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REMARKS ON MULTIPLIERS FOR BMO ON GENERAL DOMAINS

Dedicated to Profssor Nobuyuki Suita on his sixtieth birthday

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Introduction.

A measurable function φ is called a (pointwise) *BMO* multiplier if $\phi f \in BMO$ for every $f \in BMO$. The characterizations of multipliers for *BMO* spaces on n dimensional torus $\boldsymbol{T}^n$ and on n dimensional Euclidean space $\boldsymbol{R}^n$ are well known (Stegenga [9], Janson [4], Nakai-Yabuta [6]).

Although Nakai-Yabuta's characterization of $BMO(\boldsymbol{R}^n)$ multiplier is more complicate than that of $BMO(\boldsymbol{T}^n)$, these characterizations are essentially the same. Indeed we shall give a geometrically simple characterization of $BMO(D)$ multiplier for general domain D in $\boldsymbol{R}^n$ by using a metric on a space of some family of cubes in D, which is also valid for $BMO(\boldsymbol{T}^n)$.

§ 1. Preliminary and main result.

Throughout this paper we treat only 2 dimensional case for the simplicity, since the same argument holds in the case of general dimension. Let D be a domain lying in $\boldsymbol{R}^2$ and $f \in L^1_{loc}(D)$. We say $f \in BMO(D)$ if

$$\|f\|_* = \|f\|_{*,D} = \sup_Q \frac{1}{m(Q)} \int_Q |f - f_Q| \, dm < \infty ,$$

where dm is the two dimensional Lebesgue measure, $f_Q = m(Q)^{-1} \int_Q f \, dm$ and the supremum is taken for all closed squares Q in D whose sides are paralleel to the coordinate axes.

We recall some notations and results in our former paper [3]. From now on 'square' means a closed square whose sides parallel to the coordinate axes, 'dyadic square' means a square $[k2^n, (k+1)2^n] \times [l2^n, (l+1)2^n]$, $k, l, n \in \boldsymbol{Z}$, $l(Q)$ denotes the side length of a square Q, $tQ, t>0$, denotes the square having the same center as Q and $tl(Q)$ as its side length, $d(\ , \cdot)$ denotes the Euclidean distance, $A>0$ denotes a universal constant which may vary from place to place. We say that a square Q lying in D is admissible if it satisfies $d(Q, \partial D) \geqq 32 l(Q)$ and $\mathcal{A}(D)$ denotes the set of all admissible squares in D. A sequence of admissible square $Q_0, Q_1, \cdots, Q_n$ in D satisfying the condition

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$$Q_i \cap Q_{i+1} \neq \emptyset, \qquad 0 \leqq i \leqq n-1,$$
$$\frac{1}{2} \leqq \frac{l(Q_i)}{l(Q_{i+1})} \leqq 2, \quad 0 \leqq i \leqq n-1,$$

is called an admissible chain. Let Q, Q' be two admissible squares in D. We define

$$\delta_D(Q, Q') = \min \{n \geqq 1 \mid Q = Q_0, Q_1, \cdots, Q_n = Q' \text{ is an admessible chain}\}$$

and the admissible chain which attains above minimum is called geodesic admissible chain joining Q and Q'. Since we define δ_D so that $\delta_D \geqq 1$ by technical reason, δ_D is not a distance function, but the triangle inequality holds. In [3] we have used δ_D to characterize the domain with 'relative' BMO extension property. Let Q, $Q' \in \sum \mathcal{A}(\boldsymbol{R}^2)$, that is, Q, Q' be arbitrary squares lying in R^2. We define

$$\phi(Q, Q') = \log\left(1 + \frac{l(Q) + l(Q') + d(Q, Q')}{l(Q)}\right)\left(1 + \frac{l(Q) + l(Q') + d(Q, Q')}{\mathrm{WO}}\right),$$

then

PROPOSITION 1. ([3]) *Let Q, $Q' \in \mathcal{A}(D)$ then*

$$\phi(Q, Q') \leqq A\delta_D(Q, Q').$$

Conversely if there exists a square $\tilde{Q}$ such that $Q \cup Q' \subset \tilde{Q} \subset 2\tilde{Q} \subset D$ then

