Y. KAMETAN1
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TORUS SUM FORMULA OF SIMPLE INVARIANTS FOR 4-MANIFOLDS

BY YUKIO KAMETANI

1. Introduction.

The topology of moduli spaces of anti-self-dual (ASD) connections is closely related with differentiable structures on 4-manifolds. In his celebrated paper [6], Donaldson has defined the polynomial invariants to distinguish differentiable structures on a 4-manifold. Though a significant result on the vanishing has been obtained at the same time, these invariants remain to be very difficult to determine completely. In fact, many examples have been calculated using an identification of irreducible ASD connections and stable vector bundles by Donaldson ([6], [7], [9], [12], [18]). But there are another direct approaches in the case that the dimension of ASD moduli is zero and that the invariant is just a number of the points in the moduli. For example Gompf and Mrowka have defined an invariant for 4-manifolds with torus end, using 0 or 1 dimensional ASD moduli, and proved that the invariant of the glued manifolds with solid torus can be emerged as the number of ASD connections which can be extended to over the solid torus. From a topological argument on K3 surface with elliptic fibration, they calculated the above numbers for fake K3 surfaces obtained by performing logarithmic transformations on embedded 2-tori. After that, Kromheimer has observed that the ASD moduli of Kummer surface comes down to the flat moduli as all (-2) curves tends to infinity, so the invariant could be computed algebraically [13]. The invariant obtained by 0-dimensional ASD moduli is said to be a simple invariant.

In this paper we give a torus sum formula of simple invariants for 4-manifolds. Our idea and formula are simple. Suppose that we have two simply connected closed 4-manifolds which contain a 2-torus with the trivial normal bundle. We assume that the complements are simply connected and the second Stiefel-Whitney class to define the $SO(3)$ bundle does not vanish on the 2-torus. Then any ASD connection converges to some ASD connections as the 2-torus tends to infinity. On the other hand, any ASD connection over the new 4-manifold obtained by torus sum also converges to some ASD connections as the bi-collar of the intermediate 3-torus is stretched to infinity. Hence we prove that the simple invariant of the new 4-manifold is the product of that of the

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given 4-manifolds. This formula is a variant of a relation of Donaldson invariants to Floer homology in Atiyah's exposition [1].

We can apply this formula to compute the simple invariant for regular elliptic surfaces. The rational elliptic surface has the least Euler number among them. The K3 surface is obtained by gluing two rational elliptic surfaces as fiber sum [17]. Gluing more rational elliptic surfaces yields all the other regular elliptic surfaces without multiple fibers. On the other hand, the simple invariant of the K3 surface has been known to be 1 for all second Stiefel-Whitney classes ([5], [13]). Hence the value is 1 for second Stiefel-Whitney classes whose restriction to the fiber are non-zero. In particular, if the geometric genus is even, then all the value is 1.

We remark that these calculations improve two known facts the first, Sato and the author have shown that the value is non-zero for a second Stiefel-Whithey class, by using stable vector bundles [11]. The second, Ue has shown that the value of the above is independent of the choice of second Stiefel-Whitney class with the additional condition, by analyzing the action of the diffeomorphism groups on second cohomology groups [21].

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2. Review of simple invariant and main result

We recall the simple invariant γ defined by Donaldson ([6], [7, Chapter 9]) let X be an oriented closed smooth 4-manifold with the following properties;

(A1) $\pi_1(X)=1$,

(A2) $b_+(X)\geqq 3$and $b_+(X)$ is odd,

where b_+ denotes the dimension of the maximal positive subspacee for the intersection on $H^2(X)$. Then there is a well defined lift of mod 2 cup square called the Pontrjagin square $H^2(X\ \boldsymbol{Z}_2)\to H^4(X;\boldsymbol{Z}_4)$. Composing this map with evaluation on the fundamental class defines a quadratic map $H^2(X\ \boldsymbol{Z}_2)\to\boldsymbol{Z}_4$. Let $P\to X$ be the principal $SO(3)$ bundle with $w_2(P)=\eta$ and $p_1(P)=l$. They satisfy $\eta^2\equiv l\ (\mathrm{mod}\ 4)$. A theorem of Dold and Whitney [4] tells us that $SO(3)$ bundles over a compact 4-manifold are determined completely by η and /. Conversely, given η and / with $\eta^2\equiv l\ (\mathrm{mod}\ 4)$, we can easily construct the corresponding $SO(3)$ bundle over X. We give X a Riemannian metric. Then the affine space $\mathcal{A}_P$ of L_3^2 connections has a natural Banach structure from the L_3^2 norm. The L_4^2 gauge group $\mathcal{G}_P$ is a Banach Lie group. The Lie algebra of $\mathcal{G}_P$ is $L_4^2(\mathrm{Ad}\ P)$. Let $\mathcal{A}_P^*\subset\mathcal{A}_P$ denote the subset of irreducible connections. Then the quotient $\mathcal{B}_P^*=\mathcal{A}_P^*/\mathcal{G}_P$is a C^∞-Banach manifold such that the projection $\pi:\mathcal{A}_P^*\to\mathcal{B}_P^*$ defines a principal $\mathcal{G}_P$-bundle ([8], [7, 4.2]). The tangent space to $[A]\in\mathcal{B}_P^*$ is isomorphic to $\{a\in L_3^2(\Omega_X^1(\mathrm{Ad}\,P))\,|\,d_A^*a=0\}$. We say an element $[A]$ in $\mathcal{B}_P^*$ to be regular if the operator d_A^+ is surjective. Let $\mathcal{M}_X(l,\eta,g)$ denote the moduli space of g-ASD connections on P. Then the formal dimension is equal to

$-2l-3(1+b_+(X))$ [2]. We choose l_X so that $-2l_X-3(1+b_+(X))=0$. Let $\mathcal{C}_X^r$ $(r\geqq 3)$ denote the space of C^r-metrics on X. By ([8], [7, 4.3]), there is a Baire set C'_X in $\mathcal{C}_X^r$ such that for all $g\in \mathcal{C}'_X$, $\mathcal{M}_X(l_X,\eta, g)$ is a finite set consisting of irreducible regular connections. We fix $g\in \mathcal{C}'_X$. Then we can define a sign at any $[A]\in\mathcal{M}_X(l_X, \eta, g)$ using a line bundle over $\mathcal{B}_P^*$. For $A\in\mathcal{A}_P^*$, we consider the deformation complex

$$\delta_A=d_A^*\oplus d_A^+ : L_3^2(\Omega_X^1(\mathrm{Ad}\,P)) \dashrightarrow L_2^2((\Omega_X^0\oplus\Omega_X^+)(\mathrm{Ad}\,P)).$$

We choose a linear map $S: \boldsymbol{R}^N\to L_2^2(\Omega_X^+(\mathrm{Ad}P))$ so that $\delta_A\oplus S$ is surjective. Then we define the determinant line of $\delta_A\oplus S$ by

$$\Lambda_{P,A}=(\Lambda^{\max}\mathrm{Ker}\,(\delta_A\oplus S))\otimes(\Lambda^N\boldsymbol{R}^N)^* .$$

This line has an intrinsic sense by the exact sequence

$$0 \dashrightarrow \mathrm{Ker}\,\delta_A \dashrightarrow \mathrm{Ker}\,(\delta_A\oplus S)\overset{\pi_A}{\dashrightarrow}\boldsymbol{R}^N \dashrightarrow \mathrm{Coker}\,\delta_A \dashrightarrow 0 .$$

Since the surjectivity holds in a neighborhood of A, these lines are patched together to get a locally trivial line bundle over $\mathcal{A}_P^*$. It descends to the determinant line bundle $\Lambda_P\to\mathcal{B}_P^*$ by the free action of $\mathcal{G}_P$. The bundle $\Lambda_P\to\mathcal{B}_P^*$ is topologically trivial ([5], [7, 5.4]). Since δ_A is an isomorphism for any $[A]\in \mathcal{M}_X(l_X, \eta, g)$, we can define a section on Λ_P at $[A]$ by

$$(\pi_A^{-1}(e_1)\wedge\cdots\wedge\pi_A^{-1}(e_N))\otimes(e_1\wedge\cdots\wedge e_N)^* ,$$

where $e_1, \cdots, e_N$ form a basis of $\boldsymbol{R}^N$. This defines an orientation $o([A])$ of the the line bundle $\Lambda_P\to\mathcal{B}_P^*$. For a later use, we remark that this section is defined on a connected region $U([A])$consisting of irreducible regular connections about $[A]$.

On the other hand, for an orientation Ω of $H^+(X)$and an integral lift c of η, there is another orientation $o(\Omega)$ determined at a connection obtained by attaching some standard instantons over S^4 with reducible connection determined by c ([5], [7, 7.1.6]). Then the sign $\varepsilon([A])$ at $[A]\in\mathcal{M}_X(l_X,\eta, g)$ is given by $o([A])=\varepsilon([A])o(\Omega)$and the simple invariant is defined by

$$\gamma_X(\eta)=\sum_{[A]\in\mathcal{M}_X(l_X,\eta,g)}\varepsilon([A]).$$

This function γ_X is independent of the choice of the element g in $\mathcal{C}'_X$ and satisfies the following ([6], [7]): If $\phi: X\to X'$ is an orientation preserving diffeomorphism, then $\gamma_X(\phi^*(\eta'))=\varepsilon(\phi)\gamma_X(\eta')$where $\varepsilon(\phi)$ is -1 if either ϕ^* is an orientation reversing map from $H^+(X')$ onto $H^+(X)$ or $((c-\phi^*(c'))/2)^2\equiv 1 \pmod 2$ but not both and is 1 otherwise. (Here c' denotes the integral lift of η' and c is the integral lift of $\phi^*(\eta')$ used in orientaring their respective moduli spaces.) Hence the absolute value $|\gamma_X|$ can be thought of a function on

$$C_X=\{\eta\in H^2(X;\boldsymbol{Z}_2)\,|\,\eta\neq 0,\ \eta^2\equiv l_X \pmod 4\}.$$

We return to our main theorem. Let K be a compact oriented smooth 4-manifold with boundary $Z=T^3$, satisfying the following;

(B1) $\pi_1(K)=1$,

(B2) $b_+(K)\geqq 2$ and $b_+(K)$ is even.

For two such manifolds K_1, K_2 and an orientation reversing diffeomorphism $\phi: \partial K_1 \to \partial K_2$, the oriented 4-manifold $X=K_1\cup_\phi K_2$ always satisfies (A1), (A2). Let $\sigma: Z\to X$, $\sigma_i: K_i\to X$ $(i=1, 2)$ be the inclusion. For each $\eta\in C_X$ with $\sigma^*(\eta)\neq 0$, we can define an orientation reversing diffeomorphism $\phi_i: \partial K_i \to Z$ such that $\phi_i^*(\eta)$ can be extended to a class $\eta_i^*\in H^2(K_i^*; \boldsymbol{Z}_2)$. The oriented closed 4-manifold $K_i^*=K_i\cup_{\phi_i} W$ also satisfies (Al), (A2). Here W is the solid torus $T^2\times D^2$.

THEOREM 2.1. // η_i^* *satisfies* $(\eta_i^*)^2\equiv l_{K_i^*}$ (mod 4) *for each* $i=1, 2$, *then*

$$|\gamma_X(\eta)|=|\gamma_{K_1^*}(\eta_1^*)|\,|\gamma_{K_2^*}(\eta_2^*)|.$$

THEOREM 2.2. // η_i^* *does not satisfy* $(\eta_i^*)^2\equiv l_{K_i^*}$ (mod 2) *for some* $i=1, 2$, *then* $\gamma_X(\eta)=0$.

Remarks. (1) Let P. D. be the mod 2 Poincaré dual. Then η_i^*+P. D. $[T^2\times 0]$ satisfies $(\eta_i^*+\text{P. D. }[T^2\times 0])^2\equiv(\eta_i^*)^2+2$ (mod 4) and by the exact sequence

$$0 \longrightarrow H^2(W,Z \quad \boldsymbol{Z}_2)\longrightarrow H^2(K_i^* \quad \boldsymbol{Z}_2)\xrightarrow{\sigma_i^*} H^2(K_i \quad \boldsymbol{Z}_2),$$

it is only another choice for 77*. So the conditions of Theorem 2.1 and 2.2 are complementary to each other.

(2) In our application, if K_i^* and η_i^* $(i=1, 2)$ are chosen, then we will write $X=K_1^*\natural K_2^*$ and $\eta=$ tf\ηi.

3. Setting up gauge theory

We first argue the ASD moduli over a 4-manifold with torus end. According to Taubes [20], we study a gauge theory on a convenient subspace of connections to apply a known analysis and to contain all ASD connections by some gauge. The uniqueness of flat connections over the torus enable us to apply his argument directly. For $n=2, 3$, we denote by $\chi(T^n)$ the set of the conjugacy classes of representations from $\pi_1(T^n)$ to $SO(3)$. The topology of $\chi(T^n)$ has been discussed in [10, Proposition V. 2. 1]. Given a representations p, we form the associated flat $\boldsymbol{R}^3$ bundle ξ_ρ. Then we define a map

$$\omega_2: \chi(T^n) \longrightarrow H^2(T^n \quad \boldsymbol{Z}_2)$$

by $\omega_2(\rho)=\omega_2(\xi_\rho)$. This is surjective. We denote by $\chi_\alpha(T^n)$ the preimage of $\alpha\in H^2(T^n \quad \boldsymbol{Z}_2)$. Then the decomposition

$$\chi(T^n)=\bigcup_{\alpha\in H^2(T^n;\boldsymbol{Z}_2)}\chi_\alpha(T^n)$$

decomposes $\chi(T^n)$ into connected components. $\chi_0(T^n)$ is homeomorphic to $T^n/\pm 1$ and the other 7 components are isolated points whose stabilizers are isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. Any representation p in the isolated points is regular, that is $H^i(T^n;\mathrm{Ad}\,\rho)=0$ $(0\leqq i\leqq 2)$.

For the unique non-zero element $\alpha\in H^2(W;\boldsymbol{Z}_2)$, we let $\tilde{Q}$, $\tilde{\Gamma}$ and $\mathfrak{R}$ be the corresponding flat bundle, flat connection and stabilizer respectively. We write $Q=\tilde{Q}|_Z$ and $\Gamma=\tilde{\Gamma}|_Z$ Then the stabilizer of Γ is just $\mathfrak{R}$.

