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3-DIMENSIONAL SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM

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Introduction.

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite Riemannian manifold of index p and of constant curvature c, which is called *an indefinite space form of index p*. According to $c>0$, $c=0$ or $c<0$ it is denoted by $S_p^{n+p}(c)$, $\boldsymbol{R}_p^{n+p}$ or $H_p^{n+p}(c)$. A submanifold M of an indefinite space form $M_p^{n+p}(c)$ is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. Now it is pointed out by many physicians that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes get interesting in relativity theory. Also, from the differential point of view, an entire space-like hypersurface with constant mean curvature of an indefinite space form are studied by many authors (for examples: [1], [2], [3], [4] and so on). For a complete space-like submanifold M with parallel mean curvature vector of $S_p^{n+p}(c)$, it is also seen by Cheng [3] that M is totally umbilic if $n=2$ and $H^2\leqq c$ or if $n>2$ and $n^2H^2<4(n-1)c$, where H denotes the mean curvature, i.e., the norm of the mean curvature vector. On the other hand, Aiyama and Cheng [1] prove recently the following.

THEOREM. *Let M be a 3-dimensional complete space-like hypersurface with parallel mean curvature H in a Lorentzian space form $M_1^4(c)$. If* $\sup Ric(M)$ $<3(c-H^2)$, *then M is totally umbilic, and $c>H^2$.*

The purpose of this paper is to research the similar problem to the above theorem for 3-dimensional complete space-like submanifolds with parallel mean curvature vector of an indefinite space form and to prove the following.

THEOREM 1. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $S_p^{3+p}(c)$, $p\geqq 2$. If it satisfies*

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$$(1.1)\qquad \frac{8}{9}c\leqq H^2\leqq c \quad \text{and} \quad Ric(M)\leqq\delta_1<3(c-H^2),$$

then M is totally umbilic.

Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of an indefinite space form $M_p^{3+p}(c)$. We denote by S the square of the length of the second fundamental form of M. It is seen in Proposition 3.2 that if M is pseudo-umbilic and if $H^2>c$, then it satisfies

$$(1.2)\qquad S\leqq 3pH^2-3(p-1)c$$

and also in Remark 3.2 a natural example satisfying the equality of (1.2) is given. Conversely we can prove

THEOREM 2. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $M_p^{3+p}(c)$, $c\leqq 0$, $p\geqq 2$. If it satisfies*

$$(1.3)\qquad Ric(M)\leqq\delta_1<\frac{3}{2}(p-3)(H^2-c) \quad \text{and} \quad S\geqq 3pH^2-3(p-1)c,$$

then the following assertions hold,

(1) $c<0$, *and* $p\geqq 4$,

(2) *M is congruent to*

1. $H^2(c_1)\times H^1(c_2)$, $\quad p=4$,
2. $H^1(c_1)\times H^1(c_2)\times H^1(c_3)$, $\quad p=4, 5$,

where $H^n(c)$ denotes an n-dimensional hyperbolic space of constant curvature c.

2. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional indefinite Riemannian manifold of constant curvature c whose index is p, which is called *an indefinite space form of constant curvature c and with index p.* Let M be an n-dimensional submanifold of an $(n+p)$-dimensional indefinite space form $M_p^{n+p}(c)$ of index $p>0$. The submanifold M is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ adapted to the indefinite Riemannian metric of $M_p^{n+p}(c)$ and the dual coframe $\omega_1, \cdots, \omega_{n+p}$ in such a way that, restricted to the submanifold M, $e_1, \cdots, e_n$ are tangent to M. Then connection forms $\{\omega_{AB}\}$ of $M_p^{n+p}(c)$ are characterized by the structure equations

$$
(2.1)\qquad \begin{cases} d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ \Omega_{AB} = -\dfrac{1}{2} \sum \varepsilon_C \varepsilon_D R'_{ABCD} \omega_C \wedge \omega_D, \end{cases}
$$

$$
(2.2)\qquad R'_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}),
$$

where Ω_{AB} (resp. R'_{ABCD}) denotes the indefinite Riemannian curvature form (resp. the components of the indefinite Riemannian curvature tensor) of $M_p^{n+p}(c)$. Therefore the components of the Ricci curvature tensor Ric' and the scalar curvature r' of $M_p^{n+p}(c)$ are given as

$$
R'_{AB} = c(n+p-1)\varepsilon_A \delta_{AB}, \qquad r' = (n+p)(n+p-1)c.
$$

In the sequel, the following convention on the range of indices is used, unless otherwised stated:

$$
1 \leqq A, B, \cdots \leqq n+p; \ 1 \leqq i, j, \cdots \leqq n; \ n+1 \leqq \alpha, \beta, \cdots \leqq n+p.
$$

We agree that the repeated indices under a summation sign without indication are summed over the respective range. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$
(2.3)\qquad \omega_\alpha = 0 \qquad \text{for} \quad \alpha = n+1, \cdots, n+p.
$$

