1,\cdots,\Delta_s,\Xi,\cdots,\Xi)$. Thus $P=\sum_{i=1}^{s}P'_i$ is also contained in $K(\Delta_1,\cdots,\Delta_s,\Xi,\cdots,\Xi)$. As $\{P\in N;\alpha(\Delta(P;\Delta_1),\cdots,\Delta(P;\Delta_s))\geqq 0\}$ generate the boundary lattice group $K(\Delta_1,\cdots,\Delta_s)$, this shows the opposite inclusion: $K(\Delta_1,\cdots,\Delta_s)\subset K(\Delta_1,\cdots,\Delta_s,\Xi,\cdots,\Xi)$. This completes the proof.

§ 4. Appendix.

We consider arbitrary integral convex polyhedra $\Delta_1,\cdots,\Delta_k$ in $\boldsymbol{R}^m$. An integral point A of a convex polyhedron Δ is called a *vertex* of Δ if A is not on any face of Δ of dimension greater than 0. Let $\{A_{i,1},\cdots,A_{i,e_i}\}$ be the vertices of Δ_i and let $\{A_{i,e_i+1},\cdots,A_{i,d_i}\}$ be the other integral points of Δ_i for $i=1,\cdots,k$. Put

$$h_i(\boldsymbol{u},\boldsymbol{t}_i)=h_{i,\boldsymbol{t}_i}(\boldsymbol{u})=\sum_{j=1}^{d_i}t_{i,j}\boldsymbol{u}^{A_{i,j}}$$

where $\boldsymbol{t}_i=(t_{i,1},\cdots,t_{i,d_i})$. For each $\boldsymbol{t}=(\boldsymbol{t}_1,\cdots,\boldsymbol{t}_k)\in\boldsymbol{C}^{d_1+\cdots+d_k}$, we define

$$Z_{\boldsymbol{t}}^{*}=\{\boldsymbol{u}\in\boldsymbol{C}^{*m};h_1(\boldsymbol{u},\boldsymbol{t}_1)=\cdots=h_k(\boldsymbol{u},\boldsymbol{t}_k)=0\}$$

$$W_{\boldsymbol{t}}^{*}=\{\boldsymbol{u}\in\boldsymbol{C}^{*m};h_1(\boldsymbol{u},\boldsymbol{t}_1)=\cdots=h_{k-1}(\boldsymbol{u},\boldsymbol{t}_k)=0\}.$$

Let us consider the subset $\mathcal{U}:=\mathcal{U}(\Delta_1,\cdots,\Delta_k)$ of the parameter space $\boldsymbol{C}^{d_1}\times\cdots\times\boldsymbol{C}^{d_k}$ which is defined by $\boldsymbol{t}=(\boldsymbol{t}_1,\cdots,\boldsymbol{t}_k)\in\mathcal{U}$ if and only if

(1) (Stability of Newton Polyhedra) $\Delta(h_{i,\boldsymbol{t}_i})=\Delta_i$, $i=1,\cdots,k$ and

(2) (Non-degeneracy) $Z_{\boldsymbol{t}}^{*}$ is a non-degenerate complete intersection variety.

(1) is equivalent to $t_{i,j}\neq 0$ for $1\leqq j\leqq e_i$, $1\leqq i\leqq k$. Let P be an integral covector. We say that P is *trivial* on $\{\Delta_1,\cdots,\Delta_k\}$ if $\Delta(P;\Delta_i)=\Delta_i$ for each $i=1,\cdots,k$. In other words, P is trivial on $\{\Delta_1,\cdots,\Delta_k\}$ if and only if P is a constant function on $\Delta_1+\cdots+\Delta_k$. Thus

PROPOSITION (4.1). *If P is non-trivial on $\{\Delta_1,\cdots,\Delta_k\}$, we have the inequality:*

$$\dim\Big(\sum_{i=1}^{k}\Delta(P;\Delta_i)\Big)<\dim\Big(\sum_{i=1}^{k}\Delta_i\Big).$$

This is obvious from the general equality: $\sum_{i=1}^{k}\Delta(P;\Delta_i)=\Delta(P;\sum_{i=1}^{k}\Delta_i)$. For a non-trivial integral covector P, we define

$$Z_{\boldsymbol{t}}^{*}(P):=\{\boldsymbol{u}\in\boldsymbol{C}^{*m};h_{1,P}(\boldsymbol{u},\boldsymbol{t}_1)=\cdots=h_{k,P}(\boldsymbol{u},\boldsymbol{t}_k)=0\}$$

where $h_{i,P}(\boldsymbol{u},\boldsymbol{t}_1)\sum_{A_{i,j}\in\Delta(P;\Delta_i)}t_{i,j}\boldsymbol{u}^{A_{i,j}}$.