Let K denote a compact oriented smooth 4-manifold with boundary Z, satisfying (B1) and (B2). Let $\sigma: Z\to K$ be the inclusion. Define $Y=K\cup Z\times[0,\infty)$. We choose $\eta\in H^2(K;\boldsymbol{Z}_2)$ with $\tau^*(\eta)\neq 0$. Let P_0 be the $SO(3)$ bundle over K with $w_2(P_0)=\eta$ and $p_1(P_0)=0\in H^4(K\,\boldsymbol{Z})=0$. For a bundle isomorphism $\iota:\sigma^*(P_0)\to Q$, define $P=P_0\cup_\iota\pi^*Q$, where $\pi: Z\mathrm{x}[0,\infty)\to Z$ is the projection. We give Y a metric which is the product metric $h+dt^2$ on $Z\times[0,\infty)$, where h is a metric on Z. Let $\tau: Y\to\boldsymbol{R}$ be a smooth function such that $\tau(x,t)=t$ on $\boldsymbol{Z}\times[1,\infty)$ and $\tau=0$ on K. We fix a smooth connection A_0 on P such that A_0 is equal to $\pi^*\Gamma$ over $Z\chi[0,\infty)$. We write $(-1/4\pi^2)\int_Y\mathrm{Tr}(F_{A_0}\wedge F_{A_0})=l(\in\boldsymbol{Z})$. For $p\geqq 1$, $k\geqq 0$ and $\delta>0$, we define the weighted norm $L^p_{k,\delta}(Y)$ by

$$\|s\|_{L^p_{k,\delta}(Y)}=\left(\int_Y e^{\tau\delta}\sum_{l=0}^{k}|\nabla^{(l)}_{A_0}s|^p\right)^{1/p}.$$

Then we consider the following class of connections and gauge group on P

$$\mathcal{A}_P=\{A_0+a\mid a\in L^2_{3,\mathrm{loc}}(\Omega^1_Y(\mathrm{Ad}\,P)),\ \|a\|_{L^2_{3,\delta}(Y)}<\infty\},$$

$$\mathcal{G}_P=\{u\in L^2_{4,\mathrm{loc}}(\mathrm{Aut}P)\mid \|\nabla_{A_0}u\|_{L^2_{3,\delta}(Y)}<\infty\}$$

By the argument in [20, Section 7], we see that there is a well defined map $r:\mathcal{G}_P\to\mathfrak{R}$ given by

$$r(u)=\lim_{t\to\infty}u_t,$$

where the limit is in C^0-convergence. We note that the automorphisms in $\mathcal{G}_P$ are continuous by the Sobolev embedding theorem $L^2_{3,\mathrm{loc}}\to C^0$.

LEMMA 3.1. *The automorphism $a\in\mathfrak{R}$ over dK can be extended to all over K continuously if and only if $a=1$.*

Proof. The primary difference $\mathfrak{d}(1|_{\partial K},a)$ [19, §36] between the identity $1\in C^0(\mathrm{Aut}\,P_0)$ and $a\neq 1$ is a non-zero element in $H^1(\partial K;\boldsymbol{Z}_2)$. If a can be extended to K, then by the extension theorem [19, 37.11], there is an element v in $H^1(K;\boldsymbol{Z}_2)$ such that $\sigma^*(v)=\mathfrak{d}(1|_{\partial K},a)$. But this contradicts to $H^1(K;\boldsymbol{Z}_2)=0$. D

COROLLARY 3.2. *The image oj the map r is $\{1\}$ and there are no reducible connections in $\mathcal{B}_P$.*

A gauge theory on 4-manifolds with cylindrical end has been studied by Taubes [20, Section 7]. Using his argument, we can prove the following two lemmas (see also [16]). We topologize $\mathcal{A}_P$, $\mathcal{G}_P$ by the norm $\|\cdot\|_{L^2_{3,\delta}}$, $\|\nabla_{A_0}\ \|_{L^2_{3,\delta}}$ respectively. Then

LEMM 3.3. *$\mathcal{G}_P$ is a Banach Lie group. The Lie algebra of $\mathcal{G}_P$ is*

$$\mathfrak{g}_P=\{\sigma\in L^2_{4,\mathrm{loc}}(\mathrm{Ad}\,P)\,|\,\|\nabla_{A_0}\sigma\|_{L^2_{3,\delta(Y)}}<\infty\}.$$

$\mathcal{G}_P$ acts smoothly on $\mathcal{A}_P$. The set $\{u\in L^2_{4,\mathrm{loc}}(\mathrm{Aut}\,P)\ u\mathcal{A}_P=\mathcal{A}_P)$is $\mathcal{G}_P$.

LEMMA 3.4. *The quotient $\mathcal{B}_P=\mathcal{A}_P/\mathcal{G}_P$ is a C^∞-Banach manifold such that the projection $\mathcal{A}_P\to\mathcal{B}_P$ defines a principal $\mathcal{G}_P$-bundle. The tangent space to $[A]\in\mathcal{B}_P$ is isomorphic to*

$$\{a\in L^2_{3,\mathrm{loc}}(\Omega^1_Y(\mathrm{Ad}\,P))\,|\,\|a\|_{L^2_{3,\delta(Y)}}<\infty,\ e^{-\tau\delta}d^*_Ae^{\tau\delta}a=0\}.$$

LEMMA 3.5. *For bundle isomorphisms c, ι': $\sigma_*(P_0)\to Q$, we consider $SO(3)$ bundles $P=P_0\cup_\iota\pi^*Q$, $P'=P_0\cup_\iota\pi^*Q$, and C^m $(m\in\boldsymbol{N})$-connectionsA_0, A_0' on P, P' such that A_0, A_0' are equal to $\pi^*\Gamma$ over $Z\times[0,\infty)$ respectively. Then the following two conditions are equivalent.*

(1) *The automorphism $(\iota')^{-1}r\iota$ over dK can be extended to all over of K continuously for some $r\in\mathcal{R}$ with* $\mathrm{b}(1,\ (\iota')^{-1}r\iota)=0$.

(2) $(-1/4\pi^2)\int_Y\mathrm{Tr}\,(F_{A_0}\wedge F_{A_0})=(-1/4\pi^2)\int_Y\mathrm{Tr}\,(F_{A_0'}\wedge F_{A_0'})$.

Proof. We consider $SO(3)$ bundles $P^*=P_0\cup_\iota\tilde{Q}$, $(P')^*=P_0\cup_{\iota'}\tilde{Q}$ over $K\cup W$ and C^m-connections A^*, $(A')^*$extended by $\tilde{\Gamma}$ respectively. If the condition (1) holds, then the connections A_0 and A_0' have the same integral by Chern-Weil formula. Conversely if the condition (2) holds, then obviously $p_1(P^*)=p_1((P')^*)$ and, moreover, $w_2(P^*)=w_2((P')^*)$for we can write $w_2(P^*)=w_2((P')^*)+$P.D. $[T^2\times0]$, which induces $p_1(P^*)=p_1((P')^*)+2\,(\mathrm{mod}\ 4)$, a contradiction. By Dold-Whitney theorem, there exist bundle isomorphisms $f: P_0\to P_0$ and $h:\tilde{Q}\to Q$ with $\iota'f=h\iota$. We prove that A is homotopic to some $r\in\mathcal{R}$. It suffices to prove that A $_{T^2}$ is homotopic to some $r\in\mathcal{R}$. We remove an open 2-disk D^2 in T^2. By the homotopy classification theorem [19, 37.12], the assignment of the primary difference $\mathrm{b}(1,\ \cdot)$ in $H^1(T^2\setminus D^2;\boldsymbol{Z}_2)\cong H^1(T^2;\boldsymbol{Z}_2)\cong\mathcal{R}$ to each homotopy class in $[T^2\setminus D^2;\ \mathrm{Aut}\,Q]$ sets up a 1-1 correspondence between each sets. So A$|_{T^2\setminus D^2}$is homotopic to some $r\in\mathcal{R}$ by a homotopy H_t $(0\leqq t\leqq1)$. We extend H_t to all over to T^2, using the collar of ∂D^2. Since $[(D^2,\partial D^2)\ (SO(3),\mathrm{id})]=\pi_2(SO(3))=1$, $H_1|_{D^2}$ is homotopic to r relative to ∂D^2, and so $(\iota')^{-1}r\iota$ is homotopic to /. We see that $\mathrm{b}(1,\ (\iota')^{-1}r\iota)$is zero by the argument in Lemma 3.1. D

COROLLARY 3.6. *$\mathcal{B}_P$ depends only on η and /.*

We return to the moduli of ASD connections. By [15, Theorem 1.1], there exists $\bar{\delta}>0$ such that for $0<\delta<\bar{\delta}$, $p\geqq 2$ and $k\geqq 0$, if A is an ASD connection on P, then the AHS complex

$$0 \longrightarrow \Omega_Y^0(\mathrm{AdP}) \xrightarrow{d_A} \Omega_Y^1(\mathrm{AdP}) \xrightarrow{d_A^+} \Omega_Y^+(\mathrm{Ad\,P}) \longrightarrow 0$$

defines a Fredholm complex

$$0 \longrightarrow L^p_{k+2,\delta}(Y) \xrightarrow{d_A} L^p_{k+1,\delta}(Y) \xrightarrow{d_A^+} L^p_{k,\delta}(Y) \longrightarrow 0 .$$

We denote by H_A^i $(0\leqq i\leqq 2)$ the cohomology of the above complex. Then its index is given by [10, VI. 3]

$$\text{-dim } H_A^0+\dim H_A^1-\dim H_A^2=-2l-3(2+b_+(Y)).$$

Let C_Y^r $(r\geqq 3)$ be the space of all conformal classes of C^r-metrics on Y which are fixed metric $h+dt^2$ on $Z\times[0, \infty)$. If we fix a metric $[g_0]\in C_Y^r$, then C_Y^r is identified with

$$\{m: \Lambda^+\to\Lambda^- \ \ C^r\text{-bundle map, } \sup|m|<1,\ m|_{Y\setminus K^0}=0\},$$

where $\Lambda^{\pm}$ is $\pm$ self-dual space with respect to g_0 and K^0 is the interior of K ([7, 1.1.5]). We choose l_Y with $-2l_Y-3(2+b_+(Y))=0$.

PROPOSITION 3.7. *There is a Baire set $C'_Y\subset C_Y^r$ such that for all $l_Y\leqq l<0$ and $g\in C'_Y$, the ASD moduli*

$$\mathcal{M}_Y(l, \eta, g)=\left\{[A]\in\mathcal{B}_P\,|\,F_A=-*_gF_A,\ \frac{-1}{4\pi^2}\int_Y \mathrm{Tr}\,(F_A\wedge F_A)=l\right\}$$

is a finite set consisting of irreducible regular connections. Any element in $\mathcal{M}_Y(l,\eta,g)$ has a smooth representative. Its dimension is equal to $-2l-3(2+b_+(Y))$. In particular if $l_Y<l<0$, then $\mathcal{M}_Y(l, \eta, g)$ is empty. Here the regularity at $[A]\in\mathcal{M}_Y(l, \eta, g)$ means that $H_A^2=0$.

Proof. The regularity follows from the same argument as in [7, 4.3] (see also [14, 5 (iii)]). By the argument in [7, 4.2.3], we see that the representative in Coulomb gauge relative to some nearby smooth gauge is also smooth. We will prove the compactness in Appendix 2. □

Remark. The proposition above has been stated in [10, Theorem V.3.3]. But we do not know how they orient the ASD moduli $\mathcal{M}_Y(l_Y,\eta, g)$ in spite of the non-existence of reducible connections in $\mathcal{B}_P$.

Let $\eta\in C_X$, $\eta_1^*\in C_{K_1^*}$ and $\eta_2^*\in C_{K_2^*}$ satisfy the assumption of Theorem 2.1. Let P* be the $SO(3)$ bundle over X with $w_2(P^*)=\eta$ and $p_1(P_1^*)=l_{K_1^*}$, and let P* be the $SO(3)$ bundle over K_1^* with $w_2(P_1^*)=\eta$ and $p_1(P_1^*)=l_{K_1^*}$. We write $P_1^*=P_{01}\cup_{\iota_1}\bar{Q}$ for some bundle isomorphism $\iota_1: P_{01}|_Z\to Q$ and write $P^*=P_{01}\cup_{\iota_2^{-1}\iota_1}P_{02}$

for some bundle isomorphism $\iota_2: P_{02}|_Z \to Q$. We put $P_2^* = P_{02} \cup_{\iota_2} \tilde{Q}$. We fix a smooth connection A_i^* on P_i^* which is Γ on $\tilde{Q}$ $(i=1, 2)$, and a smooth connection A^* on P which is A_i^* on P_{0i}. Then

$$p_1(P_2^*) = \frac{-1}{4\pi^2}\int_{K_2} \mathrm{Tr}\,(F_{A_2^*} \wedge F_{A_2^*}) = \frac{-1}{4\pi^2}\int_X \mathrm{Tr}\,(F_A \wedge F_A) - \frac{-1}{4\pi^2}\int_{K_1} \mathrm{Tr}\,(F_{A_1^*} \wedge F_{A_2^*})$$

$$= l_X - l_{K_1^*} = l_{K_2^*},$$

and $w_2(P_2^*) = \eta_2^*$ If not, then it must be $w_2(P_2^*) = \eta_2^* + \mathrm{P.D.}\,[T^2 \times 0]$, which induces $l_{K_2^*} \equiv l_{K_2^*} + 2 \pmod 4$. This is a contradiction. We define $Y_i = K_i \cup Z \times [0, \infty)$ and $P_i = P_{0i} \cup_{\iota_i} \pi^* Q$ for each $i=1, 2$. We fix a connection A_{0i} on P_i by

$$A_{0i} = \begin{cases} A_i^* & \text{over } K_i\,, \\ \pi^*\Gamma & \text{over } Z \times [0, \infty)\,. \end{cases}$$

Using A_{0i} and $0 < \delta_0 < \delta$, we define the space $\mathcal{A}_{P_i}$, the gauge group $\mathcal{G}_{P_i}$ and the quotient $\mathcal{B}_{P_i}$.

We move on the gauge theory on 4-closed manifolds. We choose a lift $\tilde{\varphi}: Q \to Q$ of $\varphi: \partial K_1 \to \partial K_2$ so that $\tilde{\varphi}\Gamma = \Gamma$. For $n \in \boldsymbol{N}$, let X_n be the oriented closed 4-manifold defined by

$$X_n = K_1 \cup Z \times [0, n+1] \underset{\varphi_n}{\cup} Z \times [0, n+1] \cup K_2\,,$$

where φ_n is the diffeomorphism on $Z \times [n-1, n+1]$ defined by using φ and the reflection on $[n-1, n+1]$. Let P_n be the $SO(3)$ bundle over X_n defined by

$$P_n = P_{01} \underset{\iota_1}{\cup} Q \times [0, n+1] \underset{\tilde{\varphi}_n}{\cup} Q \times [0, n+1] \underset{\iota_2}{\cup} P_{02}\,,$$

where $\tilde{\varphi}_n$ is the automorphism on $Q \times [n-1, n+1]$ defined by using φ and the reflection on $[n-1, n+1]$. We put

$$A_{0n} = \begin{cases} A_{01} & \text{over } K_1 \cup Z \times [0, n+1]\,, \\ A_{02} & \text{over } K_2 \cup Z \times [0, n+1]\,. \end{cases}$$

Let $\mathcal{C}_{X_n}^r$ $(r \geqq 3)$ be the space of all conformal classes of C^r-metrics on X_n which are the fixed metric $h + dt^2$ on $Z \times [0, n+1] \cup_{\varphi_n} Z \times [0, n+1]$. If we fix a metric $[g_0]$ in $\mathcal{C}_{X_n}^r$, then $\mathcal{C}_{X_n}^r$ is identified with

$$\{m: \Lambda^+ \to \Lambda^-\,;\ C^r\text{-bundle map, } \sup|m| < 1,\ m|_{X_n \setminus (K_1^0 \amalg K_2^0)} = 0\},$$

where $\Lambda^{\pm}$ is $\pm$ self-dual space with respect to g_0. Then we have the following theorem, whose proof is also the same as in ([7, 4.3], [14, 5 (iii)]).