We see that $e_1, \cdots, e_n$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M and $\omega_1, \cdots, \omega_n$ is a local field of its dual coframes on M. It follows from (2.1), (2.3) and Cartan's Lemma that

$$
(2.4)\qquad \omega_{\alpha i} = \sum h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.
$$

The second fundamental form α and the mean curvature vector $\boldsymbol{h}$ of M are defined by

$$
\alpha = -\sum h_{ij}^\alpha \omega_i \omega_j e_\alpha, \qquad \boldsymbol{h} = -\frac{1}{n} \sum \Big(\sum_i h_{ii}^\alpha\Big) e_\alpha.
$$

The mean curvature H is defined by

$$
(2.5)\qquad H = |\boldsymbol{h}| = \frac{1}{n} \sqrt{\sum \Big(\sum_i h_{ii}^\alpha\Big)^2}.
$$

Let $S = \sum (h_{ij}^\alpha)^2$ denote the squared norm of the second fundamental form α of M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$
(2.6)\qquad \begin{cases} d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ \Omega_{ij} = -\dfrac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{cases}
$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. Therefore, from (2.1) and (2.6), the Gauss equation is given by

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})-\sum(h_{il}^{\alpha}h_{jk}^{\alpha}-h_{ik}^{\alpha}h_{jl}^{\alpha}). \tag{2.7}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk}=(n-1)c\delta_{jk}-\sum h_{ii}^{\alpha}h_{jk}^{\alpha}+\sum h_{ik}^{\alpha}h_{ij}^{\alpha}, \tag{2.8}$$

$$r=n(n-1)c-n^2H^2+\sum(h_{ij}^{\alpha})^2. \tag{2.9}$$

We also have

$$d\omega_{\alpha\beta}-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=-\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j, \tag{2.10}$$

where

$$R_{\alpha\beta ij}=-\sum(h_{il}^{\alpha}h_{jl}^{\beta}-h_{jl}^{\alpha}h_{il}^{\beta}).$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$h_{ijk}^{\alpha}-h_{ikj}^{\alpha}=0, \tag{2.11}$$

$$h_{ijkl}^{\alpha}-h_{ijlk}^{\alpha}=-\sum h_{im}^{\alpha}R_{mjkl}-\sum h_{mj}^{\alpha}R_{mikl}+\sum h_{ij}^{\beta}R_{\beta\alpha kl}, \tag{2.12}$$

where h_{ijk}^{α} and h_{ijkl}^{α} denote the components of the covariant differentials $\nabla\alpha$ and $\nabla^2\alpha$ of the second fundamental form respectively. The Laplacian Δh_{ij}^{α} of the components h_{ij}^{α} of the second fundamental form α is given by

$$\Delta h_{ij}^{\alpha}=\sum_k h_{ijkk}^{\alpha}.$$

From (2.12) we get

$$\Delta h_{ij}^{\alpha}=\sum_k h_{kkij}^{\alpha}-\sum h_{km}^{\alpha}R_{mijk}-\sum h_{mi}^{\alpha}R_{mkjk}+\sum h_{ki}^{\beta}R_{\beta\alpha jk}. \tag{2.13}$$

The following generalized maximum principle due to Omori [8] and Yau [11] will play an important role in this paper.

THEOREM 2.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from below on M, then for any $\varepsilon>0$, there exists a point p in M such that*

$$F(p)<\inf F+\varepsilon, \qquad |gradF|(p)<\varepsilon, \qquad \Delta F(p)>-\varepsilon.$$

By applying this principle the following theorem due to Nishikawa [7] is proved.

THEOREM 2.2. *Let M be an n-dimensional complete Riemannian manifold* *whose Ricci curvature is bounded from below. Let F be a non-negative C^2-function on M. If it satisfies*

$$\Delta F \geqq k F^2,$$

then $F=0$ on M, where k is a positive constant.

3. Pseudo-umbilic submanifolds.

This section is concerned with pseudo-umbilic space-like submanifolds of an indefinite space form $M_p^{n+p}(c)$. Let M be an n-dimensional space-like submanifold with parallel mean curvature vector $\boldsymbol{h} \neq 0$ of $M_p^{n+p}(c)$. Because the mean curvature vector is parallel, the mean curvature is constant. We choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then it is easily seen that we have

$$\omega_{\alpha\, n+1}=0, \qquad H=\text{constant}, \tag{3.1}$$

$$H^{\alpha} H^{n+1}=H^{n+1} H^{\alpha}, \tag{3.2}$$

$$\operatorname{tr} H^{n+1}=nH, \qquad \operatorname{tr} H^{\alpha}=0 \tag{3.3}$$

for any $\alpha \neq n+1$, where H^{α} denotes an $n \times n$ symmetric matrix (h_{ij}^{α}).