$$\delta_D(Q, Q') \leqq A\phi(Q, Q').$$

Especially, for all squares Q, $Q' \in \mathcal{A}(\boldsymbol{R}^2)$, we have

$$A^{-1}\phi(Q, Q') \leqq \delta_{R^2}(Q, Q') \leqq A\phi(\zeta 1, Q').$$

Let D be a proper subdomain of $\boldsymbol{R}^2$. There exists a decomposition of D into a countable family of dyadic squares $\mathcal{D}(D) = \{Q_\lambda\}, Q_\lambda{}^\circ \cap Q_\mu{}^\circ = \emptyset$, $(\lambda \neq \mu)$, $\cup_\lambda Q_\lambda = D$ such that

$$32 \leqq \frac{d(Q_\lambda, \partial D)}{l(Q_\lambda)} \leqq 66$$

which we call Whitney decomposition of D. We say that a sequence Q_0, $Q\backslash$, $\cdots$, $Q_n \in \mathcal{D}(D)$ is a Whitney chain if $Q_i \cap Q_{i+1} \neq \emptyset$. Since $\mathcal{D}(D) \subset \mathcal{A}(D)$, every Whitney chain is admissible. Let Q, $Q' \in \mathcal{D}(D)$. We set

$$W_D(Q, Q') = \min \{n \geqq 1 \ Q = Q_0, Q_1, \cdots, Q_n = Q' \text{ is a Whitney chain}\}$$

and the Whitney chain which attains above minimum is called geodesic Whitney chain joining Q and Q'. It holds that $\delta_D(Q, Q') \leqq W_D(Q, Q')$, Q, $Q' \in \mathcal{D}(D)$ by definition. Conversely

PROPOSITION 2. ([3]) $W_D(Q, Q') \leqq A\delta_D(Q, Q')$, Q, $Q' \in \mathcal{D}(D)$.

Let $Q_0 \in \mathcal{D}(D)$ *and set*

$$f(z)=W_D(Q, Q_0), \qquad z\in Q\in \mathcal{D}(D).$$

Then $f\in BMO(D)$ *and* $\|f\|_* \leqq A$, *which is a consequence of the following localization theorem.*

PROPOSITION 3. (cf. Reimann-Rychener [7], Jones [5]) *Let* $\lambda \geqq 1$. *Let* f *be a function in* $L^1_{loc}(D)$ *satisfying the condition*

$$\frac{1}{m(Q)}\int_Q |f-f_Q|\,dm \leqq K$$

for every square Q *in* D *such that* $d(Q, \partial D)\geqq \lambda l(Q)$ *then* $f\in BMO(D)$ *and* $\|f\|_{*,D} \leqq AK\lambda$.

$BMO(D)$ functions induce Lipschitz continuous functioins on $\mathcal{A}(D)$ as follows;

PROPOSITION 4. ([3]) *Let* D *be arbitrary domain and* $f\in BMO(D)$. *Then*

$$|f_Q - f_{Q'}| \leqq A\|f\|_{*,D}\delta_D(Q,Q'), \qquad Q,\ Q'\in \mathcal{A}(D).$$

Let D be arbitrary domain, $Q_0\in\mathcal{A}(D)$ and $f\in BMO(D)$. We say $f\in VMO(D)$ if