Let $\boldsymbol{a}=(\boldsymbol{a}_1,\cdots,\boldsymbol{a}_k)$ be a fixed parameter which satisfies the stability condition (1) and take and fix an l, $1\leqq l\leqq d_k$. Put $\boldsymbol{t}(\tau)=(\boldsymbol{a}_1,\cdots,\boldsymbol{a}_{k-1},\boldsymbol{a}_k(\tau))$ and $\boldsymbol{a}_k(\tau)=(a_{k,1},\cdots,a_{k,l}+\tau,\cdots,a_{k,d_k})$. We consider the line in the parameter space

$L_l(\boldsymbol{a})$ which is defined by $L_l(\boldsymbol{a})=\{\boldsymbol{t}(\tau);\tau\in\boldsymbol{C}\}$.

THEOREM (4.2). *Assume that we have chosen the coefficients* $\boldsymbol{t}_i=\boldsymbol{a}_i=(a_{i,1},\cdots,a_{i,d_i})$ *of* h_i *for* $i=1,\cdots,k$ *so that* $W_{\boldsymbol{a}}^*$ *and* $Z_{\boldsymbol{a}}^*(P)$ *are non-degenerate complete intersection varieties for any non-trivial covector* P *on* $\{\Delta_1,\cdots,\Delta_k\}$. *Then for any fixed* l, $1\leqq l\leqq d_k$, $L_l(\boldsymbol{a})-\mathcal{V}\cap L_l(\boldsymbol{a})$ *is a finite set where* $L_l(\boldsymbol{a})$ *is the complex line as above.*

COROLLARY (4.2.1). $\mathcal{V}$ *is a non-empty Zariski open set.*

Proof of Theorem (4.2). Let $m'=\dim(\sum_{i=1}^k\Delta_i)$. By a change of Laurent coordinates if necessary, we can assume that $m=m'$. We fix a regular simplicial cone subdivision Σ^* of $\Gamma^*(\Delta_1,\cdots,\Delta_k)$ and let X be the corresponding compactification of the torus $\boldsymbol{C}^{*m}$. As we have assumed $m=m'$, P is non-trivial for $\{\Delta_1,\cdots,\Delta_k\}$ if and only if P is a non-zero covector. Assume that the coefficients $\{t_{i,j}=a_{i,j};1\leqq j\leqq d_i, 1\leqq i\leqq k\}$ are given so that $W_{\boldsymbol{a}}^*$ and $Z_{\boldsymbol{a}}^*(P)$ are non-degenerate complete intersection varieties for any non-zero covector. We take an arbitrary l, $1\leqq l\leqq d_k$, and we consider the one-dimensional family $\{\tilde{Z}_{\boldsymbol{t}(\tau)};\tau\in\boldsymbol{C}\}$ of the divisors in $\tilde{W}_{\boldsymbol{a}}$. Recall that

$$\tilde{Z}_{\boldsymbol{t}(\tau)}-Z_{\boldsymbol{t}(\tau)}^*=\bigcup_{P\in\mathrm{Vertex}(\Sigma^*)}E_{\boldsymbol{t}(\tau)}(P) \tag{4.2.2}$$

where $E_{\boldsymbol{t}(\tau)}(P):=\hat{E}(P)\cap\tilde{Z}_{\boldsymbol{t}(\tau)}$. The base point locus of this family is the union of the divisors $E_{\boldsymbol{a}}(P)$ such that $A_{k,l}\notin\Delta(P;\Delta_k)$. The assumption that $Z_{\boldsymbol{a}}^*(P)$ is non-degenerate implies that $E_{\boldsymbol{t}(0)}(P)=E_{\boldsymbol{a}}(P)$ is also non-singular for any vertex $P\in\mathrm{Vertex}(\Sigma^*)$. Applying Bertini's theorem ([G-H]) or Curve Selection Lemma ([M]), we conclude that $\{\tilde{Z}_{\boldsymbol{t}(\tau)}^*\}$ are smooth except a finite number of exceptions $\tau=\tau_1,\cdots,\tau_\mu$. Q. E. D.

Proof of Corollary (4.2.1). By a change of Laurent coordinates if necessary, we can assume that $m=m'$. Let $\pi:\boldsymbol{C}^{d_1}-\{0\}\times\cdots\times\boldsymbol{C}^{d_k}-\{0\}\to\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ be the canonical projection and let $\bar{\mathcal{V}}=\pi(\mathcal{V})$. As $\mathcal{V}=\pi^{-1}(\bar{\mathcal{V}})$, it suffices to show that $\bar{\mathcal{V}}$ is a non-empty Zariski open set. Let

$$\mathcal{Z}^*=\{(\boldsymbol{u},\pi(\boldsymbol{t}))\in\boldsymbol{C}^{*m}\times(\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1});h_1(\boldsymbol{u},\boldsymbol{t}_1)=\cdots=h_k(\boldsymbol{u},\boldsymbol{t}_k)=0\}.$$