PROPOSITION 3.8. *There is a Baire set $\mathcal{C}'_{X_n} \subset \mathcal{C}_{X_n}^r$ such that for all $g \in \mathcal{C}'_{X_n}$, the ASD moduli $\mathcal{M}_{X_n}(l_{X_n}, \eta, g)$ is a finite set consisting of irreducible regular connections.*

For each $n\in\boldsymbol{N}$, we fix a metric g_n in $\mathcal{C}'_{X_n}$ such that g_n is independent of n on $K_2^0 \amalg K_2^0$ and the metric

$$g_\imath=\begin{cases} g_n & on\ K_\imath, \\ h+dt^2 & on\ Z\times[0,\infty), \end{cases}$$

lies in $\mathcal{C}'_{Y_\imath}$ for $\imath=1, 2$. We can find these metric tenUejv, because the intersection of countable Baire sets is again a Baire set. For $n\in\boldsymbol{N}$ we fix a function $\tau_n: X_n\to\boldsymbol{R}$ by

$$\tau_n=\begin{cases} \tau_1 & on\ \tau_1^{-1}([0, n-\varepsilon]), \\ \tau_2 & on\ \tau_2^{-1}([0, n-\varepsilon]), \end{cases}$$

to a small $0<\varepsilon<1$. We use the weighted norm $L_{k,\delta}^p(X_n)$ defined by

$$\|s\|_{L_{k,\delta}^p(X_n)}=\Big(\int_{X_n} e^{\tau_n\delta}\sum_{l=0}^{k}|\nabla_{A_{0n}}^{(l)}s|^p\Big)^{1/p}.$$

We note that if s is supported in $\tau_\imath^{-1}([0, n+1])$, then $\|s\|_{L_{k,\delta}^p(X_n)}\leqq\|s\|_{L_{k,\delta}^p(Y_\imath)}\leqq e^{2\delta}\|s\|_{L_{k,\delta}^p(X_n)}$, and if the support of s is contained in $\tau_\imath^{-1}([0, n-1])$, then $\|s\|_{L_{k,\delta}^p(X_n)}=\|s\|_{L_{k,\delta}^p(Y_\imath)}$. We fix $p>4$, $q>8$ and $0<\delta_0$, $\delta<\bar{\delta}$ by

$$\frac{1}{q}+\frac{1}{q}=\frac{1}{p}, \qquad 0<\frac{\delta_0}{2}<\frac{\delta}{p}<\frac{\lambda}{2},$$

where λ^2 is the first eigenvalue of Δ_Γ on $\operatorname{Ker} d_\Gamma^*\subset\Omega_Z^1(\operatorname{Ad} Q)$. The second inequality implies that $L_{3,\delta_0}^2(Y_\imath)\subset L_{1,\delta}^p(Y_\imath)$ [15, Lemma 7.2].

4. Decay estimate.

In [22], Uhlenbech has proved that the curvature of ASD connections controls the uniform norm of the connection matrices in Coulomb gauge. Taubes has extended the result to ASD connections on the trivial bundle over 4-manifolds ([20] see also [3, Appendix A]). We show that his argument is applicable to ASD connections on flat bundles with an unique flat connection in the same way.

LEMMA 4.1. *Let U be an oriented open noncompact Riemannian 4-manifold. Let $U'\Subset U$ be an interior domain with compact closure $\bar{U}'$. We let $Q_0\to U$ be a flat $SO(3)$ bundle and Γ_0 be a canonical flat connection. Suppose that $Q_0|_{\bar{U}'}\to\bar{U}'$ cannot admit any other flat connection topologically. Then there are constants $\varepsilon>0$ and $\zeta_{m,U'}>0$ which depend on U' and $m\in\boldsymbol{N}$ with the following significance: Let A be an ASD connection on P with $\int_U |F_A|^2<\varepsilon$. Then there exists $h\in C^\infty(\operatorname{Aut} Q_0|_{U'})$ such that*

$$\sup_{U'}\left\{\sum_{l=0}^{m}|\nabla_{\Gamma_0}^{(l)}(hA-\Gamma_0)|^2\right\}\leqq\zeta_{m,U'}\int_U|F_A|^2.$$

Proof. We fix a locally finite open cover of U by geodesic balls $\{B_i\}_{i\in N}$ such that the small balls $\{\tilde{B}_i\}_{i\in N}$ with 1/2 radius cover U and the metric on B_i is close to the Eucledian metric. Then by ([22], [7, Proposition (2.3.7), Theorem (2.3.8)]), there exists $\{h_i\in C^\infty(\mathrm{Iso}(Q_0|_{B_i}, B_i\times SO(3)))\}_{i\in N}$ such that $a_i=h_iA$ obeys $d^*a_i=0$ and

$$\sup_{B_i'}\left\{\sum_{l=0}^{m}|\nabla^{(l)}a_i|^2\right\}\leqq\zeta_i\int_{B_i}|F_A|^2. \tag{4.1}$$

Here B_i' is the ball of radius 3/4(radius (B_i)) and ∇ is the covariant derivative defined by the product structure. We denote by $\{(B_i'\cap B_j')_k\}$the connected components of $B_i'\cap B_j'$. To obtain a desired gauge, we modify h_i, inductively. First we put $h_1'=h_1$. Suppose that for $j<i$ we defined $h_j'\in C^\infty(\mathrm{Iso}(Q_0|_{B_j}, B_j\times SO(3)))$ such that $a_j'=h_j'A$ obeys

$$\sup_{B_j'}\left\{\sum_{l=0}^{m}|\nabla^{(l)}a_j'|^2\right\}\leqq\zeta_j'\int_{B_j}|F_A|^2, \tag{4.2}$$

for some modified constant $\zeta_i'>0$. On $B_i'\cap B_j'$, $h_{ij}'=h_i(h_j')^{-1}\in C^\infty(B_i'\cap B_j'\ SO(3))$ obeys

$$dh_{ij}'=h_{ij}'a_j'-a_ih_{ij}'.$$

By bootstrapping, we see that dh_{ij}' has the same C^m-bound as (4.1). If we choose $\varepsilon>0$ so small, we can write $h_{ij}'=\exp(\xi_{ijk}')z_{ijk}'$ for some constant $z_{ijk}'\in$ 50(3) on $(B_i'\cap B_i')_k$. We define $h_i'\in C^\infty(\mathrm{Iso}(Q_{0\ B_i}, B_i\times SO(3)))$by

$$h_i'=\begin{cases}\exp(-\phi_{ijk}\xi_{ijk}')h_i & \text{on } (B_i'\cap B_i')_k,\\ h_i & \text{on} B_i'\setminus B_j',\end{cases}$$

where ϕ_{ijk} is a cut-off function equal to 1 on $(\tilde{B}_i\cap\tilde{B}_j)_k$ and is supported in $(B_i'\cap B_j')_k$. Then $a_i'=h_i'A$ obeys the estimate (4.2) for a modified constant $\zeta_i'>0$. In the construction above, we also obtained that $h_i'(h_j')^{-1}=z_{ijk}'$on $(\tilde{B}_i\cap\tilde{B}_j)_k$ The data $\{\hat{B}_i\cap\bar{U}',z_{ijk}'\}$ define a flat bundle $Q_0'\to\bar{U}'$ and a flat connection Γ_0' on Q_0'. The data $\{\tilde{B}_i\cap\bar{U}',h_i'|_{\tilde{B}\cap\bar{U}'}\}$ define an isomorphism h': $Q_0|_{\bar{U}}\to Q_0'$. By the assumption, there is an isomorphism h'': $Q_0'\to Q_0|_{\bar{U}}$ such that $h''\Gamma_0'=\Gamma_0$. (4.2) guarantees that $h-h''h'$ obeys the desired estimate. □

PROPOSITION 4.2. *There are constants* $\varepsilon>0$ *and* $\zeta=\zeta_m>0$, *independent of* $8\leqq T\leqq\infty$, *with the following significance Let* A *be an ASD connection on* $Q\times(0, T)$ *uith* $\int_{Z\times(0,T)}|F_A|^2<\varepsilon$, *then there exists* $h\in C^\infty(\mathrm{Aut}(Q\times(1, T-1)))$ *suchthat for* $8\leqq t\leqq T-8$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(hA-\pi^*\Gamma)|^2\right\}\leqq\zeta\int_{Z\times[t-8,t+8]}|F_A|^2.$$

Proof. The proof is essentially the same as that of [20, Lemma 10.5]. We apply Lemma 4.1 with $U=Z\times(j-1, j+8)$ and $U'=Z\times[j, j+7]$ for each $j\in 4\boldsymbol{N}$. Then there exists $h_j\in C^\infty(\mathrm{Aut}(Q\times[j,j+7]))$ such that

$$\sup_{Z\times[j,j+7]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(h_jA-\pi^*\Gamma)|^2\right\}\leqq\zeta\int_{Z\times[j-1,j+8]}|F_A|^2 \tag{4.3}$$

On $Z\times[j+4, j+7]$, $h_{j,j+4}=h_jh_{j+4}^{-1}\in C^\infty(\mathrm{Aut}(Q\times[j+4, j+7]))$ obeys

$$d_{\pi^*\Gamma}h_{j,j+4}=h_{j,j+4}(h_{j+4}A-\pi^*\Gamma)-(h_jA-\pi^*\Gamma)h_{j,j+4}.$$

By bootstrapping, we see that $h_{j,j+4}$ has the same estimate as (4.3). If we choose $\varepsilon>0$ so small, the argument in Lemma 4.1 can be repeated with the data $\{h_{j,j+4}\}$ to produce $h'_j\in C^\infty(\mathrm{Aut}(Q\times[j, j+7]))$ such that $h'_j(h'_{j+4})^{-1}=z_{j,j+4}\in\mathfrak{R}$ on $Z\times[j+5, j+6]$ and for some $\zeta'>0$,

$$\sup_{Z\times[j,j+7]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(h'_jA-\pi^*\Gamma)|^2\right\}\leqq\zeta'\int_{Z\times[j-1,j+8]}|F_A|^2.$$

Now we change h'_j to

$$\tilde{h}_j=z_{0,4}\cdots z_{j-4,j}h'_j\,.$$

Then $\tilde{h}_j$ obeys the same bound as above and $\tilde{h}_j(\tilde{h}_{j+4})^{-1}=1$. We define $h\in C^\infty(\mathrm{Aut}(Q\times[1, T)))$ by $h=\tilde{h}_j$ on $Z\times[j+1, j+6]$. Then A satisfies the desired estimate. □

We prove the decay estimate of the curvature of ASD connections over the cylinder $Z\times[0, \infty)$ by the parallel discussion in [7, 7.3]. The following three lemmas can be proved by replacing the trivial connection in the argument of [7, 2.3.4, 2.3.5, 2.3.6, 2.3.7, 2.3.8] by the flat connection Γ. We prove the first only.

LEMMA 4.3. *There are constants N, $\eta>0$ such that if B is a connection on Q in Coulomb gauge relative to Γ (i.e. $d^*_\Gamma(B-\Gamma)=0$) and satisfies $\|B-\Gamma\|_{L^4}<\eta$, then* $\|B-\Gamma\|_{L^2_1}=\|B-\Gamma\|_{L^2}+\|\nabla_\Gamma(B-\Gamma)\|_{L^2}\leqq N\|F_B\|_{L^2}$.

Proof. Since $H^1(Z\ \mathrm{Ad}\ Q)=0$, the basic elliptic estimate for the operator $d^*_\Gamma+d_\Gamma$ on 1-forms gives a bound

$$\|B-\Gamma\|_{L^2_1}\leqq c_1\|d_\Gamma(B-\Gamma)\|_{L^2}\,.$$

Using the Sobolev multiplication theorem, we get

$$\|B-\Gamma\|_{L^2_1}\leqq c_1\|F_B\|_{L^2}+c_1c_2\|B-\Gamma\|_{L^4}\|B-\Gamma\|_{L^2_1}\,.$$

If $\|B-\Gamma\|_{L^4}<1/(2c_1c_2)$, then we can rearrange it as

$$\|B-\Gamma\|_{L_1^2}\{1-c_1c_2\|B-\Gamma\|_{L^4}\}\leqq c_1\|F_B\|_{L},$$

to get $\|B-\Gamma\|_{L_1^2}\leqq 2c_1\|F_B\|_{L^2}$. □

For a connection B on Q and $l\geqq 1$, put:

$$Q_l(B)=\|F_B\|_{L^\infty}+\sum_{j=0}^{l}\|\nabla_B^{(i)}F_B\|_{L^2}.$$

LEMMA 4.4. *There is a constant* $\eta'>0$ *such that if the connection* B *of Lemma* 4.3 *has* $\|B-\Gamma\|_{L^4}<\eta'$, *then for each* $l\geqq 1$, *a bound,*

$$\|B-\Gamma\|_{L_{l+1}^2}=\sum_{k=0}^{l}\|\nabla_\Gamma^{(k)}(B-\Gamma)\|_{L^2}\leqq f_l(Q_l(B))$$

holds for a universal continuous function f_l, *independent of* B, *with* $f_l(0)=0$.

In the lemma below, by a one-parameter family we mean that they are smooth in the Z variable, and all partial derivatives are continuous in both variables.

PROPOSITION 4.5. *There is a constant* $\varepsilon>0$ *such that if* B'_t $(0\leqq t\leqq 1)$ *is a one-parameter family of connections on* Q *with* $\|F_{B'_t}\|_{L^2}<\varepsilon$ *for all* t *and with* $B'_0=\Gamma$, *then for each* t *there exists a one-parameter family of gauge transformations* u_t *such that* $u_0=1$ *and* $u_t(B'_t)=B_t$ *satisfies*

$$d_\Gamma^*(B_t-\Gamma)=0,$$

$$\|B_t-\Gamma\|_{L_1^2}<2N\|F_{B_t}\|_{L^2},$$

where N *is the constant in Lemma* 4.3.