A submanifold M is said to be *pseudo-umbilic*, if it is umbilic with respect to the direction of the mean curvature vector $\boldsymbol{h}$, that is,

$$h_{ij}^{n+1}=H\delta_{ij}. \tag{3.4}$$

We denote by μ an $n \times n$ symmetric matrix with components defined by $\mu_{ij}=h_{ij}^{n+1}-H\delta_{ij}$. We then have

$$\operatorname{tr}\mu=0, \qquad |\mu|^2=\operatorname{tr}(\mu)^2=\sum \mu_{ij}^2=\operatorname{tr}(H^{n+1})^2-nH^2. \tag{3.5}$$

So the pseudo-umbilic submanifolds are characterized by the property $\mu=0$. A non-negative function τ is denoted by $\tau^2=\sum_{\beta \neq n+1}(h_{ij}^{\beta})^2$. Then we have

$$S=|\mu|^2+\tau^2+nH^2, \tag{3.6}$$

which means that $S \geqq nH^2$, where the equality holds at a point if and only if the point is umbilic. Hence it is seen that $|\mu|^2$ as well as τ^2 are independent of the choice of the frame fields and they are functions globally defined on M. It is also seen that if the pseudo-umbilic submanifold satisfies $\tau=0$, then it is totally umbilic.

Now, in general, it is asserted by Cheng [3] that a complete $n(\geqq 3)$-dimensional space-like submanifold with parallel mean curvature vector $\boldsymbol{h}$ of $S_p^{n+p}(c)$ is totally umbilic if it satisfies

$$H^2<\frac{4(n-1)}{n^2}c.$$

PROPOSITION 3.1. *Let M be an n-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $S_p^{n+p}(c)$, $p\geqq 2$. If M is pseudo-umbilic and if it satisfies*

$$\frac{4(n-1)}{n^2}c\leqq H^2\leqq c, \tag{3.7}$$

then M is totally umbilic.

Proof. From (2.13) and the Gauss equation (2.7) and (2.10) we get

$$\begin{aligned}\Delta h_{ij}^{\alpha}=nch_{ij}^{\alpha}-c\sum h_{kk}^{\alpha}\delta_{ij}+\sum h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}-2\sum h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}\\+\sum h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}-\sum h_{kk}^{\beta}h_{im}^{\alpha}h_{mj}^{\beta}+\sum h_{ik}^{\beta}h_{mj}^{\alpha}h_{mk}^{\beta}\end{aligned} \tag{3.8}$$

for any index α. Moreover we see

$$\frac{1}{2}\Delta\tau^2=\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+\sum_{\alpha\neq n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}.$$

Accordingly it follows from (3.8) and the above equation that we get

$$\begin{aligned}\frac{1}{2}\Delta\tau^2=&\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+nc\tau^2+\sum_{\alpha\neq n+1}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha}\\&-2\sum_{\alpha\neq n+1}h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}\\&-nH\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}+\sum_{\alpha\neq n+1}h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{\alpha}h_{ij}^{\alpha},\end{aligned}$$

and hence we obtain

$$\begin{aligned}\frac{1}{2}\Delta\tau^2=&\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+nc\tau^2\\&+\sum_{\alpha,\beta\neq n+1}h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha}-2\sum_{\alpha,\beta\neq n+1}h_{ik}^{\beta}h_{km}^{\alpha}h_{mj}^{\beta}h_{ij}^{\alpha}\\&+\sum_{\alpha,\beta\neq n+1}h_{im}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}+\sum_{\alpha,\beta\neq n+1}h_{ik}^{\beta}h_{km}^{\beta}h_{mj}^{\alpha}h_{ij}^{\alpha}\\&+\sum_{\alpha\neq n+1}h_{km}^{\alpha}h_{mk}^{n+1}h_{ij}^{n+1}h_{ij}^{\alpha}-2\sum_{\alpha\neq n+1}h_{ik}^{n+1}h_{km}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}\\&+\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mk}^{n+1}h_{kj}^{n+1}h_{ij}^{\alpha}-nH\sum_{\alpha\neq n+1}h_{im}^{\alpha}h_{mj}^{n+1}h_{ij}^{\alpha}\\&+\sum_{\alpha\neq n+1}h_{jm}^{\alpha}h_{mk}^{n+1}h_{ki}^{n+1}h_{ij}^{\alpha}.\end{aligned} \tag{3.9}$$