$$\frac{1}{m(Q)}\int_Q |f-f_Q|\,dm \longrightarrow 0$$

as $Q\in\mathcal{A}(D)$, $\delta_D(Q, Q_0)\to\infty$. Every continuous function on D with compact support belongs to $VMO(D)$. $VMO(D)$ is a closed subspace of $BMO(D)$ and its definition is independent of the choice of $Q_0\in\mathcal{A}(D)$. When D is bounded, $f\in VMO(D)$ if and only if $f\in BMO(D)$ and $m(Q)^{-1}\int_Q |f-f_Q|\,dm\to 0$ as $Q\in\mathcal{A}(D)$, $l(Q)\to 0$. $BMO(D)$ and $VMO(D)$ are invariant under quasi-conformal mappings. Remark that in case of $D=\boldsymbol{R}^2$, our $VMO(\boldsymbol{R}^2)$ space does not coinside with the usual VMO space which consists of function $f\in BMO(\boldsymbol{R}^2)$ such that $m(Q)^{-1}\int_Q |f-f_Q|\,dm\to 0$ as $Q\in\mathcal{A}(D)$, $l(Q)\to 0$. It is easy to show that our VMO space on $\boldsymbol{R}^2$ is contained in the usual VMO space on $\boldsymbol{R}^2$, but the converse is not true. For example, $\log^+|z|$ belongs to the usual VMO space on R^2 but it does not belong to our VMO space on $\boldsymbol{R}^2$.

By using the same method as the proof of Proposition 4, we have

PROPOSITION 4′. *Let* D *be arbitrary domain and* $f\in VMO(D)$. *Then*

$$|f_Q - f_{Q_0}| = o(\delta_D(Q, Q_0))$$

as $Q\in\mathcal{A}(D)$, $\delta_D(Q, Q_0)\to\infty$.

We say a measurable function ϕ is a $BMO(D)$ (resp. $VMO(D)$) multiplier if $\phi f\in BMO(D)(VMO(D))$ for every $f\in BMO(D)$ $(VMO(D))$. To consider BMO or

VMO multiplier it is convenient to introduce the norm

$$\|f\|_{**}=\|f\|_{**,D}=\|f\|_{*,D}+|f|_{Q_0}, \qquad f\in BMO(D)\ (VMO(D))$$

where Q_0 is a fixed square in $\mathcal{A}(D)$ and $|f|_{Q_0}=m(Q_0)^{-1}\int_{Q_0}|f|\,dm$. Then closed graph theorem shows that the operator $T_\phi: f\mapsto\phi f$ on $BMO(D)$ ($T'_\phi: f\mapsto\phi f$ on $VMO(D)$)is bounded. Let $\|T_\phi\|$ ($\|T'_\phi\|$) denotes its operator norm. Our main result is

THEOREM 1. *Let D be arbitrary domain. For a measurable function* ϕ *on D, the following three conditions are equivalent to each other*

(1) φ *is a* $BMO(D)$ *multiplier.*

(2) φ *is a* $VMO(D)$ *multiplier.*

(3) *There exists a constant* $M\geqq 0$ *such that*

$$\|\phi\|_\infty\leqq M,$$

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq\frac{M}{\delta_D(Q, Q_0)}, \qquad Q\in\mathcal{A}(D).$$

In this case $\|T_\phi\|\leqq AM$, $\|T'_\phi\|\leqq AM$ *holds. Conversely if* φ *is a* $BMO(D)$ *(resp.* $VMO(D)$*) multiplier then we can choose the constant M so that* $M\leqq A\|T_\phi\|$ ($M\leqq A\|T'_\phi\|$).

COROLLARY 1. (cf. Nakai-Yabuta [6]) *For a measurable functions* φ *on* $\boldsymbol{R}^2$ *the following conditions are equivalent to each other*

(1) φ *is a* $BMO(\boldsymbol{R}^2)$ *multiplier.*

(2) φ *is a* $VMO(\boldsymbol{R}^2)$ *multiplier.*

(3) $\phi\in L^\infty(\boldsymbol{R}^2)$ *and there exists a constant* $M\geqq 0$ *such that*

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq\frac{M}{\phi(Q, Q_0)}$$

for every square Q in $\boldsymbol{R}^2$.

Moreover we can replace $\boldsymbol{R}^2$ with arbitrary inner NTA domain, especially arbitrary uniform domain (see §3), in this corollary.

Let $\Delta=\{|z|<1\}$. Since $\delta(Q, Q_0)$, $Q\in\mathcal{A}(\Delta)$ is comparable with $\log(2+(1/l(Q))$, we have the following by Theorem 1 and Proposition 3 (of disk version). Its correspondence for holomorphic *BMO* function, which is usually called Bloch function, is well known (Brown-Shields [1]).