We fix a regular simplicial cone subdivision Σ^* of $\Gamma^*(\Delta_1,\cdots,\Delta_k)$ and let X be the corresponding compactification of the torus $\boldsymbol{C}^{*m}$. Let $\mathcal{X}=X\times\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ and let $p:\mathcal{X}\to\boldsymbol{P}^{d_0-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ be the projection. Let $\mathcal{Z}$ be the compactification (=closure in $\mathcal{X}$) of $\mathcal{Z}^*$ in $\mathcal{X}$. For each vertex P, let $\hat{\mathcal{E}}(P)=\hat{E}(P)\times\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ and let $\mathcal{E}(P)=\tilde{\mathcal{Z}}\cap\hat{\mathcal{E}}(P)$. Let $S(P)$ be the set of singular points of $\mathcal{E}(P)$ as a complete intersection variety in $\hat{\mathcal{E}}(P)$ and let S be the union $\bigcup_{P\in\mathrm{Vertex}(\Sigma^*)}S(P)$. Let $D=p(S)$ and $D'=\bigcup_{i=1}^k\bigcup_{j=1}^{e_i}\{t_{i,j}=0\}$. By the proper mapping theorem ([Re]), $p(\mathcal{Z})$ and D are analytic subsets of $\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$. Thus they are also algebraic by Chow's theorem. If $\alpha(\Delta_1,\cdots,\Delta_k)<0$ and Z^*

is non-degenerate, $Z^*(P)=\emptyset$ for any covector P. Thus $\mathcal{U}=\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}-p(\tilde{\mathcal{Z}})$. If $\alpha(\Delta_1,\cdots,\Delta_k)\geqq 0$ and Z^* is non-degenerate, $Z^*\neq\emptyset$ (Proposition (5.4), [Ok4]). Therefore $\mathcal{U}=p(\tilde{\mathcal{Z}})-(D\cup D')$. Assume that $\mathcal{U}\neq\emptyset$. Then the transversality argument shows that $\mathcal{U}$ is an open set in the strong topology. Thus if $\mathcal{U}\neq\emptyset$ and $\alpha(\Delta_1,\cdots,\Delta_k)\geqq 0$, $p(\mathcal{Z})=\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ as $\boldsymbol{P}^{d_1-1}\times\cdots\times\boldsymbol{P}^{d_k-1}$ is irreducible. Thus in any case $\mathcal{U}$ is a Zariski open set. Therefore it suffices to show that $\mathcal{U}\neq\emptyset$. Now the non-emptiness $\mathcal{U}\neq\emptyset$ follows easily from Theorem (4.2) using the induction on k and m'.

References

[Be] D. N. Bernshtein, The number of roots of a system of equations, Funktsional'nyi Analiz i Ego Prilozheniya 9, No. 3 (1975), 183–185.

[Dn1] V. I. Danilov, The geometry of toric varietics, Russian Math. Surveys **33**: **2** (1978), 97–154.

[Dn2] V. I. Danilov, Newton polyhedra and vanishing cohomology, Funct. Anal. Appl. **13** (1979), 103–115.

[G-H] P. Griffiths and J. Harris, "Principles of Algebraic Geometry," A Wiley-Interscience Publication, New York-Chichester-Brisbane-Toronto, 1978.

[Eh] F. Ehlers, Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einer isolierter Singularitäten, Math. Ann. **218** (1975), 127–.56.

[Kh1] A. G. Khovanskii, Newton polyhedra and toral varieties, Funkts. Anal. Prilozhen. **11**, No. 4 (1977), 56–67.

[Kh2] A. G. Khovanskii, Newton polyhedra and the genus of complete intersections, Funkts. Anal. Prilozhen. **12**, No. 1 (1977), 51–61.

[K-K-M-S] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, "Toroidal Embeddings," Lecture Notes in Math., Springer, Berlin-Heidelberg-New York, 1973.

[M] J. Milnor, "Singular Points of Complex Hypersurface," Annals Math. Studies, Princeton Univ. Press, Princeton, 1968.

[Od1] T. Oda, "Lectures on torus embeddings and applications," Tata Inst. Fund. Research, Springer-Verlag, Berlin- Heidelberg-New York, 1978.

[Od2] T. Oda, "Convex Bodies and Algebraic Geometry," Springer, Berlin-Heidelberg-New York, 1987.

[Ok1] M. Oka, On the Resolution of Hypersurface Singularities, Advanced Study in Pure Mathematics **8** (1986), 405–436.

[Ok2] M. Oka, On the deformation of a certain type of algebraic varieties, Contemporary Mathematics **78** (1988), 593–603.

[Ok3] M. Oka, Examples of algebraic surfaces, in "A Fete of Topology," Academic Press, Boston-San Diego-New York, 1988, pp. 355–363.

[Ok4] M. Oka, Principal zeta-function of non-degenerate complete intersection singularity, J. Fac. Sci., Univ. of Tokyo **37**, No. 1 (1990), 11–32.

[Ok5] M. Oka, On the topology of full non-degenerate complete intersection variety, Nagoya Math. J. **121** (1991), 137–148.

[Ok6] M. Oka, A Lefshetz type theorem in a torus, to Appear in Proceeding of College of Singularity at Trieste 1991.

[Re] R. REMMERT, Holomorphe und meromorphe Abbildingen komplexer Räume, Math. Ann. **133** (1957), 328–370.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU, TOKYO
152, JAPAN