We put $S_{\alpha\beta}=\sum h_{ij}^{\alpha}h_{ij}^{\beta}$ for any $\alpha,\beta\neq n+1$. Then $(S_{\alpha\beta})$ is a $(p-1)\times(p-1)$ symmetric matrix. It can be assumed to be diagonal for a suitable choice of $e_{n+2},\cdots,e_{n+p}$. Set $S_\alpha=S_{\alpha\alpha}$. We then have $\tau^2=\sum S_\alpha$. In general, for a matrix $A=(a_{ij})$, we define $N(A)=\mathrm{tr}(A\,{}^tA)$. Then the above equation can be reduced to

$$\frac{1}{2}\Delta\tau^2=\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+\sum_{\alpha,\beta\neq n+1}\{(S_{\alpha\beta})^2-2\,\mathrm{tr}H^{\alpha}H^{\beta}H^{\alpha}H^{\beta}+2\,\mathrm{tr}H^{\alpha}H^{\alpha}H^{\beta}H^{\beta}\}$$
$$+\sum_{\alpha\neq n+1}\{\sum h_{km}^{\alpha}h_{mk}^{n+1}h_{ij}^{n+1}h_{ij}^{\alpha}-2\,\mathrm{tr}H^{\alpha}H^{n+1}H^{\alpha}H^{n+1}$$
$$+2\,\mathrm{tr}H^{\alpha}H^{\alpha}H^{n+1}H^{n+1}-nH\,\mathrm{tr}H^{\alpha}H^{n+1}H^{\alpha}\}.$$

By (3.2), (3.3) and (3.4) and the definition of the function τ, we have

$$\frac{1}{2}\Delta\tau^2=\sum_{\alpha\neq n+1}(h_{ijk}^{\alpha})^2+nc\tau^2+\sum_{\alpha\neq n+1}(S_{\alpha})^2$$
$$+\sum_{\alpha,\beta\neq n+1}N(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})-nH^2\tau^2. \tag{3.10}$$

Obviously we see

$$\sum_{\alpha,\beta\neq n+1}N(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})\geqq 0. \tag{3.11}$$

Suppose $p\geqq 2$. Let

$$(p-1)\sigma_1=\tau^2=\sum S_{\alpha},$$
$$(p-1)(p-2)\sigma_2=2\sum_{\alpha<\beta,\ \alpha,\beta\neq n+1}S_{\alpha}S_{\beta}.$$

Then we have

$$\sum S_{\alpha}^2=(p-1)\sigma_1^2+(p-1)(p-2)(\sigma_1^2-\sigma_2),$$
$$\sum_{\alpha<\beta,\ \alpha,\beta\neq n+1}(S_{\alpha}-S_{\beta})^2=(p-1)^2(p-2)(\sigma_1^2-\sigma_2).$$

Hence we obtain

$$\sum_{\alpha\neq n+1}(S_{\alpha})^2\geqq(p-1)\sigma_1^2=\frac{1}{p-1}\tau^4. \tag{3.12}$$

Accordingly it follows from (3.10), (3.11) and (3.12) that we have

$$\frac{1}{2}\Delta\tau^2\geqq nc\tau^2+\frac{1}{p-1}\tau^4-nH^2\tau^2$$
$$=\frac{1}{p-1}\tau^2\{\tau^2-n(p-1)(H^2-c)\}. \tag{3.13}$$

By the assumption of the proposition we get

$$\Delta\tau^2\geqq\frac{2}{p-1}\tau^4.$$

By (2.8), (3.2) and (3.4) the Ricci curvature is bounded from below by a constant $-(n-1)(H^2-c)$, we can apply Theorem 2.2 to the non-negative function τ^2 and we get

$$\tau^2=0.$$

Thus M is totally umbilic. ■

Remark 3.1. Proposition 3.1 is essentially proved by Cheng [3].

Next the case of $H^2>c$ is investigated.

PROPOSITION 3.2. *Let M be an n-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{n+p}(c)$, $p\geqq 2$. If M is pseudo-umbilic and if $H^2>c$, then it satisfies*

$$nH^2\leqq S\leqq npH^2-n(p-1)c\,. \tag{3.14}$$

Proof. Since M is pseudo-umbilic by the assumption, we have $\mu=0$, which implies $S=\tau^2+nH^2$ by (3.6). This means that

$$\begin{aligned}\tau^2-n(p-1)(H^2-c)&=S-nH^2-n(p-1)(H^2-c)\\&=S+n(p-1)c-npH^2\,.\end{aligned}$$

By (3.13) we have

$$\frac{1}{2}\Delta S\geqq\frac{1}{p-1}(S-nH^2)\{S+n(p-1)c-npH^2\}. \tag{3.15}$$

Given any positive number a, a function F is defined by $F=1/\sqrt{S+a}$, which is bounded from above by $1/\sqrt{a}$ and is bounded from below by 0. Since the Ricci curvature of M is bounded from below and since M is complete and space-like, we can apply the Generalized Maximum Principle (Theorem 2.1) to the function F. For any given positive number $\varepsilon>0$, there exists a point p at which F satisfies

$$\inf F>F(p)-\varepsilon,\qquad |\operatorname{grad}F|(p)<\varepsilon,\qquad \Delta F(p)>-\varepsilon\,. \tag{3.16}$$