COROLLARY 2. *For a measurable functions* φ *on A the following conditions are equivalent to each other*

(1) φ *is a* $BMO(\Delta)$ *multiplier.*

(2) φ *is a* $VMC(\Delta)$ *multiplier.*

(3) $\phi\in L^\infty(\Delta)$ *and there exists a constant* $M\geqq 0$ *such that*

$$\frac{1}{m(B)}\int_B|\phi-\phi_B|\,dm\leqq M\left(\log\left(2+\frac{1}{\mathrm{rad}(B)}\right)\right)^{-1}$$

for every disk B in Δ, where rad(B) *denotes the radius of B.*

Moreover we can also replace Δ with arbitrary Holder domain, especially arbitrary bounded uniform domain (see §3), in this corollary.

Let S be the unit sphere in R^3, σ the normalized surface measure on S and $BMO_\sigma(\boldsymbol{S})$ the *BMO* space on S with respect to σ. We fix a disk B_0 on S. Then the distance between B_0 and arbitrary ball B on S, which corresponds to δ_D, is comparable with $\log(2+(1/\mathrm{rad}(B))$. Hence we have the following result, its one dimensional version for the *BMO* space on the unit circle is well known (Stegenga [9], Janson [4]), as the same way.

COROLLARY 3. *For a measurable functions ϕ on S the following conditions are equivalent to each other*

(1) *φ is a $BMO_\sigma(\boldsymbol{S})$ multiplier.*

(2) *φ is a $VMO_\sigma(\boldsymbol{S})$multiplier.*

(3) *$\phi\in L^\infty(\boldsymbol{S})$ and there exists a constant $M\geqq 0$ such that*

$$\frac{1}{\sigma(B)}\int_B|\phi-\phi_{B,\sigma}|\,d\sigma\leqq M\left(\log\left(2+\frac{1}{\mathrm{rad}(B)}\right)\right)^{-1}$$

for every disk B in $\boldsymbol{S}$.

Since we can identify $BMO(\boldsymbol{R}^2)$ and $VMO(\boldsymbol{R}^2)$with $BMO_\sigma(\boldsymbol{S})$and $VMO_\sigma(\boldsymbol{S})$ respectively throughout the stereographic projection (See Reimann-Rychener [7] for BMO, and the similar method proves this for VMO.), Corollary 3 gives another characterization of $BMO(\boldsymbol{R}^2)$ multiplers.

§ 2. Proof of Theorem 1.

LEMMA 1. (cf. Stegenga [9], Nakai-Yabuta [6]) *// φ is a $BMO(D)$ (resp. $VMO(D)$) multiplier then $\phi\in L^\infty(D)$ and $\|\phi\|_\infty\leqq 3\|T_\phi\|$ ($\|\phi\|_\infty\leqq 3\|T'_\phi\|$).*

Proof. Let φ be a $VMO(D)$multiplier. We fix a point $z\in D$. Let Q be the square having z as its center and $l(Q)=t$. Let h be a function on D such that

$$|h|=\frac{1}{m(Q)}\chi_Q,\qquad \int_D h\,dm=0$$

and set $k=\overline{\mathrm{sgn}}(\phi h)$. Let k_n be a sequence of continuous function with compact support which converges to k a.e. and $\|k_n\|_\infty\leqq\|k\|_\infty$. Then $k_n\in VMO(D)$ and since $\|k_n\|_\infty\leqq 1$ we have $\|k_n\|_{**}\leqq 3$. Hence

$$\int_D k_n\phi h\,dm=\int_D\{\phi k_n-(\phi k_n)_Q\}\,h\,dm$$

$$\leqq \frac{1}{m(Q)}\int_Q |\phi k_n-(\phi k_n)_Q|\,dm \leqq \|k_n\phi\|_* \leqq 3\|T'_\phi\|.$$

And so by $n\to\infty$,

$$\frac{1}{m(Q)}\int_Q |\phi|\,dm=\int_D k\phi h\,dm \leqq 3\|T'_\phi\|.$$

Letting $t\to 0$ we have $\|\phi\|_\infty \leqq 3\|T'_\phi\|$ by **Lebesgue's** theorem. This proves the assertion since above proof is valid for BMO. Q. E. D.