PROPOSITION 4.6. *There are constants* $\varepsilon>0$ *and* $\zeta>0$ *independent of* T *such that if* A *is an ASD connection on* $Q\times(-T,T)$ *with* $\int_{Z\times(-T,T)}|F_A|^2\leqq\varepsilon$, *then for all* $(x,t)\in Z\times[-T+8,T-8]$,

$$|F_A|_{(x,t)}\leqq\zeta e^{-\lambda(T-|t|)}\left(\int_{Z\times(-T,T)}|F_A|^2\right)^{1/2}.$$

Proof. We apply Proposition 4.2 over $Z\times(-T,T)$ to obtain a gauge transformation $h\in C^\infty(\mathrm{Aut}(Q\times(-T+1,T-1)))$ such that for $-T+8\leqq t\leqq T-8$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}^{1}|\nabla_{\pi^*\Gamma}^{(l)}(hA-\pi^*\Gamma)|^2\right\}\leqq\zeta\int_{Z\times[t-8,t+8]}|F_A|^2. \tag{4.4}$$

We henceforth omit h for simplicity. We write A_t for the restriction of A to

$Z\times\{t\}$ $(-T+8\leqq t\leqq T-8)$. We take the path

$$A_{t,s}=\Gamma+s(A_t-\Gamma) \qquad (0\leqq s\leqq 1)$$

from Γ to A_t. By (4.4) we can assume that $\|F_{A_{t,s}}\|_{L^2}<\varepsilon$ for small $\zeta>0$. Then we apply Proposition 4.5 to get a gauge transformation l_t on Q which is homotopic to the identity and satisfies

$$\|l_tA_t-\Gamma\|_{L_1^2}<2N\|F_{A_t}\|_{L^2}\,. \tag{4.5}$$

We use the 'Chern-Simons invariant relative to Γ' defined by

$$T_Z(\Gamma,\ A_t)=\int_Z \mathrm{Tr}\,(d_\Gamma(A_t-\Gamma)\wedge(A_t-\Gamma)+\frac{2}{3}(A_t-\Gamma)\wedge(A_t-\Gamma)\wedge(A_t-\Gamma)).$$

A direct calculation shows that

$$T_Z(\Gamma,\,l_tA_t)=T_Z(\Gamma,\,A_t)+\frac{1}{3}\deg l_t\ . \tag{4.6}$$

Here the last term

$$\deg l_t=\int_Z \mathrm{Tr}\,(d_\Gamma l_tl_t^{-1}\wedge d_\Gamma l_tl_t^{-1}\wedge d_\Gamma l_tl_t^{-1})$$

is a homotopy invariant by Stoke's theorem. So we have $\deg l_t=0$. For $-T+8\leqq t\leqq T-8$, we write

$$\nu(t)=\int_{Z\times(-t,\,t)}|F_A|^2.$$

Then we can easily verify that

$$\frac{d\nu}{dt}=2(\|F_{A_t}\|_{L^2}^2+\|F_{A_{-t}}\|_{L^2}^2)$$

and

$$\nu(t)=T_Z(A_t,\ \Gamma)-T_Z(A_{-t},\Gamma)$$

We need a simple lemma; any $a\in\Omega_Z^1(\mathrm{Ad}\,Q)$ satisfies

$$\int_Z\mathrm{Tr}\,(d_\Gamma a\wedge a)\leqq\frac{1}{\lambda}\int_Z|d_\Gamma a|^2. \tag{4.7}$$

We prove it quickly. If a is replaced by $a+d_\Gamma f$ for some $f\in\Omega^0(\mathrm{Ad}\,Q)$, each of the integrals is unchanged, so we may assume that a satisfies $d_\Gamma^*a=0$. Then

$$\int_Z\mathrm{Tr}\,(d_\Gamma a\wedge a)\leqq\|a\|_{L^2}\|d_\Gamma a\|_{L^2}$$

and $\int_Z |d_\Gamma a|^2=\langle a,\,\Delta_\Gamma a\rangle$. So $\|d_\Gamma a\|_{L^2}\geqq\lambda\|a\|_{L^2}$. Since $H^1(Z\,;\mathrm{Ad}\,Q)=0$, we see that $\int_{JZ} \mathrm{Tr}\,(d_\Gamma a \wedge a) \leqq \lambda^{-1} \int_{JZ} |d_\Gamma a|^2$.

Using (4.5), (4.6), (4.7) and the Sobolev embedding theorem $L_1^2 \to L^4$, we get

$$|T_Z(\Gamma, A_t)| \leqq \frac{1}{\lambda} \|F_{A_t}\|_{L^2}^2 + c\|l_t A_t - \Gamma\|_{L_1^2}^3$$

$$\leqq \frac{1}{\lambda} \|F_{A_t}\|_{L^2}^2 + 8cN^3 \|F_{A_t}\|_{L^2}^3 ,$$

where the constant c depends only on Q. So we have

$$\frac{d\nu}{dt} \geqq 2\lambda\nu - c\left(\frac{d\nu}{dt}\right)^{3/2}. \tag{4.8}$$

We also know that ν and $d\nu/dt$ are small by (4.4). We use an elementary inequality; if $y + Cy^{3/2} \geqq 2\lambda x$ for some C, and x and y are small, then $y \geqq 2\lambda x - C' x^{3/2}$ for another constant C' '. Hence (4.8) gives

$$\frac{d\nu}{dt} \geqq 2\lambda\nu - c\nu^{3/2} .$$

We choose $\varepsilon > 0$ so small that $\delta = \varepsilon^{1/2} \leqq (1/2)\lambda$. Then the inequality above gives $d\nu/dt \geqq (2\lambda - \delta)\nu$, which we can integrate to get an exponential bound

$$\nu(t) \leqq e^{(2\lambda-\delta)(t-T)} \nu(T) .$$

Feeding back this into the differential inequality, we get

$$\frac{d\nu}{dt} \geqq 2\lambda\nu - c\delta e^{(2\lambda-\delta)(t-T)/2} \nu$$

$$\geqq 2\lambda\nu - \frac{c\delta e^{(2\lambda-\delta)(t-T)/2}}{2\lambda-\delta} \frac{d\nu}{dt} .$$

It follows that

$$\log \nu(T) - \log \nu(t) \geqq 2\lambda \int_t^T \frac{d\tau}{1 + ce^{(2\lambda-\delta)(t-T)/2}}$$

$$\geqq 2\lambda \int_t^T (1 - ce^{(2\lambda-\delta)(t-T)/2}) d\tau$$

$$\geqq 2\lambda(T-t) - \frac{4c\lambda}{2\lambda-\delta} .$$

Taking exponentials, we have the bound

$$\nu(t) \leqq K\nu(T) e^{2\lambda(t-T)} ,$$

with $K = \exp(4c\lambda/(2\lambda-\delta))$. Finally we use the following for any element f in $\Omega^2(\mathrm{Ad}(Q \times (-1, 1)))$ with $d_A^* f = d_A f = 0$,we have an elliptic estimate:

$$\sup_{Z\times\{0\}} |f|^2 \leqq c\int_{Z\times(-1,1)} |f|^2 .$$

Since A obeys the uniform C^1-estimate by (4.4), we can take c which is independent of A. Applying this to $f=F_A$ on $Z\times(-T+8, T-8)$, we get

$$|F_A|_{(x,t)} \leqq c\nu(t)^{1/2} \leqq ce^{\lambda(|t|-T)}\nu(T)^{1/2}$$

and the proof is completed. □

We apply the above proposition over $Z\times(n_0, 2t-n_0)$ to obtain that

COROLLARY 4.7. *If A is an ASD connection on π^*Q with $\int_{Z\times[n_0,\infty)} |F_A|^2<\varepsilon$ for some $n_0\in \boldsymbol{N}$, then for all $(x, t)\in Z\times[n_0+1, \infty)$,*

$$|F_A|_{(x,t)} \leqq \zeta e^{-\lambda t}\Big(\int_{Z\times[n_0,\infty)} |F_A|^2\Big)^{1/2}.$$

5. Gluing ASD connections.

According to [7, 7.1], we shall construct a gluing map from $\mathcal{M}_{Y_1}(l_{Y_1}, \sigma_1^*(\eta), g_1) \times \mathcal{M}_{Y_2}(l_{Y_2}, \sigma_2^*(\eta), g_2)$ to $\mathcal{M}_{X_n}(l_{X_n}, \eta, g_n)$. To obtain the gluing map globally, we need a technical lemma below. In this section, we use c for a constant independent of n and use c_i for a constant with respect to Y_i.

LEMMA 5.1. *There are constants $\varepsilon>0$, $t_0=t_{0,m}>0$ and $\rho=\rho_m>0$ with the following significance: Let A be an ASD connection on π^*Q. Suppose that $n_0\in \boldsymbol{N}$ exists such that $\int_{Z\times[n_1,\infty)} |F_A|^2<\varepsilon$. Then there exists $h\in C^\infty(\mathrm{Aut}(Q\times[n_0+1, \infty)))$ such that for $t\geqq n_0+t_0$,*

$$\sup_{Z\times[t-1,t+1]}\Big\{\sum_{l=0}^{m} |\nabla_{\pi^*\Gamma}^{(l)}(hA-\pi^*\Gamma)|^2\Big\} \leqq \rho e^{-2\lambda(t-n_0)},$$

$$i_{\partial/\partial t}(hA-\pi^*\Gamma)=0 .$$

Such a h is unique up to $\mathcal{R}$. Moreover, if A satisfies

$$\int_{Z\times(0,\infty)} \sum_{l=0} |\nabla_{\pi^*\Gamma}^{(l)}(A-\pi^*\Gamma)|^2<\infty ,$$

then we can choose h so that $h|_{t=n_0}$ is homotopic to the identity, and such a h is unique.

Proof. By Proposition 4.2 and Corollary 4.7, there exists $h\in C^\infty(\mathrm{Aut}(Q\times[n_0+1, \infty)))$ such that for $t\geqq n_0+8$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(hA-\pi^*\Gamma)|^2\right\}\leqq ce^{-2\lambda(t-n_0)}.$$

Solve the ordinary differential equation for $\tilde{h}\in C^\infty(\mathrm{Aut}(Q\times[n_0+1,\infty)))$:

$$i_{\partial/\partial t}(\tilde{h}(hA)-\pi^*\Gamma)=-\frac{\partial}{\partial t}\tilde{h}\tilde{h}^{-1}+\tilde{h}i_{\partial/\partial t}(hA-\pi^*\Gamma)\tilde{h}^{-1}=0$$

with initial condition $\lim_{t\to\infty}\tilde{h}=1$. From the differential equation we have

$$-\frac{\partial}{\partial t}(d_\Gamma\tilde{h})=d_\Gamma\tilde{h}i_{\partial/\partial t}(hA-\pi^*\Gamma)+\tilde{h}d_\Gamma(i_{\partial/\partial t}(hA-\pi^*\Gamma))\cdot \tag{5.1}$$

This gives us

$$-\frac{\partial}{\partial t}|d_\Gamma\tilde{h}|^2\leqq ce^{-\lambda(t-n_0)}(|d_\Gamma\tilde{h}|^2+|d_\Gamma\tilde{h}|),$$

which integrates over $[t-1,\infty)$ to get an inequality

$$\|d_\Gamma\tilde{h}\|^2_{C^0(Z\times[t-1,\infty))}\leqq ce^{-\lambda(t-n_0)}(\|d_\Gamma\tilde{h}\|^2_{C^0(Z\times[t-1,\infty))}+\|d_\Gamma\tilde{h}\|_{C^0(Z\times[t-1,\infty))}).$$

For $t>n_0+t_0$ with $ce^{-2\lambda t_0}<1/2$, we can rearrange this to get

$$\|d_\Gamma\tilde{h}\|_{C^0(Z\times[t-1,t+1])}\leqq\|d_\Gamma\tilde{h}\|_{C^0(Z\times[t-1,\infty))}\leqq 2ce^{-\lambda(t-n_0)}.$$

Now we can bootstrap the equation

$$\tilde{h}(hA)-\pi^*\Gamma=-d_\Gamma\tilde{h}\tilde{h}^{-1}-\tilde{h}\iota_{\partial/\partial t}(hA-\pi^*\Gamma)\tilde{h}^{-1}+\tilde{h}(hA-\pi^*\Gamma)\tilde{h}^{-1}. \tag{5.2}$$

Then we see that $h'=\tilde{h}h$ satisfies the desired properties.

If h' also satisfies the properties of Lemma 5.1, then $h'h^{-1}$ is independent of t and satisfies

$$d_\Gamma(h'h^{-1})=h'h^{-1}i_{\partial/\partial t}(hA-\pi^*\Gamma)-i_{\partial/\partial t}(h'A-\pi^*\Gamma)h'h^{-1}.$$

This implies that $h'h^{-1}$ converges to a flat gauge as $t\to\infty$. So $h'h^{-1}$ must be an element in $\mathfrak{R}$.

If A satisfies the additional condition, then by the equation

$$d_{\pi^*\Gamma}h=h(A-\pi^*\Gamma)-(hA-\pi^*\Gamma)h$$

and the Sobolev embedding theorem $L^2_3\to C^0$, we get

$$\|d_{\pi^*\Gamma}h\|_{C^0([t,t+1])}\longrightarrow 0\ (t\to\infty).$$

Hence A factors as $h=r(h)\exp\xi$ with $r(h)\in\mathfrak{R}$ and $\xi\in C^\infty(\mathrm{Ad}(Q\times[n_1,\infty)))$ for some $n_1\in\boldsymbol{N}$. So $h'=r(h)^{-1}h$ has the desired properties and it is unique, since the primary difference $\mathfrak{b}(1, \mathrm{r})$ is non-zero for any non-identity element $r\in\mathfrak{R}$. □

The above unique gauge is said to be exponential gauge. We combine Lemma 5.1 with Lemma 3.1 to deduce

COROLLARY 5.2. *If B_i is an ASD connection in $\mathcal{A}_{P_i}$ then there are $n_0 \in N$ and $u_i \in \mathcal{G}_{P_i}$ such that $A_i = u_i B_i$ satisfies the following conditions.*

(i) *A is smooth over $\tau_i^{-1}([n_0+1, \infty))$*

(ii) *For $t \geqq n_0 + t_0$,*

$$\sup_{Z\times[t-1,t+1]} \left\{ \sum_{l=0}^{m} |\nabla_{\pi^*\Gamma}^{(l)}(A_i - \pi^*\Gamma)|^2 \right\} \leqq \rho e^{-2\lambda(t-n_0)},$$

$$i_{\partial/\partial t}(A_i - \pi^*\Gamma) = 0 .$$

If A'_i also satisfies the above condition. then $A_i = A'_i$ over $\tau_i^{-1}([n_0+t_0, \infty))$

Let β_i, γ_i and μ_i be smooth cut off functions satisfying

$$\beta_i(x, t) = \begin{cases} 1 & \text{on } \tau_i^{-1}([0, n+1]), \\ (n+N+1-t)/N & \text{on } \tau_i^{-1}([n+1+\varepsilon, n+N+1-\varepsilon]), \\ 0 & \text{on } \tau_i^{-1}([n+N+1, \infty)), \end{cases}$$

$$\gamma_i = \begin{cases} 1 & \text{on } \tau_i^{-1}([0, n-1]), \\ 0 & \text{on } \tau_i^{-1}([n+1, \infty)), \end{cases}$$

$$\mu_i = \begin{cases} 1 & \text{on } \tau_i^{-1}([0, n-N-2]), \\ 0 & \text{on } \tau_i^{-1}([n-N-1, \infty)), \end{cases}$$

$$\gamma_1(x, t) + \gamma_2(n-x, t) = 1 \qquad \text{if } (x, t) \in \tau_i^{-1}([n-1, n+1]),$$

for a small $0 < \varepsilon < 1$. Then a direct calculation shows that

$$\|\nabla \beta_i\|_{L^q(Y_i)} \leqq K N^{-(q-1)/q},$$

where K is independent of a and N. For $A_i \in \mathcal{A}_{P_i}$, we define a connection A'_i on P_i by $A'_i = \mu_i(A_i - \pi^*\Gamma) + (1-\mu_i)\pi^*\Gamma$ and a connection A' on P_n by

$$J_n(A_1, A_2) = A' = \begin{cases} A'_1 & \textit{over } \tau_1^{-1}([0, n+1]), \\ A'_2 & \textit{over } \tau_2^{-1}([0, n+1]). \end{cases}$$

LEMMA 5.3. *If A_i is an ASD connection in $\mathcal{A}_{P_i}$ satisfying the conditions* (i) (ii), *then there is a constant $c > 0$ such that*

(1) $\|A_i - A'_i\|_{L^q(Y_i)} < c e^{-\lambda(n-N)}$

(2) $\|A_i - A'_i\|_{L^p_\delta(Y_i)} + \|\nabla_{\pi^*\Gamma}(A_i - A'_i)\|_{L^p_\delta(Y_i)} < c e^{-(\lambda - \delta/p)(n-N)}$,

(3) $\|F^+_{A'_i}\|_{L^p_\delta(Y_i)} < c e^{-(\lambda-\delta/p)(n-N)}$.