Consequently the following relationship

$$\frac{1}{2}F(p)^4\Delta S(p)<3\varepsilon^2+F(p)\varepsilon \tag{3.17}$$

can be derived by the simple and direct calculations. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m\to 0(m\to\infty)$ and $\varepsilon<0$, there exists a point sequence $\{p_m\}$ such that $\{F(p_m)\}$ converges to $F_0=\inf F$ by (3.16). On the other hand, it follows from (3.17) that we have

$$\frac{1}{2}F(p_m)^4\Delta S(p_m)<3\varepsilon_m{}^2+F(p_m)\varepsilon_m\,. \tag{3.18}$$

The right hand side of (3.18) converges to 0 because F is bounded. Accordingly, for any positive number $\varepsilon(<2)$ there exists a sufficiently large integer m_0 for which we have

$$F(p_m)^4\Delta S(p_m)<\frac{\varepsilon}{p-1}\qquad\text{for}\quad m>m_0\,.$$

This inequality and (3.15) yield

$$2\{S(p_m)-nH^2\}\{S(p_m)+n(p-1)c-npH^2\}<\{S(p_m)+a\}^2\varepsilon,$$

and hence we get

$$(2-\varepsilon)S^2(p_m)+2\{n(p-1)c-n(p+1)H^2-a\varepsilon\}S(p_m)$$
$$-2nH^2\{n(p-1)c-npH^2\}-a^2\varepsilon<0,$$

which implies that the sequence $\{S(p_m)\}$ is bounded. Thus the infimum F_0 of F satisfies $F_0\neq 0$ by the definition of F and hence the inequality (3.18) implies that $\limsup \Delta S(p_m)\leqq 0$. This means that the supremum $\sup S$ of the squared norm S satisfies

$$nH^2\leqq \sup S\leqq npH^2-n(p-1)c.$$

Remark 3.2. Let M be a maximal space-like submanifold of $H_{p-1}^{n+p-1}(c')$ and let $H_{p-1}^{n+p-1}(c')$ be a totally umbilic hypersurface of $H_p^{n+p}(c)$ $(0>c>c')$, whose mean curvature is denoted by H. Then M can be regarded as a submanifold of $H_p^{n+p}(c)$. It is a pseudo-umbilic submanifold with non-zero parallel mean curvature vector $\boldsymbol{h}$ and the squared norm S is given by $S=S'+nH^2$, where S' is denoted the squared norm of M in $H_{p-1}^{n+p-1}(c')$. According to Proposition 3.2, we have $S\leqq npH^2-n(p-1)c$ in $H_p^{n+p}(c)$. The last equality $S=npH^2-n(p-1)c$ is equivalent to $S'=n(p-1)(H^2-c)$. This is the second estimation of S' obtained by Ishihara [5].

4. 3-dimensional space-like submanifolds.

In this section, for a 3-dimensional space-like submanifold M we shall give a sufficient condition for M to be pseudo-umbilical. Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{3+p}(c)$. From (2.13) we have

$$\Delta h_{ij}^{\alpha}=-\sum h_{km}^{\alpha}R_{mijk}-\sum h_{mi}^{\alpha}R_{mkjk}+\sum h_{ik}^{\beta}R_{\beta\alpha jk} \tag{4.1}$$

for any indices α, i and j. By the similar discussion to that in Section 3 we choose e_4 in such a way that its direction coincides with that of the mean curvature vector. Furthermore, for any fixed point p in M we choose also a local frame field e_1, e_2, e_3 such that

$$h_{ij}^4=\lambda_i\delta_{ij} \tag{4.2}$$

for any i and j. By (4.1) we have

$$\frac{1}{2}\Delta|\mu|^2\geqq\sum(h_{ijk}^4)^2-\sum h_{ij}^4(h_{km}^4R_{mijk}+h_{mi}^4R_{mkjk}),$$

from which combining with (4.2) it follows that

$$\frac{1}{2}\Delta|\mu|^2\geqq\sum(h_{ijk}^4)^2+\frac{1}{2}\sum(\lambda_i-\lambda_j)^2R_{ijji}. \tag{4.3}$$

On the other hand, since M is a 3-dimensional submanifold, its Weyl conformal curvature tensor vanishes identically on M, i.e.,

$$R_{ijkl}=R_{il}\delta_{jk}-R_{ik}\delta_{jl}+\delta_{il}R_{jk}-\delta_{ik}R_{jl}-\frac{r}{2}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}).$$

Hence we get

$$R_{ijji}=R_{ii}+R_{jj}-\frac{r}{2}$$

for any distinct indices. By $R_{11}+R_{22}+R_{33}=r$, we have

$$R_{ijji}=\frac{r}{2}-R_{kk}$$

for any distinct indices. Thus the following equation

$$\frac{1}{2}\Delta|\mu|^2\geqq\Sigma(h^4_{ijk})^2+\frac{1}{2}\Sigma\left(\frac{r}{2}-R_{kk}\right)(\lambda_i-\lambda_j)^2 \tag{4.4}$$

is derived.