LEMMA 2. (cf. Stegenga [9], Nakai-Yabuta [6]) *Let* $f\in L^1_{loc}(D)$ *and* $\phi\in L^\infty(D)$ *then*

$$\left| |f_Q|\frac{1}{m(Q)}\int_Q |\phi-\phi_Q|\,dm-\frac{1}{m(Q)}\int_Q |\phi f-(\phi f)_Q|\,dm\right| \leqq 2\|\phi\|_\infty \frac{1}{m(Q)}\int_Q |f-f_Q|\,dm.$$

holds for every square Q *lying in* D.

Proof.

$$\left| |f_Q|\frac{1}{m(Q)}\int_Q |\phi-\phi_Q|\,dm-\frac{1}{m(Q)}\int_Q |f\phi-(f\phi)_Q|\,dm\right|$$

$$\leqq \frac{1}{m(Q)}\int_Q (|(f-f_Q)\phi|+|f_Q\phi_Q-(f\phi)_Q|)\,dm$$

$$\leqq \|\phi\|_\infty \frac{1}{m(Q)}\int_Q |f-f_Q|\,dm+\left|\frac{1}{m(Q)}\int_Q (f-f_Q)\phi\,dm\right|$$

$$\leqq 2\|\phi\|_\infty \frac{1}{m(Q)}\int_Q |f-f_Q|\,dm.$$

Q. E. D.

The following lemma shows that the estimation of Proposition 4 is best possible.

LEMMA 3. *Let* D *be arbitrary domain and* $Q_0,\ Q_1\in\mathcal{A}(D)$. *Then there exists a function* $f\in VMO(D)$ *such that*

$$\delta_D(Q_0,\ Q_1)\leqq A|f_{Q_1}|+A,\quad \|f\|_{*,D}\leqq A,\quad |f|_{Q_0}\leqq A.$$

Proof. First, assume there exists a square $\tilde{Q}$ in D such that $Q_0\cup Q_1\subset\tilde{Q}\subset 2\tilde{Q}\subset D$. Let z_i, $i=0,\ 1$ be the center of Q_i. In this case the first inequality reduces to $\phi_D(Q_0,\ Q_1)\leqq A|f_{Q_1}+A$ by Proposition 1, and so the function

$$f(z)=\min\left\{\log^+\left(\frac{l(Q_0)+l(Q_1)+d(Q_0,\ Q_1)}{|z-z_1|}\right),\ \log\left(\frac{l(Q_0)+l(Q_1)+d(Q_0,\ Q_1)}{l(Q_1)}\right)\right\}$$

if $l(Q_1)<l(Q_0)$ and

$$f(z)=\min\left\{\log^+\frac{|z-z_0|}{l(Q_0)},\ \log\left(\frac{l(Q_0)+l(Q_1)+d(Q_0,\ Q_1)}{l(Q_0)}\right)\right\}$$

if $l(Q_1)\geqq l(Q_0)$, satisfies the required condition since $\log|z|\in BMO(\boldsymbol{R}^2)$.

Next assume there exists no such square $\tilde{Q}$. In this case $D\neq\boldsymbol{R}^2$. Let Q_i', $i=0, 1$ be a square in $\mathscr{D}(D)$ such that $Q_i\cap Q_i'\neq\emptyset$ and z_i the center of Q_i. Then

$$\delta_D(Q_0, Q_1)\leqq\delta_D(Q_0, Q_0')+\delta_D(Q_0', Q_1')+\delta_D(Q_1', Q_1)$$

$$\leqq A\log^+\frac{l(Q_0')}{l(Q_0)}+AW_D(Q_0', Q_1')+A\log^+\frac{l(Q_1')}{l(Q_1)}+A\,.$$