Proof. This is obvious. □

We are in the position to argue the right inverse to $d^+_{A'}$. For an ASD connection A_i in $\mathcal{A}_{P_i}$, we take the Laplacian

$$\Delta_{A_i}=d^+_{A_i}e^{-\tau_i\delta}(d^+_{A_i})^*e^{\tau_i\delta}: L^p_{2,\delta}(Y_i)\longrightarrow L^p_\delta(Y_i).$$

The condition $H^2_{A_i}=0$ implies that Δ_{A_i} is invertible [15, Lemma 7.3]. Then there is the right inverse P_i to the operator d^+_A

$$P_i=e^{-\tau_i\delta}(d^+_{A_i})^*e^{\tau_i\delta}(\Delta_{A_i})^{-1}: L^p_\delta(Y_i)\longrightarrow L^p_{1,\delta}(Y_i),$$

which satisfies $\|P_i\xi\|_{L^p_{1,\delta}(Y_i)}\leqq c_i\|\xi\|_{L^p_\delta(Y_i)}$ for some constant c_i. Composing with the Sobolev embedding

$$L^p_{1,\delta}(Y_i)\longrightarrow L^q_{q\delta/p}(Y_i)$$

[15, Lemma 7.2], we have $\|P_i\xi\|_{L^q_{q\delta/p}(Y_i)}\leqq c_i\|\xi\|_{L^q_\delta(Y_i)}$ We also need Hölder's inequality, for 1-forms a, ft,

$$\|(a\wedge b)^+\|_{L^p_\delta}\leqq\sqrt{2}\,\|a\|_{L^q_{q\delta/p}}\|b\|_{L^q}\leqq\sqrt{2}\,\|a\|_{L^q_{q\delta/p}}\|b\|_{L^q_{q\delta/p}},$$

over X_n or Y_i. Now let $Q_i: L^p_\delta(X_n)\to L^p_{1,\delta}(X_n)$ be the operator defined by

$$Q_i\xi=\beta_iP_i\gamma_i(\xi).$$

Then we obtain a bound $\|Q_i\xi\|_{L^p_{1,\delta}(X_n)}\leqq c_ie^{2\delta}\|\xi\|_{L^p_\delta(X_n)}$.

LEMMA 5.4. *There is a constant* $\varepsilon_i=\varepsilon_i(N,n)\to 0$ as $n\to\infty$ *and* $N\to\infty$ *in order such that*

$$\|\gamma_i\xi-d^+_{A'_i}Q_i\xi\|_{L^p_\delta(X_n)}\leqq\varepsilon_i(N,n)\|\xi\|_{L^p_\delta(X_n)}$$

Proof. If we write $A'_i=A_i+a_i$, then

$$\begin{aligned}d^+_{A'_i}(Q_i\xi)&=d^+_{A_i+a_i}(\beta_iP_i(\gamma_i\xi))\\&=\beta_i(d^+_{A_i}P_i(\gamma_i\xi))+(\nabla\beta_i)P_i(\gamma_i\xi)+[\beta_ia_i,\ P_i(\gamma_i\xi)]\\&=\gamma_i\xi+(\nabla\beta_i)P_i(\gamma_i\xi)+[\beta_ia_i,\ P_i(\gamma_i\xi)].\end{aligned}$$

By Lemma 5.3, we get

$$\begin{aligned}\|(\nabla\beta_i)P_i(\gamma_i\xi)\|_{L^p_\delta(X_n)}&\leqq\|(\nabla\beta_i)P_i(\gamma_i\xi)\|_{L^p_\delta(Y_i)}\\&\leqq\sqrt{2}\,\|\nabla\beta_i\|_{L^q(Y_i)}\|P_i(\gamma_i\xi)\|_{L^q_{q\delta/p}(Y_i)}\\&\leqq c_iKN^{-(q-1)/q}e^{2\delta}\|\xi\|_{L^p_\delta(X_n)},\end{aligned}$$

$$\begin{aligned}\|[\beta_ia_i,\ P_i(\gamma_i\xi)]\|_{L^p_\delta(X_n)}&\leqq\|[\beta_ia_i,\ P_i(\gamma_i\xi)]\|_{L^p_\delta(Y_i)}\\&\leqq\sqrt{2}\,\|\beta_ia_i\|_{L^q(Y_i)}\|P_i(\gamma_i\xi)\|_{L^q_{q\delta/p}(Y_i)}\\&\leqq c_ie^{-\lambda(n-N)+2\delta}\|\xi\|_{L^p_\delta(X_n)}.\end{aligned}$$

The result now follows by letting $\varepsilon_i=c_ie^{2\delta}(e^{-\lambda(n-N)}+KN^{-(q-1)/q})$. □

We put

$$Q=Q_1+Q_2:\ L^p_\delta(X_n) \longrightarrow L^p_{1,\delta}(X_n).$$

The operator $R=d^+_A Q-1$ obeys a bound

$$\|R(\xi)\|_{L^p_\delta(X_n)} \leqq (\varepsilon_1(N, n)+\varepsilon_2(N, n))\|\xi\|_{L^p_\delta(X_n)}.$$

We choose N_0 and $n_0=n_0(N_0)$ so that $\varepsilon_i(N_0, n)\leqq 1/3$ for all $n\geqq n_0$. Then the operator norm of R is at most 2/3. So $1+R$ is invertible and the norm of the inverse is at most 3. Thus the right inverse $P=Q(1+R)^{-1}$ to d^+_A satisfies

$$\begin{aligned}\|P\xi\|_{L^p_{1,\delta}(X_n)} &\leqq \|Q_1(1+R)^{-1}\xi\|_{L^p_{1,\delta}(X_n)}+\|Q_2(1+R)^{-1}\xi\|_{L^p_{1,\delta}(X_n)}\\ &\leqq c_1e^{2\delta}\|(1+R)^{-1}\xi\|_{L^p_\delta(X_n)}+c_2e^{2\delta}\|(1+R)^{-1}\xi\|_{L^p_\delta(X_n)}\\ &\leqq c\|\xi\|_{L^p_\delta(X_n)} \qquad (5.3)\end{aligned}$$

with $c=3e^{2\delta}(c_1+c_2)$. Combining with the Sobolev embedding, we get $\|P\xi\|_{L^q_{q\delta/p}} ax_n) \leqq c\|\xi\|_{L^p_\delta(X_n)})$. We seek a solution $A'+a$ to the ASD equations in the form $a=P(\xi)$. If we write $q(\xi)=(P\wedge P\xi)^+$, the ASD equation becomes

$$\xi+q(\xi)=-F^+_A\ , \qquad (5.4)$$

By Hölder's inequality,

$$\|q(\xi_1)-q(\xi_2)\|_{L^p_\delta(X_n)} \leqq \sqrt{2}\,c^2\|\xi_1-\xi_2\|_{L^p_\delta(X_n)}\{\|\xi_1\|_{L^p_\delta(X_n)}+\|\xi_2\|_{L^p_\delta(X_n)}\}.$$

LEMMA 5.5. ([7, Lemma (7.2.23)] *Let* $5: B\to B$ *be a smooth map on a Banach space and* $\|S\xi_1-S\xi_2\| \leqq k\{\|\xi_1\|+\|\xi_2\|\}\|\xi_1-\xi_2\|$ *for some* $k>0$ *and all* $\xi_1, \xi_2\in B$. *Then for each* $\eta\in B$ *with* $\|\eta\|<1/(10k)$ *there is a unique* ξ *with* $\|\xi\|\leqq 1/(5k)$ *such that*

$$\xi+S(\xi)=\eta.$$

We apply Lemma 5.5 to the above equation with $S=q$, $\eta=-F^+_{A'}$ and $k\geqq \sqrt{2}\,c^2$. Then

PROPOSITION 5.6. *Let* A_i *be an ASD connection in* $\mathcal{A}_{P_i}$ *satisfying the conditions* (i), (ii). *Then there exists* N_0 *and* $n_0=n_0(N_0)$ *such that for all* $n\geqq n_0$, *we can find an ASD connection* $I_n(A_1,A_2)=A'+a$ *on* P_n *with* $\|a\|_{L^q_{q\delta/p}(X_n)} \leqq c\|a\|_{L^p_{1,\delta}(X_n)} \leqq ce^{-(\lambda-\delta/p)(n-N_0)}$. *a is the unique such solution which can be written in the form* $P\xi$. *If* u_iA_i *also satisfies the conditions* (i), (ii) *for some* $u_i\in\mathcal{G}_{P_i}$, *then*

$$I_n(u_1A_1,\ u_2A_2)=(u_1\cup u_2)I_n(A_1,\ A_2).$$

Moreover we can choose N_0 *and* $n_0=n_0(N_0)$ *so that for all* $n\geqq n_0$, $A_t=A'+ta$, $(0\leqq t\leqq 1)$ *is regular, that is* $[A']\in U(I_n(AA_2))$.

Proof. By bootstrapping (5.4), we see that the solutions we solved are in

C^∞ if so are A_ι. The second part is obvious from the gauge equivalence of the above construction. We prove the last part. By (5.3), we have

$$\|(d^+_{A_t}-d^+_{A'})\xi\|_{L^p_\delta(X_n)}=\|t[a,\ \xi]\|_{L^p_\delta(X_n)}\leqq ce^{-(\lambda-\delta/p)(n-N_0)}\|\xi\|_{L^p_{1,\delta}(X_n)}.$$

If we choose n_0 so large that $ce^{-(\lambda-\delta/p)(n_0-N_0)}\leqq 1/2$, then for $n\geqq n_0$, the operator norm of $(d^+_{A_t}-d^+_{A'})P$ is at most 1/2 and $d^+_{A_t}P=1+(d^+_{A_t}-d^+_{A'})P$ is invertible. So $A't$ is regular. □

Now we obtain the gluing map

$$I_n:\ \mathcal{M}_{Y_1}(l_{Y_1},\ \rho^*_1(\eta),\ g_1)\times\mathcal{M}_{Y_2}(l_{Y_2},\ \sigma^*_2(\eta),\ g_2)\longrightarrow\mathcal{M}_{X_n}(l_{X_n},\ \eta,\ g_n),$$

for large $N\geqq N_0$ and $n\geqq n_0(N_0)$. Here N_0, $n_0(N_0)$ are the maximal values of all $[A_\iota]$ in $\mathcal{M}_Y(l_{Y_i}, \sigma^*_i(\eta), g_i)$. We may suppose that $N_0=0$.

We prove that I_n is injective for large n. Let A_ι, B_ι be as above. Suppose that $I_n(A_1, A_2)=A'+a$ is gauge equivalent to $I_n(B_1, B_2)=B'+b$ by some gauge $u_n\in\mathcal{G}_{X_n}$. We expand the equation $u_n(A'+a)=B'+b$ over $\tau_\iota^{-1}([n-3,\ n-2])$. Then

$$d_{\pi^*\Gamma}u_n=u_n(A_i-\pi^*\Gamma)+u_na-(B_i-\pi^*\Gamma)u_n-bu_n.$$

By Lemma 5.3,

$$\|d_{\pi^*\Gamma}u_n\|_{C^0(\tau_\iota^{-1}([n-3,\ n-2]))}\leqq c\|d_{\pi^*\Gamma}u_n|_{\tau_\iota^{-1}([n-3,\ n-2])}\|_{L^p_{1,\delta}(Y_i)}\leqq ce^{-(\lambda-\delta/p)n}.$$

Hereafter $\|\ \ \mathrm{I}\ \|_{L^p_{k,\delta}(\cdot)}$ is the integral over the restriction. This implies that for large n, u_n factors as $u_n=r(u_n)\exp h_{\iota n}$ over $\tau_\iota^{-1}([n-3,\ n-2])$, where $r(u_n)\in\mathcal{R}$ satisfies

$$\|r(u_n)^{-1}u_n-1\|_{C^0(\tau_\iota^{-1}([n-3,\ n-2]))}\leqq c\|d_{\pi^*\Gamma}u_n\|_{C^0(\tau_\iota^{-1}([n-3,\ n-2]))}\leqq ce^{-(\lambda-\delta/p)n}.$$

Lemma 3.1 asserts that $r(u_n)$ is the identity. Since the exponential map on $\mathfrak{so}(3)$ is a local diffeomorphism, we get

$$\|h_{\iota n}\|_{C^0(\tau_\iota^{-1}([n-3,\ n-2]))}+\|d_{\pi^*\Gamma}h_{\iota n}\|_{C^0(\tau_\iota^{-1}([n-3,\ n-2]))}\leqq ce^{-(\lambda-\delta/p)n}.$$

We put

$$u_{\iota n}=\begin{cases} u_n & \text{over}\quad \tau_\iota^{-1}([0,\ n-3]),\\ \exp(\delta_i h_{\iota n}) & \text{over}\quad \tau_\iota^{-1}([n-3,\ \infty)),\end{cases}$$

where δ_i is a cut off function such that $\delta_i=1$ on $\tau_\iota^{-1}([0,\ n-3])$ and $\delta_i=0$ on $\tau_\iota^{-1}([n-2,\ \infty))$. By estimating all the term in the right hand side of the equation

$$u_{\iota n}A_i-B_i=\begin{cases} -u_nau_n^{-1}+b & \text{over}\quad \tau_\iota^{-1}([0,\ n-3]),\\ -d_{\pi^*\Gamma}(\exp(\delta_ih_{in}))u_{\iota n}^{-1} & \\ \quad +u_{\iota n}(A_i-\pi^{*}\ \text{[illegible]} & \text{over}\quad \tau_\iota^{-1}([n-3,\ n-2]),\\ (A_i-\pi^*\Gamma)-(B_i-\pi^*\Gamma) & \text{over}\quad \tau_\iota^{-1}([n-2,\ \infty)),\end{cases}$$

we see that $u_{in}A_i$ converges to B_i in $L^p_\delta(Y_i)$ as $n\to\infty$. On the other hand, $\mathcal{M}_{Y_i}(l_{Y_i},\sigma_i^*(\eta),g_i)$ consists of isolated points with respect to the metric (c. f. [7, Lemma (4.2.4)])

$$d_{P_i}([A],[B])=\inf_{u\in\mathcal{G}_{P_i}}\|A-uB\|_{L^p_\delta(Y_i)}.$$

This implies that $[A_i]=[B_i]$

We will prove that I_n is surjective for larger n. For a moment, we work with the space of L^p_1 connections $\mathcal{A}'_{P_n}$, L^p_2 gauge group $\mathcal{G}'_{P_n}$ and the quotient $\mathcal{B}'_{P_n}=\mathcal{A}'_{P_n}/\mathcal{G}'_{P_n}$. Let d_{P_n} be the metric in $\mathcal{B}'_{P_n}$ given by [7, Lemma (4.2.4)]

$$d_{P_n}([A],[B])=\inf_{u\in\mathcal{G}'_{P_n}}\|A-uB\|_{L^q_{q\delta/p}(X_n)}.$$

We define

$$J_n:\ \mathcal{M}_{Y_1}(l_{Y_1},\ \sigma_1^*(\eta),\ g_1)\times\mathcal{M}_{Y_2}(l_2,\ \sigma_2^*(\eta),\ g_2)\longrightarrow\mathcal{B}'_{P_n}$$

by $J_n([A_1],[A_2])=[J_n(A_1,A_2)]=[A']$ For $\nu>0$, let $U(\nu)\subset\mathcal{B}'_{P_n}$ be the open set

$$U(\nu)=\{[A]\in\mathcal{B}'_{P_n}\,|\,d_{P_n}([A],\operatorname{Im}J_n)<\nu,\ \|F_A^+\|_{L^p_\delta(X_n)}<\nu^{3/2}\}.$$

The solutions we have constructed lie in $U(\nu)$, if $n\geqq n_0=n_0(\nu)$ with $ce^{-(\lambda-\delta/p)n_0}<\nu$. Conversely.