PROPOSITION 4.1. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{3+p}(c)$. If it satisfies*

$$Ric(M)\leqq\delta_1<3(c-H^2), \tag{4.5}$$

then M is pseudo-umbilic.

Proof. In order to prove this property it suffices to show that the function $|\mu|^2$ vanishes identically. By (4.4) and (4.5) we have

$$\frac{1}{2}\Delta|\mu|^2\geqq\frac{1}{4}\Sigma(r-2\delta_1)(\lambda_i-\lambda_j)^2,$$

which is equivalent to

$$\Delta|\mu|^2\geqq3(r-2\delta_1)|\mu|^2. \tag{4.6}$$

From (2.9) we have

$$\Delta|\mu|^2\geqq3|\mu|^2\{|\mu|^2+6(c-H^2)-2\delta_1\},$$

from which together with the assumption we have

$$\Delta|\mu|^2\geqq3|\mu|^4.$$

Since the Ricci curvature of M is bounded from below and M is complete and space-like and moreover since the function $|\mu|^2$ is smooth, Theorem 2.2 yields $|\mu|^2=0$, which means that M is pseudo-umbilic. ■

Remark 4.1. Proposition 4.1 is a higher codimensional version of a theorem due to Aiyama and Cheng [1] for a space-like hypersurface.

Proof of Theorem 1. Since the assumption of Theorem 1 satisfies (4.5), M is pseudo-umbilic by Proposition 4.1. Accordingly we can apply Proposition 3.1 to this case and we see that M is totally umbilic. ∎

Next we consider the case of $H^2>c$.

PROPOSITION 4.2. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{3+p}(c)$. If it satisfies $H^2>c$ and if*

$$Ric(M)\leqq\delta_1<\frac{3}{2}(p-3)(H^2-c), \tag{4.7}$$

then we get

$$|\mu|^2\leqq 3(p-1)(H^2-c). \tag{4.8}$$

Proof. From (2.9) the scalar curvature r is given by $r=6c-9H^2+S$ and hence we get by (3.6) and (4.7)

$$\begin{aligned} r-2\delta_1 &> |\mu|^2+\tau^2-6(H^2-c)-3(p-3)(H^2-c) \\ &\geqq |\mu|^2-3(p-1)(H^2-c). \end{aligned}$$

Accordingly (4.6) and the above inequality yield

$$\Delta|\mu|^2\geqq 3|\mu|^2\{|\mu|^2-3(p-1)(H^2-c)\}.$$

Given any positive number a, a function F is defined by $1/\sqrt{|\mu|^2+a}$. Then, by the similar method to that in the proof of Proposition 3.2, we obtain the conclusion. ∎

5. Proof of Theorem 2.

In this section Theorem 2 is proved. Let M be an $n(=3)$-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{n+p}(c)$, $p\geqq 2$. We assume $H^2\geqq c$ and

$$Ric(M)\leqq\delta_1<\frac{3}{2}(p-3)(H^2-c) \quad\text{and}\quad S\geqq 3pH^2-3(p-1)c. \tag{5.1}$$

Then the scalar curvature r is given by $r=3(p-3)(H^2-c)$ and hence

$$r-2\delta_1\geqq 3(p-3)(H^2-c)-2\delta_1=\delta$$

is a positive constant. From (4.6) we have

$$\Delta|\mu|^2\geqq 3\delta|\mu|^2. \tag{5.2}$$

Given any positive number a, a function F is defined by $F=1/\sqrt{|\mu|^2+a}$, which is bounded from above by $1/\sqrt{a}$ and is bounded from below by 0. Since the Ricci curvature of M is bounded from below and since M is complete and space-like, we can apply the Generalized Maximum Principle (Theorem 2.1) to the function F. For any given positive number ε, there exists a point p at which F satisfies (3.16). Consequently the following relationship

$$\frac{1}{2}F(p)^4\Delta|\mu|^2(p)<3\varepsilon^2+F(p)\varepsilon \tag{5.3}$$

can be derived by the simple and direct calculations. For any convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m\to 0$ $(m\to\infty)$ and $\varepsilon_m>0$, there exists a point sequence $\{p_m\}$ such that $\{F(p_m)\}$ converges to $F_0=\inf F$ by (3.16). On the other hand, it follows from (5.3) that we have

$$\frac{1}{2}F(p_m)^4\Delta|\mu|^2(p_m)<3\varepsilon_m{}^2+F(p_m)\varepsilon_m. \tag{5.4}$$

The right hand side of (5.4) converges to 0, because the function F is bounded. Accordingly, for any positive number ε there exists a sufficiently large integer m_0 for which we have

$$F(p_m)^4\Delta|\mu|^2(p_m)<\varepsilon \qquad \text{for} \quad m>m_0. \tag{5.5}$$