Let

$$f_1(z)=\min\left\{\log^+\frac{|z-z_0|}{l(Q_0)},\ \log^+\frac{l(Q_0')}{l(Q_0)}\right\},\qquad z\in D\,,$$

$$f_2'(z)=mm\ \{W_D(Q_0', Q),\ W_D(Q_0', Q_1')\}\,,\qquad z\in Q\in\mathscr{D}(D)\,,$$

$$f_3(z)=\min\left\{\log^+\frac{l(Q_1')}{|z-z_0|},\ \log^+\frac{l(Q_1')}{l(Q_1)}\right\},\qquad z\in D\,,$$

We slightly modify f_2' into a continuous functions f_2 (or we may define f_2 by

$$f_2(z)=\min\ \{k_D(z_0, z),\ k_D(z_0, z_1)\}$$

where z_i be the center of Q_i and k_D is the distance function obtained by the quasi-hyperbolic metric $|dz|/d(z,\partial D))$ and set $f=f_1+f_2+f_3$ Then $f\in VMO(D)$ and by the remark below Proposition 2, we have $\|f\|_{*,D}\leqq\sum_{i=1}^3\|f_i\|_{*,D}\leqq A$, $|f|_{Q_0}\leqq\sum_{i=1}^3|f_i|_{Q_0}\leqq A$ and

$$\log\frac{l(Q_0')}{l(Q_0)}\leqq A(f_1)_{Q_1}+A\,,\quad W_D(Q_0', Q_1')\leqq A(f_2)_{Q_1}+A\,,\quad \log\frac{l(Q_1')}{l(Q_1)}A(f_3)_{Q_1}+A\,.$$

Summerizing above inequalities we have $\delta_D(Q_0, Q_1)\leqq A|f_{Q_1}|+A$. Q. E. D.

Proof of Theorem 1. We will prove only (1)$\leftrightarrow$(3) since we can show (2)$\leftrightarrow$(3) similarly by appealing to Proposition 4$'$ instead of Proposition 4. Let φ satisfy the condition (3). Let $f\in BMO(D)$ and $Q\in\mathcal{A}(D)$. Proposition 4 shows that

$$|f_Q|\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq\{|f_{Q_0}|+A\|f\|_{*,D}\delta_D(Q, Q_0)\}\frac{M}{\delta_D(Q, Q_0)}$$

$$\leqq AM\{|f|_{Q_0}+\|f\|_{*,D}\}\leqq AM\|f\|_{**,D}\,.$$

Hence by Lemma 2,

$$\frac{1}{m(Q)}\int_Q|\phi f-(\phi f)_Q|\,dm\leqq AM\|f\|_{**,D}+2\|\phi\|_\infty\|f\|_*\leqq AM\|f\|_{**,D}\,.$$

Applying localization theorem we have $\|\phi f\|_{*,D}\leqq AM\|f\|_{**,D}$. Since $|\phi f|_{Q_0}\leqq\|\phi\|_\infty|f|_{Q_0}\leqq M\|f\|_{**,D}$it follows that $\|\phi f\|_{**,D}\leqq AM\|f\|_{**,D}$.

Conversely let φ be a $BMO(D)$ multiplier. Let $Q_1\in\mathcal{A}(D), f$ the function satisfying the condition of Lemma 3. Then Lemmas 1 and 2 show that

$$\delta_D(Q_1, Q_0)\frac{1}{m(Q_1)}\int_{Q_1}|\phi-\phi_{Q_1}|dm\leqq(A|f_{Q_1}|+A)\frac{1}{m(Q_1)}\int_{Q_1}|\phi-\phi_{Q_1}|dm$$
$$\leqq A(\|T_\phi f\|_{*,D}+2\|\phi\|_\infty\|f\|_*)+A\|T_\phi 1\|_{*,D}\leqq A\|T_\phi\|.$$

which implies the assertion. Q. E. D.

§ 3. Some consequences.

Let $D\neq \boldsymbol{R}^2$. We say a function F on $\mathcal{D}(D)$ is admissible if it satisfies $F\geqq M^{-1}$ and

$$M^{-1}\leqq\frac{F(Q)}{F(Q')}\leqq M,\quad Q\cap Q'\neq\emptyset,\quad Q, Q'\in\mathcal{D}(D).$$

for some constant $M>0$. Let F be an admissible function on $\mathcal{D}(D)$. We set

$$\hat{F}(Q)=F(\tilde{Q})+\log\left(2+\frac{l(\tilde{Q})}{l(Q)}\right),\qquad Q\in\mathcal{A}(D)$$

where $\tilde{Q}$ is one of the square in $\mathcal{D}(D)$ such that $\tilde{Q}\cap Q\neq\emptyset$. We fix a square $Q_0\in\mathcal{A}(D)$.