PROPOSITION 5.7. *There is a constant $\nu_0>0$ such that for $0<\nu<\nu_0$, some $n_0=n_0(\nu)\in\boldsymbol{N}$ satisfies the following If $n>n_0$, then $[A]\in U(\nu)$ can be represented by a connection A of the form $A'+P\xi$ with $\xi\in L^p_\delta(X_n)$ and $\|\xi\|_{L^p_\delta(X_n)}<1/(5k)$ $(k=\sqrt{2}c^2)$.*

Proof. Let B be an element of $U(\nu)$. Then there is a connection $[A']\in\operatorname{Im}J_n$ with

$$\|A'-B\|_{L^q_{q\delta/p}(X_n)}<\nu.$$

We write $B=A'+b$ and consider the path

$$B_t=A'+tb\ \ (0\leqq t\leqq 1),$$

then $\|A'-B_t\|_{L^q_{q\delta/p}(X_n)}<\nu$and

$$F^+_{B_t}=(1-t)F^+_{A'}+tF^+_B+(t^2-t)(b\wedge b)^+.$$

So

$$\begin{aligned}\|F^+_{B_t}\|_{L^q_{q\delta/p}(X_n)}&\leqq(1-t)\|F^+_{A'}\|_{L^q_{q\delta/p}(X_n)}+t\|F^+_B\|_{L^q_{q\delta/p}(X_n)}+\sqrt{2}(t-t^2)\|b\|^2_{L^q_{q\delta/p}(X_n)}\\&\leqq(1-t)ce^{-(\lambda-q\delta/p)n}+t\nu^{3/2}+\sqrt{2}(t-t^2)\nu^2.\end{aligned}$$

Thus we can find $\nu_0>0$ with the following: For $0<\nu<\nu_0$, there is a constant $n_0=n_0(\nu)\in\boldsymbol{N}$ such that if $n>n_0$, then $[B_t]$ $(0\leqq t\leqq 1)$ is contained in $U(\nu)$. *We* define $S\subset[0, 1]$ to be the set of times for which there are $u_t\in\mathcal{G}'_{P_n}$ and $[A']\in\operatorname{Im}J_n$ such that

$$u_tB_t=A'+P\xi$$

with $\|\xi\|_{L^p_\delta(X_n)}<1/(5k)$. *We* will prove that S is closed and open. Suppose that t is in S. *We* may take $u_t=1$. Then the representation $B_t=A'+P\xi$ gives

$$F^+_{B_t}=F^+_{A'}+\xi+(P\xi\wedge P\xi)^+.$$

This gives a bound

$$\|\xi\|_{L^p_\delta(X_n)}\leqq\|F^+_{B_t}\|_{L^p_\delta(X_n)}+\|F^+_{A'}\|_{L^p_\delta(X_n)}+\sqrt{2}\|P\xi\|^2_{L^q_{q\delta/p}(X_n)}$$
$$\leqq\nu^{3/2}+ce^{-(\lambda-\delta/p)n}+\sqrt{2}c^2\|\xi\|^2_{L^p_\delta(X_n)}.$$

Arranging this, we get

$$\|\xi\|_{L^p_\delta(X_n)}\leqq\frac{5}{4}(\nu^{3/2}+ce^{-(\lambda-\delta/p)n}).$$

This implies that for larger $n_0\in\boldsymbol{N}$ and smaller $\nu_0>0$, $\|\xi\|_{L^p_\delta(X_n)}\leqq1/(10k)$. That is, $\|\xi\|_{L^p_\delta(X_n)}<1/(5k)$ implies $\|\xi\|_{L^p_\delta(X_n)}\leqq1/(10k)$, so this open condition is also closed. *We* will prove that S is closed. Suppose that $\{t_i\}$ is in S with $t_i\to t$. Then we have connections $A_{t_i}=A'+P\xi_i$ with $\|\xi_i\|_{L^p_\delta(X_n)}$, and $u_i\in\mathcal{G}'_{P_n}$ with $u_iB_{t_i}=A_{t_i}$. By the uniform convergence bound on the ξ_i above, we may suppose that, taking a subsequence, the ξ_i converge to a limit ξ, weakly in $L^p_\delta(X_n)$, with $\|\xi\|_{L^p_\delta(X_n)}<1/(5k)$ Then the connections A_{t_i} converge weakly in $L^p_{1,\delta}(X_n)$ and the equation

$$d_{B_t}u_i=u_i(B_{t_i}-B_t)-(A_{t_i}-B_t)u_i$$

implies that, after taking a subsequence, u_i converges to a limit u weakly in $L^p_{2,\delta}(X_n)$. The gauge relation is preserved under weak limit, so

$$uB_t=A'+P\xi$$

and t is in 5. *We* will prove that S is open. Suppose that t is in S. Then $B_t=A'+P\xi$ with $\|\xi\|_{L^p_\delta(X_n)}<1/(10k)$. *We* define a map

$$M\colon \Omega^0_{X_n}(\operatorname{Ad}P_n)\times\Omega^+_{X_n}(\operatorname{Ad}P_n)\longrightarrow\Omega^1_{X_n}(\operatorname{Ad}P_n)$$

by

$$M(\chi, \eta)=(\exp(\chi)(A'+P(\xi+\eta))-B_t.$$

Let B_1 be the completion of $\Omega^0_{X_n}(\operatorname{Ad}P_n)\times\Omega^+_{X_n}(\operatorname{Ad}P_n)$ in the norm :

$$\|(\chi, \eta)\|_{B_1}=\|d_{A'}\chi\|_{L^q_{q\delta/p}(X_n)}+\|\eta\|_{L^p_\delta(X_n)}.$$

Since A' is irreducible by the unique continuation theorem [7, Lemma (4.3.2)], we have an elliptic estimate

$$\|\chi\|_{C^0(X_n)} \leqq c_n \|d_{A'}\chi\|_{L^q(X_n)} \leqq c_n \|d_{A'}\chi\|_{L^q_{q\delta/p}(X_n)}$$

for some constant $c_n > 0$ (c.f. [7, (7.2.30)]). So B_1 is a norm. Let B_2 be the completion of $\Omega^1_{X_n}(\mathrm{Ad}\, P_n)$in the **norm:**

$$\|\alpha\|_{B_2} = \|\alpha\|_{L^q_{q\delta/p}(X_n)} + \|d^+_{A'}\alpha\|_{L^p_\delta(X_n)}.$$

Then M can be extended to a map from B_1 to B_2 and the derivative at (0, 0) is given by

$$T(\chi, \eta) = DM_{(0,0)}(\chi, \eta) = d_{A'}\chi + P\eta.$$

By definition, T is a bounded map from B_1 to B_2.

LEMMA 5.8. *There is a larger $n_0 \in \boldsymbol{N}$ such that if $n > n_0$ then*

$$\|(\chi, \eta)\|_{B_1} \leqq 4\|T(\chi, \eta)\|_{B_2}.$$

Proof. Let $\alpha = d_{A'}\chi + P\eta$, so that rfJ»α=[FJs χ]$+\eta$. Since

$$d_{\pi^*\Gamma}\chi = d_{A'}\chi + \gamma_i[(A_i - \pi^*\Gamma), \chi]$$

on $\tau_i^{-1}([n-2, n-1])$, we have an elliptic estimate

$$\begin{aligned} &\|\chi\|_{C^0(\tau_n^{-1}([n-2,n-1]))} \\ &\leqq c\|d_{\pi^*\Gamma}\chi|_{\tau_n^{-1}([n-2,n-1])}\|_{L^q(X_n)} \\ &\leqq c\|d_{A'}\chi|_{\tau_n^{-1}([n-2,n-1])}\|_{L^q(X_n)} + ce^{-\lambda n}\|\chi\|_{C^0(\tau_n^{-1}([n-2,n-1]))}. \end{aligned}$$

For $n > n_0$ with $ce^{-\lambda n_0} < 1/2$, we can rearrange this to get

$$\|\chi\|_{C^0(\tau_n^{-1}([n-2,n-1]))} \leqq 2c\|d_{A'}\chi|_{\tau_n^{-1}([n-2,n-1])}\|_{L^q(X_n)}.$$

Noting that the support of $[F^+_{A'}, \chi]$ is contained in $\tau_n^{-1}([n-2, n-1])$, we see that

$$\begin{aligned} \|\eta\|_{L^p_\delta(X_n)} &\leqq \|\alpha\|_{B_2} + \|[F^+_{A'}, \chi]\|_{L^p_\delta(X_n)} \\ &\leqq \|\alpha\|_{B_2} + \|F^+_{A'}\|_{L^p_\delta(X_n)}\|\chi\|_{C^0(\tau_n^{-1}([n-2,n-1]))} \\ &\leqq \|\alpha\|_{B_2} + ce^{-(\lambda-\delta/p)n}\|d_{A'}\chi|_{\tau_n^{-1}([n-2,n-1])}\|_{L^q_{q\delta/p}(X_T)} \\ &\leqq \|\alpha\|_{B_2} + ce^{-(\lambda-\delta/p)n}\|\alpha - P\eta\|_{L^q_{q\delta/p}(X_n)} \\ &\leqq \|\alpha\|_{B_2} + ce^{-(\lambda-\delta/p)n}(\|\alpha\|_{B_2} + \|\eta\|_{L^p_\delta(X_n)}). \end{aligned}$$

Thus, for $n > n_0$ with $ce^{-(\lambda-\delta/p)n_0} < 1/2$, we obtain

$$\|\eta\|_{L^p_\delta(X_n)} \leqq 3\|\alpha\|_{B_2}.$$

This gives us then

$$\|(\chi, \eta)\|_{B_1} \leqq 4\|\alpha\|_{B_2}. \quad \square$$

On the other hand, the operator P is a pseudo-differential operator and the symbol is homotopic to that of $(d_{A'}^{+})^{*}(1+\Delta_{A'})^{-1}$. It follows that $d_{A'}+P$ is Fredholm and its index equals that of $d_{A'}+(d_{A'}^{+})^{*}$. So

$$\text{index}(T)=2l_{X_n}+3(1+b_{+}(X_n))=0.$$

Thus T is an isomorphism from B_1 to B_2 with operator norm $\|T^{-1}\|\leqq 4$. This implies that M is invertible near (0, 0) and 5 is open. It follows that $S=[0, 1]$ and the proof is completed. D

COROLLARY 5.9. *For* $0<\nu<\nu_0$ *and* $n>n_0(\nu)$, *the intersection* $U(\nu)\cap\mathcal{M}_{X_n}(l_{X_n}, \eta, g_n)$ *is equal to the image of* I_n.

Suppose that I_n is not surjective for any large n. Then there are a subsequence *{n}* (now we relabeled) with $n\to\infty$ and a sequence of g_n-ASD connections $\{A_n\}$ not coming from the map I_n. Uhlenbeck's compactness principle ([22], [7, 4.4]) and the preservation of $w_2(P)$ under weak limit imply that, after taking a subsequence, the following data exists:

(1) A bundle $P'\to Y_1\amalg Y_2$ with $w_2(P'|_{Y_1})=\sigma_1^*(\eta)$, $w_2(P'|_{Y_2})=\sigma_2^*(\eta)$,

(2) ASD connections A_1 on $P'|_{Y_1}$ and A_2 on $P'|_{Y_2}$ with $(-1/4\pi^2)\int_{Y_1}\text{Tr}(F_{A_1}\wedge F_{A_1})=l_1$ and $(-1/4\pi^2)\int_{Y_2}\text{Tr}(F_{A_2}\wedge F_{A_2})=l_2$,

(3) A collection of points $\{x_1,\cdots, x_a\}\in Y_1$, $\{x_{a+1},\cdots, x_{a+b}\}\in Y_2$,

(4) C^∞-gauge transformations $\{k_n\}$ over $X_n\backslash\{x_1,\cdots, x_{a+b}\}$,

(5) fen/L^ converges to A_1, A_2 in C^∞ on compact subsets of $Y_1\backslash\{x_1, \cdots, x_a\}\amalg Y_2\backslash\{x_{a+1},\cdots, x_{a+b}\}$,

(6) $4a+4b-l_1-l_2\leqq -l_{X_n}$.

Since $\sigma_1^*(\eta)\neq 0$ and $\sigma_2^*(\eta)\neq 0$, there are no flat connections on $P'|_{Y_1}$, P'_{Y_2}, which implies that $l_1<0$ and $l_2<0$. Lemma 5.1 and Lemma 3.5 supply $h_i\in C^\infty$ (Aut P) such that $[h_iA_i]$ lies in $\mathcal{M}_Y(l_i, \sigma_i^*(\eta), g_i)$. By Proposition 3.7, we obtain $l_1=l_{Y_1}$ $l_2=l_{Y_2}$ and $a=b=0$.

LEMMA 5.10. *There are constants* $\varepsilon>0$, $t_0=t_{0,m}>0$ *and* $\rho=\rho_m>0$, *independent of* T, *with the following significance Let* A *be an ASD connection on* $Q\times(-T, T)$ *with* $\int_{Z\times(-T,T)}|F_A|^2<\varepsilon$. *Then there exists* $h_T\in C^\infty(\text{Aut}(Q\times(-T+1, T-1))$ *such that for* $-T+t_0\leqq t\leqq T-t_0$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}|\nabla_{\pi^*\Gamma}^{(l)}(h_TA-\pi^*\Gamma)|^2\right\}\leqq\rho e^{-2\lambda(T-|t|)},$$

$$i_{\partial/\partial t}(h_TA-\pi^*\Gamma)=0.$$

Proof. The proof is very similar to that of Lemma 5.1. *We* apply Proposition 4.2 over $Z\times(-T, T)$ to obtain a gauge transformation $h\in C^\infty(\text{Aut}(Q\times[-T+1, T-1]))$ such that for $-T+8\leqq t\leqq T-8$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(hA-\pi^*\Gamma)|^2\right\}<ce^{-2\lambda(T-|t|)}.$$

Solve the ordinary differential equation for $\tilde{h}\in C^\infty(\mathrm{Aut}(Q\times[-T+1,\ T-1]))$:

$$i_{\partial/\partial t}(\tilde{h}(hA)-\pi^*\Gamma)=0$$

with initial condition $\tilde{h}|_{t=0}=1$. Then by (5.1) we have

$$\frac{\partial}{\partial t}|d_\Gamma\tilde{h}|^2\leqq ce^{-\lambda(T-|t|)}(|d_\Gamma\tilde{h}|^2+|d_\Gamma\tilde{h}|).$$

If $t\geqq 0$, then we integrate over $[0,\ t+1]$ to get an inequality

$$\|d_\Gamma\tilde{h}\|^2_{C^0(Z\times[0,t+1])}\leqq ce^{-\lambda(T-t)}(\|d_\Gamma\tilde{h}\|^2_{C^0(Z\times[0,t+1])}+\|d_\Gamma\tilde{h}\|_{C^0(Z\times[0,t+1])}).$$

For $t_0>0$ with $ce^{-\lambda t_0}<1/2$ and $0\leqq t\leqq T-t_0-1$, we obtain

$$\|d_\Gamma\tilde{h}\|_{C^0(Z\times[t-1,t+1])}\leqq\|d_\Gamma\tilde{h}\|_{C^0(Z\times[0,t+1])}\leqq 2ce^{-\lambda(T-|t|)}.$$