Since it is seen by Proposition 4.1 that the function $|\mu|^2$ is bounded, the infimum F_0 of the function F satisfies $F_0\neq 0$ and hence the inequality (5.5) yields that $\limsup\Delta|\mu|^2(p_m)\leqq 0$. This means that the supremum of $|\mu|^2$ is equal to 0 by (5.2), because δ is the positive constant. So we obtain $\mu=0$, i.e., M is pseudo-umbilic, which yields that the equality of (3.14) in Proposition 3.2 holds. Then the equalities of all inequalities in Section 3 have to hold. Consequently, from (3.4) and (3.13) it is seen that we have

$$h_{ijk}^{\alpha}=0 \tag{5.6}$$

for any i, j, k and α. Also from (3.2) and (3.11) it follows that we get

$$H^{\alpha}H^{\beta}=H^{\beta}H^{\alpha} \tag{5.7}$$

for any α and β. The equations imply that all of H^{α} are simultaneously diagonalizable and the normal connection in the normal bundle of M is flat. Hence we can choose a suitable basis $\{e_i\}$ such that

$$h_{ij}^{\alpha}=\lambda_i^{\alpha}\delta_{ij} \tag{5.8}$$

for any i, j and α. The submanifold M is said to be *isoparametric* [9] if the normal connection is flat and the charactristic polynomial of the shape operator A_ξ for any local parallel normal field ξ is constant over the domain.

LEMMA 5.1. *M is isoparametric.*

Proof. Since the normal connection is flat, it is seen that there exist locally p mutually orthogonal unit normal vector fields which are parallel in the normal bundle. So we can choose a suitable parallel basis $\{e_\alpha\}$ and then we have $\omega_{\alpha\beta}=0$. Hence, since we have

$$\sum h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} - \sum h_{kj}^{\alpha}\omega_{ki} - \sum h_{ik}^{\alpha}\omega_{kj} + \sum h_{ij}^{\beta}\omega_{\beta\alpha}, \tag{5.9}$$

setting $i=j$ in the above equation and using (5.6) we get $dh_{ii}^{\alpha}=0$. Hence h_{ii}^{α} is constant and M is isoparametric. ■

LEMMA 5.2. *M is of non-positive curvature.*

Proof. Suppose first that there exist indices i, j and α such that $h_{ii}^{\alpha}\neq h_{jj}^{\alpha}$. From the equation (5.9) we get

$$\sum h_{kj}^{\alpha}\omega_{ki} + \sum h_{ik}^{\alpha}\omega_{kj} = (h_{ii}^{\alpha} - h_{jj}^{\alpha})\omega_{ij} = 0,$$

from which it follows that $\omega_{ij}=0$. For any index i we denote by $[i]$ the set of indices k such that $h_{kk}^{\alpha}=h_{ii}^{\alpha}$. Under this notation the above property shows

$$\omega_{ik}=0 \qquad \text{for any} \quad k\notin[i]. \tag{5.10}$$

Accordingly, we obtain

$$\sum \omega_{ik}\wedge\omega_{kj}=0.$$

In fact, the left hand side of the above equation can be regarded as

$$\sum \omega_{ik}\wedge\omega_{kj} = \sum_{k\in[i]} \omega_{ik}\wedge\omega_{kj} + \sum_{k\in[j]} \omega_{ik}\wedge\omega_{kj} + \sum_{k\notin[i]\cup[j]} \omega_{ik}\wedge\omega_{kj},$$

each term of which vanishes identically, because of (5.10). Thus, from the structure equation

$$d\omega_{ij} + \sum \omega_{ik}\wedge\omega_{kj} = -\frac{1}{2}\sum R_{kijl}\omega_k\wedge\omega_l,$$

we obtain

$$R_{ijji} = c - \sum_{\beta}\lambda_i^{\beta}\lambda_j^{\beta} = 0.$$

Next, suppose that $h_{ii}^{\alpha}=h_{jj}^{\alpha}$ for distinct indices i and j and for any α. Then the Gauss equation implies

$$R_{ijji} = c - \sum_{\alpha}(h_{ii}^{\alpha})^2 = c - \sum_{\alpha}(\lambda_i^{\alpha})^2 = c - H^2 - \sum_{\alpha\neq 4}(\lambda_i^{\alpha})^2 \leqq 0,$$

because of $H^2-c\geqq 0$.

Thus M is of non-positive curvature. ■

Proof of Theorem 2. First of all, we notice that M is not totally umbilic under the condition (5.1). In fact, suppose that M is totally umbilic. The equation (3.6) means that M is totally umbilic if and only if $S=nH^2$, from which combining with the second equation of (5.1) it follows that we have $H^2=c=0$. So M is totally geodesic and it satisfies $Ric(M)=0$. On the other hand, by the first equation of (5.1), we get $Ric(M)<0$, a contradiction.