THEOREM 2. *Let $D\neq\boldsymbol{R}^2$. The following conditions are equivalent for an admissible function F on $\mathcal{D}(D)$*

(1) *There exists a constant $M>0$ such that*

$$\delta_D(Q, Q_0)\leqq M\hat{F}(Q),\qquad Q\in\mathcal{A}(D).$$

(2) *Let ϕ be an $L^\infty(D)$ function on D satisfying the condition*

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|dm\leqq\frac{M}{\hat{F}(Q)},\qquad Q\in\mathcal{A}(D)$$

for some constant $M\geqq 0$, then φ is a $BMO(D)$ multiplier.

(3) *Let φ be an $L^\infty(D)$ function on D satisfying the same condition as* (2) *then φ is a $VMO(D)$ multiplier.*

Proof. (2)↔(3) and (1)→(2) are the consequence of Theorem 1. Now will prove (2)→(1). We can assume $Q_0\in\mathcal{D}(D)$ from the beginning since the condition (1) is independent of the choice of $Q_0\in\mathcal{A}(D)$. Then $\delta_D(Q, Q_0)$, $Q\in\mathcal{A}(D)$ is comparable with $W_D(\tilde{Q}, Q_0)+\log(2+(l(\tilde{Q})/l(Q))$ where $\tilde{Q}$ is one of the squares in $\mathcal{D}(D)$ such that $Q\cap\tilde{Q}\neq\emptyset$. Let h be a fixed non-zero C^∞ function supported on the square of side length 1 and center the origin such that $\int h\,dm=0$. Let $Q\in\mathcal{D}(D)$ and z_0 its center. We set a function φ on D by

$$\phi(z)=\frac{1}{F(Q)}h\left(\frac{z-z_0}{l(Q)}\right)\qquad z\in Q\in\mathcal{D}(D).$$

ϕ is a bounded $C^\infty(D)$ function on D and it holds that

$$|\nabla\phi(z)|\leqq\frac{A}{F(Q)l(Q)}, \qquad z\in Q\in\mathcal{D}(D).$$

Let $Q\in\mathcal{A}(D)$. Let $\tilde{Q}$ one of the square in $\mathcal{D}(D)$ such that $Q\cap\tilde{Q}\neq\emptyset$ and z_0 its center then by above estimate we have

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq\frac{2}{m(Q)}\int_Q|\phi-\phi(z_0)|\,dm$$

$$\leqq\frac{2}{m(Q)}\int_Q\frac{A}{F(\tilde{Q})l(\tilde{Q})}l(Q)dm\leqq\frac{A}{F(\tilde{Q})}\frac{l(Q)}{l(\tilde{Q})}\leqq\frac{A}{\hat{F}(Q)}.$$

Hence ϕ is a BMO multiplier by the assumption. Further let $Q\in\mathcal{D}(D)$ then

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm=\frac{1}{m(Q)}\int_Q|\phi|\,dm\geqq\frac{A}{F(Q)}$$

and so theorem 1 implies the assertion. Q. E. D.

Let $Q_0\in\mathcal{A}(D)$. We say a domain D is an inner NTA domain if there exists a constant $M>0$ such that

$$\delta_D(Q,\,Q_0)\leqq M\psi(Q,\,Q_0), \qquad Q\in\mathcal{A}(D).$$

This deefinition is somewhat different from the original one (cf. Shimomura [8]). $\boldsymbol{R}^2$ is inner NTA by Proposition 1. More generally every uniform domain is inner NTA (cf. Gehring [2]). We say also a domain D is a Holder domain (cf. Shimomura [8]) if there exists a constant $M>0$ such that

$$\delta_D(Q,\,Q_0)\leqq M\log\left(2+\frac{1}{l(Q)}\right), \qquad Q\in\mathcal{A}(D).$$