In the case $t\leqq 0$, we can also prove the above by the parallel discussion. So by bootstrapping (5.2), we see that $h_T=\tilde{h}h$ satisfies the desired properties for some $\rho>0$. D

We choose $n_0\in\boldsymbol{N}$ so that

$$\int_{\tau_i^{-1}([n_0,\infty))}|F_{A_i}|^2<\frac{1}{2}\varepsilon.$$

Since

$$\lim_{n\to\infty}\int_{\tau_n^{-1}([n_0,n])}|F_{A_n}|^2=\int_{\tau_n^{-1}([n_0,\infty))}|F_{A_1}|^2+\int_{\tau_n^{-1}([n_0,\infty))}|F_{A_2}|^2,$$

we have

$$\int_{\tau_n^{-1}([n_q,n])}|F_{A_n}|^2<\varepsilon\ \ (n>n_1)$$

for some $n_1>n_0$. By Lemma 5.10, we can find $h_n\in C^\infty(\mathrm{Aut}(Q\times[n_0+1,n+1]\cup_{\tilde{\varphi}_n} Q\times[n_0+1,\ n+1])$ such that for $n_0+t_0\leqq t\leqq n$,

$$\sup_{Z\times[t-1,t+1]}\left\{\sum_{l=0}^{m}|\nabla^{(l)}_{\pi^*\Gamma}(h_nA_n-\pi^*\Gamma)|^2\right\}\leq\rho e^{-2\lambda(t-n_0)}.$$

$$i_{\partial/\partial t}(h_nA_n-\pi^*\Gamma)=0.$$

So the connection $(h_nA_n)'=\gamma_i(h_nA_n-\pi^*\Gamma)+(1-\gamma_i)\pi^*\Gamma$ over $\tau_i^{-1}([n_0+1,\ \infty))$ also satisfies the same condition as above for $n_0+t_0\leqq t\leqq n-1$. Now we can apply Ascoli-Arzela's theorem with diagonal argument to deduce that, after taking a subsequence, $(h_nA_n)'$ converges to an ASD connection A_i'' in C^{m-1} over compact sets in $\tau_i^{-1}([n_0+t_0,\ \infty))$. So A_i'' satisfies

$$\sup_{\tau_\iota^{-1}([t-1,\, t+1])}\left\{\sum_{l=0}^{m-1} |\nabla_{\pi^*\Gamma}^{(l)}(A_\iota''-\pi^*\Gamma)|^2\right\} \leqq c_i e^{-2\lambda(t-n_0)}, \tag{5.5}$$

$$i_{\partial/\partial t}(A_\iota''-\pi^*\Gamma)=0$$

for $t \geqq n_0+t_0$ By Lebesgue convergence theorem we have

$$\|((h_nA_n)'-A_\iota'')|_{\tau_n^{-1}([n_0+t_0,\infty))}\|_{L^q_{q\delta/p}(Y_\iota)} \longrightarrow 0\,(n\to\infty). \tag{5.6}$$

We bootstrap the equation

$$d_{A_\iota}(h_nk_n^{-1})=h_nk_n^{-1}(k_nA_n-A_\iota)-(h_nA_n-A_\iota)h_nk_n^{-1},$$

to deduce that there is a subsequence $\{h_nk_n^{-1}\}$ (now we relabeled) such that $h_nk_n^{-1}$ converges to some u_ι in C^m on $\tau_\iota^{-1}((n_0+t_0,\, n_0+t_0+1))$ By Lemma 3.5, if we replace $\{h_n\}$ by $\{r_ih_n\}$ for some $r_i\in\mathfrak{R}$, we can suppose that $\tilde{u}_\iota$ can be extended to u_ι^* over $\tau_\iota^{-1}([0,\, n_0+t_0+1))$ and $\mathfrak{b}(1,\, \tilde{u}_\iota)=0$. Then for large $n\in\boldsymbol{N}$, $\mathfrak{b}(1,\, r_ih_nk_n^{-1})=\mathfrak{b}(1,\, \tilde{u}_\iota)=0$. So we have $\mathfrak{b}(1,\, r_2r_1^{-1})=\mathfrak{b}(1,\, r_2h_nk_n^{-1}(r_1h_nk_n^{-1})^{-1})=\mathfrak{b}(1,\, r_2h_nk_n^{-1})+\mathfrak{b}(1,\, r_1h_nk_n^{-1})=0$. Since $(r_2r_1^{-1})\Gamma=\Gamma$, we see that $r_1=r_2$. Now we can apply the argument of [7, Lemma (4.4.5)] (see also Appendix 1) to patch gauge transformations k_n and r_ih_n over $\tau_\iota^{-1}((n_0+t_0,\, n_0+t_0+1))$ Then taking a subsequence, we can find C^{m-1} gauge transformations $\{u_n\}$ on X_n such that $u_n=h_n$ on $\tau_n^{-1}([n_0+t_0+1,\, n])$ and u_nA_n converges to an ASD connection $\hat{A}_\iota$ on P_ι in C^{m-2} over compact subsets of Y_ι, which is in the exponential gauge. If we choose $m\geqq 5$, then $\tilde{A}_\iota$ is gauge equivalent to a C^∞-connection and the uniqueness in Lemma 5.1 implies that $\tilde{A}_\iota$ is smooth over $Z\times[n_0+t_0,\, \infty)$ and satisfies the condition (ii) for $t\geqq n_0+t_0$. So we obtain

$$\|u_nA_n-J_n(\tilde{A}_1A_2)\|_{L^q_{q\delta l_{r_n}}(X_n)} \longrightarrow 0\ (n\to\infty).$$

Now by Corollary 5.9, $[A_n]-[I_n(A_1,\tilde{A}_2)]\in\mathcal{M}_{X_n}(l_{X_n}\eta,\, g_n)$ for large n, which is a contradiction.

Next for $n\in\boldsymbol{N}$, we consider the oriented 4-manifold K_{in}^* defined by

$$K_{in}^*=K_i\cup Z\times[0,\, n+1]\underset{\phi_{in}}{\cup} Z\times[0,\, n+1]\cup W,$$

and the $SO(3)$ bundle $P_{\iota n}$ over K_{in}^* defined by

$$P_{\iota n}=P_{0i}\underset{\iota_i}{\cup}Q\times[0,\, n+1]\underset{\tilde{\phi}_{in}}{\cup} Q\times[0,\, n+1]\cup\tilde{Q}$$

for each $\iota=1,\, 2$. Here ϕ_{in} and $\tilde{\phi}_{in}$ are defined by using ϕ_i and a lift $\tilde{\phi}_i\colon Q\to Q$ with $\tilde{\phi}_i\Gamma=\Gamma$ as before. Let $\mathcal{C}^r_{K_{in}^*}(r\geqq 3)$ be the space of all conformal classes of C^r-metrics on K_{in}^* which are a fixed metric on W and $h+dt^2$ on $Z\times[0,\, n+1]\cup_{\phi_{in}}Z\times[0,\, n+1]$. If we fix a metric $[g_{i0}]$ in $\mathcal{C}^r_{K_{in}^*}$, then $\mathcal{C}^r_{K_{in}^*}$ is identified with

$$\{m: \Lambda^+\to\Lambda^-;\ C^r\text{-bundle map},\ \sup|m|<1,\ m|_{K_{in}^*\setminus K_i^0}=0\},$$

where $\Lambda^{\pm}$ is $\pm$ self-dual space with respect to g_{i0}. Then we have the following theorem, whose proof is also the same as in ([7, 4.3], [14, 5 (iii)]).

PROPOSITION 5.11. *There is a Baire set $C'_{K^*_{in}} \subset C^r_{K^*_{in}}$ such that for all $g \in C'_{K^*_{in}}$, the ASD moduli $\mathcal{M}_{K^*_{in}}(l_{K^*_{in}}, \eta^*_i, g)$ is a finite set consisting of irreducible regular connections.*

For each $n \in \boldsymbol{N}$, we fix a metric g_{in} in $C'_{K^*_{in}}$ such that g_{in} is independent of n on K^0_i and the metric

$$g_i = \begin{cases} g_{in} & on \quad K_i, \\ h + dt^2 & on \quad Z \times [0, \infty), \end{cases}$$

lies in C'_{Y_i} *for* $i=1, 2$.

We extend Q and f naturally over $W \cup Z \times [0, \infty)$, which we also denote by $\tilde{\Gamma}$ and $\tilde{Q}$ respectively. We replace Y_2 by $W \cup Z \times [0, \infty)$ and set a function $\tau_2 : Y_2 \to \boldsymbol{R}$ as before. Then we can define the gluing map

$$I_{1n} : \mathcal{M}_{Y_1}(l_{Y_1}, \sigma^*_1(\eta), g_1) \times \{\tilde{\Gamma}\} \longrightarrow \mathcal{M}_{K^*_{1n}}(l_{K^*_{1n}}, \eta^*_1, g_{1n})$$

for large $n \in \boldsymbol{N}$ just as before. We can prove that I_{1n} is bijective in the same way as that of /. But in the proof, we need to correct it at some points. First, after Corollary 5.9, we treat a sequence of connections $\{A_n\}$. Then we obtain $l_1 = l_{Y_1}$, $l_2 = 0$ and $a = b = 0$. So $u^*_2 A_2 = \tilde{\Gamma}$ for some C^∞-gauge u^*_2 on $W \cup Z \times [0, \infty)$ and $u^*_2 k_n A_n$ converges to $\tilde{\Gamma}$ in C^∞ on compact subsets of $W \cup Z \times [0, \infty)$. Second, after (5.6), on $\tau_1^{-1}((n_0+t_0, n_0+t_0+1))$, if we replace $\{h_n\}$ by $\{r_1 h_n\}$ for some $r_1 \in \mathfrak{R}$, we can suppose that $r_1 h_n k_n^{-1}$ converges to u_1 over $\tau_1^{-1}((n_0+t_0, n_0+t_0+1))$ We bootstrap the equation

$$d_{\pi^*\Gamma}(r_1 h_n (u^*_2 k_n)^{-1}) = r_1 h_n (u^*_2 k_n)^{-1} (u^*_2 k_n A_n - \pi^*\Gamma) - r_1 (h_n A_n - \pi^*\Gamma) h_n (u^*_2 k_n)^{-1}$$

to obtain that, if we choose n_0 so large, then there is a subsequence $\{r_1 h_n (u^*_2 k_n)^{-1}\}$ (now we relabeled) such that $r_1 h_n (u^*_2 k_n)^{-1}$ converges to some u_2 in C^{m-1} on $\tau_2^{-1}((n_0+t_0, n_0+t_0+1))$ and, since the right hand side is so small, u_2 can be extended to $\tilde{u}^*_2$ over $\tau_2^{-1}([0, n_0+t+1))$. Now we can apply the argument of [7, Lemma (4.4.5)] to patch gauge transformations k_n *(resp.* $u^*_2 k_n)$ and $r_1 h_n$ over $\tau_1^{-1}((n_0+t_0, n_0+t_0+1))$ *(resp.* $\tau_2^{-1}((n_0+t_0, n_0+t_0+1))$). Then as was shown there, we see that I_{1n} is surjective for large $n \in \boldsymbol{N}$. (This bijection has been stated in [10, Theorem V. 3.4].) Of course, the same holds for Y_2 and I_{2n} defined as before. Composing three bijections, we have a bijection

$$K_n : \mathcal{M}_{K^*_{1n}}(l_{K^*_{1n}}, \eta^*_1, g_{1n}) \times \mathcal{M}_{K^*_{2n}}(l_{K^*_{2n}}, \eta^*_2, g_{2n}) \longrightarrow \mathcal{M}_{X_n}(l_{X_n}, \eta, g_n)$$

for large $n \in \boldsymbol{N}$. Hence Theorem 2.1 follows from the lemma below.

LEMMA 5.12. *If $[B_{i0}], [B_{i1}] \in \mathcal{M}_{K^*_{in}}(l_{K^*_{in}}, \eta^*_i, g_{in})$ satisfies $o([B_{i1}]) = \varepsilon_i o([B_{i0}])$ $(i=1, 2)$, then $o(K_n([B_{11}], [B_{21}])) = \varepsilon_1 \varepsilon_2 o(K_n([B_{10}], [B_{20}]))$.*

Proof. We write $[B_{\iota 0}]=[I_{\iota n}(A_{\iota 0},\tilde{\Gamma})]$, $[B_{\iota 1}]=[I_{\iota n}(A_{\iota 1},\text{f})]$ for some $[A_{\iota 0}]$, $[A_{\iota 1}]\in\mathcal{M}_{Y_i}(l_{Y_i},\sigma_i^*(\eta),g_\iota)$. We choose any path $A_{\iota t}$ $(0\leqq t\leqq 1)$ in $\mathcal{A}_{P_\iota}$ from $A_{\iota 0}$ to $A_{\iota 1}$. Define a connection $A'_{\iota t}$ on $P_{\iota n}$ by

$$A'_{\iota t}=\begin{cases}\mu_i(A_{\iota t}-\pi^*\Gamma)+(1-\mu_i)\pi^*\Gamma & \text{over} \quad \tau_\iota^{-1}([0,\ n+1]),\\ \tilde{\Gamma} & \text{over} \quad W\cup Z\times[0,\ n+1],\end{cases}$$

and a connection A'_t on P_n by $A'_t=J_n(A_{1t},A_{2t})$. Since $[A'_{\iota 0}]$ (resp. $[A'_{\iota 1}]$) lies in $U([A_{\iota 0}])$(resp. $U([A_{\iota 0}])$) for large $n\in\boldsymbol{N}$, we have a nowhere zero section s_ι over $[A'_{\iota t}]$ $(0\leqq t\leqq 1)$ such that

$$s_i([A'_{\iota 0}])=\varepsilon_i o([B_{\iota 0}]),\qquad s_i([A'_{\iota 1}])=o([B_{\iota 1}]).$$

Since $\operatorname{Ker}\delta^*_{A'_{\iota t}}$ is supported in $\tau_\iota^{-1}([0,n-1])$, we can choose a linear map $S_\iota:\boldsymbol{R}^{N_\iota}\to L^2_2((\Omega^0_{K^*_{\iota n}}\oplus\Omega^+_{K^*_{\iota n}})(\operatorname{Ad}P_{\iota n}))$ so that for all $0\leqq t\leqq 1$, $\delta_{A'_{\iota t}}\oplus S_\iota$ is surjective and the image of S_ι is supported in $\tau_\iota^{-1}([0,n-1])$. So $\operatorname{Ker}(\delta_{A'_{\iota t}}\oplus S_\iota)$ is supported in $\tau_\iota^{-1}([0,\ n-1])\times\boldsymbol{R}^{N_\iota}$ $(i=1,\ 2)$ and $\operatorname{Ker}(\delta_{A'_t}\oplus S_1\oplus S_2)$ is supported in $\tau_n^{-1}([0,\ n-1])\times\boldsymbol{R}^{N_1+N_2}$. Then we have a natural isomorphism

$$\theta_t:\ \operatorname{Ker}(\delta_{A'_{1t}}\oplus S_1)\oplus\operatorname{Ker}(\delta_{A'_{2t}}\oplus S_2)\longrightarrow\operatorname{Ker}(\delta_{A'_t}\oplus S_1\oplus S_2)$$

$$(t_1,\ t_2)\longmapsto t_1+t_2,$$

by which we obtain a nowhere zero section $(\Lambda^{\max}\theta_t)(s_1\otimes s_2)$ on Λ_{P_n} over $[A'_t]$ $(0\leqq t\leqq 1)$. Since $\operatorname{Ker}\delta^*_{A'_t}$ is supported in $\tau_n^{-1}([0,n-1])$, $\delta_A\oplus S_1\oplus S_2$ is surjective. Now we get

$$\begin{aligned}&(\Lambda^{\max}\theta_0)(s_1\otimes s_2)([A'_0])=(\Lambda^{\max}\theta_0)(s_1([A'_{10}])\otimes s_2([A'_{20}]))\\&=\varepsilon_1\varepsilon_2(\Lambda^{\max}\theta_0)(o([B_{10}])\otimes o([B_{20}]))=\varepsilon_1\varepsilon_2 o([I_n(A_{10},\ A_{20})])\\&=\varepsilon_1\varepsilon_2 o(K_n([B_{10}],\ [B_{20}])),\\&(\Lambda^{\max}\theta_1)(s_1\otimes s_2)([A'_1])=o(K_n([B_{11}],\ [B_{21}])).\end{aligned}$$