These definitions are independent of the choice of $Q_0\in\mathcal{A}(D)$. Remark that the inverse inequality holds for every domain in either case. In case of $D\neq\boldsymbol{R}^2$, D is inner NTA if and only if there exists a square $Q_0\in\mathcal{D}(D)$ and a constant $M>0$ such that

$$W_D(Q,\,Q_0)\leqq M\psi(Q,\,Q_0), \qquad Q\in\mathcal{D}(D)$$

and D is Holder if and only if there exists a square $Q_0\in\mathcal{D}(D)$ and a constant $M>0$ such that

$$W_D(Q,\,Q_0)\leqq M\log\left(2+\frac{1}{l(Q)}\right) \qquad Q\in\mathcal{D}(D).$$

There is a simple relation between inner NTA domains and Holder domains.

LEMMA 4. *A domain D is a Holder domain if and only if it ɪs a bounded inner NTA domain.*

Proof. It suffices to show that Holder domains are bounded. Let D be a Holder domain. Since $l(Q)\to 0$ as $Q\in\mathcal{A}(D)$, $Q\to\infty$, $D\neq\boldsymbol{R}^2$ and we can assume that Q_0 is the biggest square in $\mathcal{D}(D)$ and $l(Q_0)=1$ Let $Q\in\mathcal{D}(D)$ and set $l(Q)=2^{-N}$. Let $Q_0, Q_1, \cdots, Q_n=Q$ be a geodesic Whitney chain and set $n_0=0$, $n_k=\max\{n\,|\,l(Q_n)=2^{-k}\}$, $1\leqq k\leqq N$. Let z_k be the center of Q_{n_k} and set $d_k=|z_k-z_{k-1}|$. Since $n_k-n_{k-1}\geqq A2^k d_k$ we have

$$\sum_{k=1}^{m}2^k d_k\leqq A\sum_{k=1}^{m}(n_k-n_{k-1})\leqq An_m\leqq AW_D(Q_m, Q_0)\leqq C\log\left(2+\frac{1}{l(Q_{n_m})}\right)\leqq Cm.$$

Hence

$$2^N\sum_{k=1}^{N}d_k=\sum_{k=1}^{N}2^k d_k+\sum_{m=1}^{N-1}\left(2^{N-m-1}\sum_{k=1}^{m}2^k d_k\right)\leqq C2^N.$$

Thus $d(Q, Q_0)\leqq A\sum_{k=1}^{N}d_k\leqq C$which implies the assertion. Q.E.D.

Applying Theorem 1 in case of $D=\boldsymbol{R}^2$ and applying Theorem 2 to the functions $F(Q)=\phi(Q, Q_0)$ or $F(Q)=\log(2+(1/l(Q))$ in case of $D\neq\boldsymbol{R}^2$, we have

COROLLARY 4. *The following conditions are equivalent for a domain D*

(1) *D is an inner NT A domain.*

(2) *Let ϕ be a $L^\infty(D)$ function on D satisfying the condition*

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq\frac{M}{\phi(Q, Q_0)},\qquad Q\in\mathcal{A}(D)$$

for some constant $M\geqq 0$, then φ is a BMO(D) multiplier.

(3) *Let φ be a $L^\infty(D)$ function on D satisfying the same condition as* (2) *then φ is a VMO(D) multiplier.*

COROLLARY 5. *The following conditions are equivalent for a domain D*

(1) *D is a Holder domain.*

(2) *Let φ be a $L^\infty(D)$ function on D satisfying the condition*

$$\frac{1}{m(Q)}\int_Q|\phi-\phi_Q|\,dm\leqq M\left(\log\left(2+\frac{1}{l(Q)}\right)\right)^{-1},\qquad Q\in\mathcal{A}(D)$$

for some constant $M\geqq 0$, then φ is a BMO(D) multiplier.

(3) *Let φ be a $L^\infty(D)$ function on D satisfying the same condition as* (2) *then φ is a VMO(D) multiplier.*

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