This implies the desired equality. □

Proof of Theorem 2.2. Suppose that $\gamma_X(\eta)\neq 0$. Then for each n there is an ASD connection $[A_n]$ in $\mathcal{M}_{X_n}(l_{X_n},\eta,g_n)$. (Here, in the construction of P_n, ι_1 and ι_2 are chosen so that $w_2(P_n)=\eta, p_1(P_n)=l_{K_n}$ are only satisfied.) Then after taking a subsequence, we have the data from (1) to (6) in the above. As was shown there, we see that $l_1=l_{Y_1}$ and $l_2=l_{Y_2}$. The resulting connection A_ι on P_ι can be thought of an element in $\mathcal{M}_{Y_i}(l_{Y_i},\sigma_i(\eta),g_\iota)$. The following lemma contradicts to the assumption. Π

LEMMA 5.13. *For any connection* A *on* P *which is isomorphic to* $\pi^*\Gamma$ *on* $Z\times[n,\ \infty)$ *for some* $n\in\boldsymbol{N}$,

$$\frac{-1}{4\pi^2}\int_Y \mathrm{Tr}\,(F_A\wedge F_A)\equiv(\eta^*)^2 \pmod 2.$$

Proof. Define $\tilde{P}=P|_{\tau^{-1}([0,n])}\cup\tilde{Q}$. $\sigma^*(w_2(\tilde{P}))=\sigma^*(\eta^*)$ implies that $w_2(\tilde{P})=\eta^*$ or $w_2(\tilde{P})=\eta^*+\mathrm{P.D.}[T^2\times 0]$. In any case, we obtain

$$\frac{-1}{4\pi^2}\int_Y \mathrm{Tr}\,(F_A\wedge F_A)=p_1(\tilde{P})\equiv(w_2(\tilde{P}))^2\equiv(\eta^*)(\mathrm{mod}\ 2). \qquad \square$$

Remark. The vanishing of $\gamma_X(\eta)$ can be observed for more general elements $\eta\in C_X$. In a forthcoming paper, we will treat it.

6. Explicit calculations on elliptic surfaces.

In this section we calculate values of the simple invariant for the regular elliptic surfaces without multiple fibers. Let $\pi: S_k\to \boldsymbol{CP}^1$ be a regular minimal elliptic surface with $p_g=k-l$ and without multiple fibers. Then S_k satisfies $\pi_1(S_k)=1$, $b_+(S_k)=2k-1$ and $l_{S_k}--3k$. It admits a differentiable section Σ_k . $\boldsymbol{CP}^1\to S_k$, which has the self-intersection number $(\Sigma_k)^2=-k$. *We* take a general fiber / in S_k. Then we can interpret this surface S_k as a fiber sum up to fiber preserving diffeomorphism as follows [17]: Given S_i and $S_{k-i}(1\leqq i\leqq k-1)$, identify the tubular neighborhood of a general fiber in each with $T^2\times D^2$ so that the fibrations correspond to projection onto D^2. Remove the interior of tubular neighborhoods from S_i and S_{k-i}, and glue the two remaining manifolds together by an orientation reversing and fiber preserving diffeomorphism on the boundaries. Then we get an oriented manifold $S_i\natural S_{k-i}$, a fibration π: $S_i\natural S_{k-i}\to\boldsymbol{CP}^1$, and a section $\Sigma_i\natural\Sigma_{k-i}$: $\boldsymbol{CP}^1\to S_i\natural S_{k-i}$. *We* note that S_2 contains the Kummer surface, which is one of the K3 surface.

We use a well known result by Donaldson ([6], [7], [13]).

PROPOSITION 6.1. *(Donaldson) If we fix the orientation of $H^+(S_2)$ determined by the complex structure of S_2, then $\gamma_{S_2}(\eta)=1$ for any $\eta\in C_{S_2}$.*

LEMMA 6.2. $|\gamma_{S_3}(\mathrm{P.D.}([\Sigma_3]+[f]))|=1$.

Proof. Because $S_6=S_3\natural S_3=S_2\natural S_2\natural S_2$ and $\Sigma_6=\Sigma_3\natural\Sigma_3=\Sigma_2\natural\Sigma_2\natural\Sigma_2$: $\boldsymbol{CP}^1\to S_6$, we apply Theorem 2.1 three times to deduce that

$$|\gamma_{S_6}(\mathrm{P.D.}[\Sigma_6])|=|\gamma_{S_6}(\mathrm{P.D.}([\Sigma_6]+2[f]))|=|\gamma_{S_3}(\mathrm{P.D.}([\Sigma_3]+[f]))|^2$$
$$=|\gamma_{S_2}(\mathrm{P.D.}[\Sigma_2])|^3=1.$$

So $|\gamma_{S_3}(\mathrm{PD}([\Sigma_3]+[f]))|=1$. G

COROLLARY 6.3. *For integer $k\geqq 2$,*

$$|\gamma_{S_k}(\mathrm{P.D.}[\Sigma_k])|=1 \qquad \textit{if } k \textit{ is even},$$

$$|\gamma_{S_k}(\mathrm{P.D.}([\Sigma_k]+[f]))|=1 \qquad \textit{if } k \textit{ is odd.}$$

Remark. In a recent paper [11], we found that for integer $k\geqq 2$,

$$|\gamma_{S_k}(\mathrm{P.D.}[\Sigma_k])|\neq 0 \qquad \text{if } k \text{ is even,}$$

$$|\gamma_{S_k}(\mathrm{P.D.}([\Sigma_k]+[f]))|\neq 0 \qquad \text{if } k \text{ is odd.}$$

using the moduli of stable vector bundles.

For k odd, we can determine the image of $|\gamma_{S_k}|$.

LEMMA 6.4. $|\gamma_{S_3}(\eta)|=1$ *for any* $\eta\in C_{S_3}$.

Proof. Since the characteristic element is $w_2(S_3)\equiv\mathrm{P.D.}[f]$ (mod 2), $\langle\eta, [f]\rangle\equiv\eta\cdot\eta\equiv 1\,(\mathrm{mod}\,2)$. We construct $\eta\natural\eta\in C_{S_6}$, by identifying a tubular neighborhood of a general fiber in two copies of S_3. If we identify S_3 with $S_2\natural S_1$, then $\eta\in C_{S_3}$ factors as $\eta=\eta_2\natural\eta_1$ for some $\eta_2\in H^2(S_2;\boldsymbol{Z}_2)$ and $\eta_1\in H^2(S_1;\boldsymbol{Z}_2)$ with $\eta^2\equiv(\eta_2)^2+(\eta_1)^2$ (mod 4). If $(\eta_2)^2\equiv 0\,(\mathrm{mod}\,4)$, then $(\eta_2+\mathrm{P.D.}[f])^2\equiv(\eta_2)^2+2\equiv 2\,(\mathrm{mod}\,4)$. So we may assume that $\eta_2\in C_{S_2}$, $\eta_1\in C_{S_1}$. In the same way as above we have an element $\eta_1\natural\eta_1\in C_{S_2}$. Now by Theorem 2.1 and Proposition 6.1, we get

$$|\gamma_{S_6}(\eta\natural\eta)|=|\gamma_{S_3}(\eta)|^2=|\gamma_{S_2}(\eta_2)||\gamma_{S_2}(\eta_1\natural\eta_1)||\gamma_{S_2}(\eta_2)|=1. \quad \square$$

We can apply the above argument on $S_k=S_2\natural\cdots\natural S_2\natural S_3$ to deduce that

COROLLARY 6.5. *If k is odd and $k\geqq 3$, then $|\gamma_{S_k}(\eta)|=1$ for any $\eta\in C_{S_k}$.*

COROLLARY 6.6. *If k is even, then $|\gamma_{S_k}(\eta)|=1$ for $\eta\in C_{S_k}$ with $\langle\eta, [f]\rangle\equiv$* $1\,(\mathrm{mod}\,2)$.

Remark. In fact, Ue has obtained that the value $|\gamma_{S_k}(\eta)|$ is independent of $\eta\in C_k$ with $\langle\eta, [f]\rangle=1$ (mod 2), by analyzing the action of diffeomorphism group of S_k on C_{S_k} [21].

Appendix 1

Lemma (4.4.5) in [7] omits a necessary hypothesis on extension of gauge group. We write a precise statement, but omit its proof, since it is same as that of Lemma (4.4.5).

Lemma (4.4.5)′. *We fix $m\in\boldsymbol{N}$. Suppose that A_n is a sequence of unitary C^m-connections on a $SO(3)$ bundle P over a base manifold Ω (possibly non-compact), and let $\tilde{\Omega}\Subset\Omega$ be an interior domain. Suppose that there are gauge transformations $u_n\in C^{m+1}(\mathrm{Aut}\,P)$ and $\tilde{u}_n\in C^{m+1}(\mathrm{Aut}\,P|_{\tilde{\Omega}})$ such that u_nA_n converge in C^m over Ω and $\tilde{u}_nA_n$ converges in C^m over $\tilde{\Omega}$. Then we may assume that; taking a subsequence $\{n'\}$, the $\tilde{u}_{n'}u_{n'}^{-1}$ converge in C^m over $\tilde{\Omega}$ to a limit $\tilde{u}$.*

// u extends over Ω, then for any compact set $K\subset\tilde{\Omega}$ we can find gauge transformations $w_{n'}\in C^m(\mathrm{Aut}\,P)$ such that $w_{n'}-\tilde{u}_{n'}$ in a neighborhood of K and the connections $w_{n'}A_{n'}$ converge in C^{m-1} over Ω.

In Section 5, we apply the above lemma with $\Omega=\tau_i^{-1}([0, n_0+t_0+1))$, $\tilde{\Omega}=\tau_i^{-1}((n_0+t_0, n_0+t_0+(3/4)))$ and $K=\tau_i^{-1}([n_0+t_0+(1/4), n_0+t_0+(1/2)])$.

Appendix 2.

We prove the compactness in Proposition 3.7. For $l_Y<l<0$, it is vacuous. Let $\{[A_i]\}$ be a sequence in $\mathcal{M}_Y(l_Y, \eta, g)$. Then after taking a subsequence, the following data exists:

(1) A bundle $P'\to Y$ with $w_2(P')=\eta$,

(2) An ASD connection A on P' with $(-1/4\pi^2)\int_Y \mathrm{Tr}(F_A\wedge F_A)=l$,

(3) A collection of points $\{x_1, \cdots, x_a\}\in Y$,

(4) C^∞-gauge transformations $\{k_n\}$ over $Y\backslash\{x_1, \cdots, x_a\}$,

(5) k_nA_n converges to A in C^∞ on compact subsets of $Y\backslash\{x_1,\cdots, x_a\}$,

(6) $4a-l\leqq -l_Y$.

Since $\eta\neq 0$, there are no flat connections on P' ', and $l<0$. By Lemma 5.1 and Lemma 3.5, there is $h\in C^\infty(\mathrm{Aut}\,P)$ such that $[hA]$ lies in $\mathcal{M}_Y(l, \eta, g)$. So we have $l=l_Y$, $a=0$. We choose $t_0\in \boldsymbol{N}$ so that

$$\int_{\tau^{-1}([t_0,\infty))}|F_A|^2<\varepsilon,$$

where we choose $\varepsilon>0$ by Lemma 5.1. Since

$$\lim_{n\to\infty}\int_{\tau^{-1}([t_0,\infty))}|F_{A_n}|^2=\int_{\tau^{-1}([t_0,\infty))}|F_A|^2,$$

we have

$$\int_{\tau^{-1}([t_0,\infty))}|F_{A_n}|^2<\varepsilon \quad (n>n_0)$$

for some $n_0\in \boldsymbol{N}$. Then there exists $h_n\in C^\infty(\mathrm{Aut}(Q\times[t_0,\infty)))$ such that

$$\sup_{\tau^{-1}([t-1,\, t+1])}\left\{\sum_{l=0}^{m}|\nabla_{\pi^*\Gamma}^{(l)}(h_nA_n-\pi^*\Gamma)|^2\right\}\leqq \rho e^{-2\lambda(t-t_0)}$$

for $t\geqq t_0$. Now we apply Ascoli-Arzela's theorem with diagonal argument to deduce that h_nA_n converges to an ASD connection A' in C^{m-1} over compact sets in $\tau^{-1}([t_0, {}^{\mathrm{TO}}))$ So

$$\sup_{\tau^{-1}([t-1,\, t+1])}\left\{\sum_{l=0}^{m-1}|\nabla_{\pi^*\Gamma}^{(l)}(A'-\pi^*\Gamma)|^2\right\}\leqq \rho e^{-2\lambda(t-t_0)}$$

for $t\geqq t_0$. By bootstrapping the equation

$$d_A(h_n k_n^{-1}) = h_n k_n^{-1}(k_n A_n - A) - (h_n A_n - A) h_n k_n^{-1},$$

we see that there is a subsequence $\{h_n k_n^{-1}\}$ (now we relabeled) such that $h_n k_n^{-1}$ converges to some u in C^{m-1} on $\tau^{-1}((t_0, t_0+1))$ By Lemma 3.5, if we replace $\{h_n\}$ by $\{rh_n\}$ for some $r \in \mathcal{R}$, we can suppose that $\tilde{u}$ can be extended to u^* over $\tau^{-1}([0, t_0+1))$ So we can apply the argument of [7, Lemma (4.4.5)] (see also Appendix 1) to patch gauge transformations k_n and rh_n over $\tau^{-1}((t_0, t_0+1))$ Then taking a subsequence, we can find C^{m-1}-gauge transformations $\{u_n\}$ on Y such that $u_n = h_n$ on $\tau^{-1}([t_0+1, \infty))$ and $u_n A_n$ converges to an ASD connection A'' on P in C^{m-2} on compact subsets of Y. If we choose $m \geqq 5$, then

$$\|u_n A_n - A''\|_{L^2_{3,\delta}(Y)} \longrightarrow 0 \ (n \to \infty),$$

and $[u_n A_n], [A''] \in \mathcal{M}_Y(l_Y, \eta, g)$. Now by Lemma 3.3, $\{u_n\}$ is in $\mathcal{G}_P$. □

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DEPARTMENT OF MATHEMATICS,
TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OHSAWA 1-1,
HACHIOJI-SHI, TOKYO 192-03,
JAPAN