Now we consider an n-dimensional space-like submanifold M of $\boldsymbol{R}_p^{n+p}$. By a theorem due to Koike [6] and Lemmas 5.1 and 5.2 it is seen that M is locally congruent to the product submanifold

$$H^{n_1}(c_1)\times \cdots \times H^{n_q}(c_q)\times \boldsymbol{R}^m \tag{5.11}$$

of $\boldsymbol{R}_q^{n+q}$ whose mean curvature vector is parallel in the normal bundle of M in $\boldsymbol{R}_q^{n+q}$, where $\sum_{r=1}^q n_r+m=n$, $q\geqq 0$, $m\geqq 0$ and $\boldsymbol{R}_q^{n+q}$ is a totally geodesic submanifold of $\boldsymbol{R}_p^{n+p}$. Then M can be naturally regarded as the space-like submanifold of $\boldsymbol{R}_p^{n+p}$.

The condition for the codimension is next given. For the purpose the squared norm S of the second fundamental form and the mean curvature H of M in $\boldsymbol{R}_q^{n+q}$ and hence in $\boldsymbol{R}_p^{n+p}$ are calculated. In fact, the product manifold is constructed as follows: Without loss of generality, an $(n+q)$-dimensional semi-Euclidean space $\boldsymbol{R}_q^{n+q}$ of index $q\geqq 0$ can be first regarded as a product manifold of

$$\boldsymbol{R}_1^{n_1+1}\times \cdots \times \boldsymbol{R}_1^{n_q+1}\times \boldsymbol{R}^m,$$

where $\sum_{r=1}^q n_r+m=n$. With respect to the standard orthonormal basis of $\boldsymbol{R}_q^{n+q}$ a class of space-like submanifolds

$$H^{n_1}(c_1)\times \cdots \times H^{n_q}(c_q)\times \boldsymbol{R}^m$$

of $\boldsymbol{R}_q^{n+q}$ is defined as the Pythagorean product

$$\begin{aligned} &H^{n_1}(c_1)\times \cdots \times H^{n_q}(c_q)\times \boldsymbol{R}^m \\ &\quad =\left\{(x_1, \cdots, x_{q+1})\in \boldsymbol{R}_q^{n+q}=\boldsymbol{R}_1^{n_1+1}\times \cdots \times \boldsymbol{R}_1^{n_q+1}\times \boldsymbol{R}^m:\ |x_r|^2=-\frac{1}{c_r}>0\right\}, \end{aligned}$$

where $r=1, \cdots, q$ and $|\ |$ denotes the norm defined by the product on the Minkowski space $\boldsymbol{R}_1^{k+1}$ which is given by $\langle x, x\rangle=-(x_0)^2+\sum_{j=1}^k (x_j)^2$. The mean curvature vector $\boldsymbol{h}$ of M in $\boldsymbol{R}_q^{n+q}$ and hence in $\boldsymbol{R}_p^{n+p}$ is given by

$$\boldsymbol{h}=-\frac{1}{n}(n_1c_1x_1+\cdots+n_qc_qx_q)$$

at $x=(x_1, \cdots, x_{q+1})\in M$, which is parallel in the normal bundle of M. So, the squared norm S of the second fundamental form and the mean curvature H of M in $\boldsymbol{R}_p^{n+p}$ are given by

$$S=-\sum_{r=1}^q n_rc_r, \qquad n^2H^2=-\sum_{r=1}^q n_r{}^2c_r,$$

which yields

$$S-pnH^2=\frac{1}{n}\sum_{r=1}^q n_r(pn_r-n)c_r=0. \tag{5.12}$$

Suppose that $p\leqq 3$. Then we see $Ric(M)\leqq\delta_1<0$ by (5.1). Since M is 3-dimensional and it is congruent to the product submanifold (5.11), the negative definiteness of the Ricci curvature means that M is totally umbilic, a contradiction. We next suppose $p\geqq 4$. This means that M is totally umbilic by (5.12), a contradiction.

Hence the case of $c=0$ can not occur.

Suppose next that $c<0$. By means of Koike's theorem and Lemmas 5.1 and 5.2 again, M is locally congruent to the product submanifold $H^{n_1}(c_1)\times\cdots\times H^{n_{q+1}}(c_{q+1})$ in $H_q^{n+q}(c')$, where $\sum_{r=1}^{q+1}n_r=n$, $q\geqq 0$, and $\sum_{r=1}^{q+1}(1/c_r)=(1/c')\geqq(1/c)$, and $H_q^{n+q}(c')$ is a totally umbilic submanifold of $H_p^{n+p}(c)$.

We investigate the relation between the mean curvature H and the squared norm S of M in $H_p^{n+p}(c)$. We consider an n-dimensional space-like submanifold with parallel mean curvature vector of $H_q^{n+q}(c')$. Without loss of generality, an $(n+q+1)$-dimensional indefinite Euclidean space $\boldsymbol{R}_{q+1}^{n+q+1}$ of index $(q+1)$ can be regarded as a product manifold of

$$\boldsymbol{R}_1^{n_1+1}\times\cdots\times\boldsymbol{R}_1^{n_{q+1}+1}$$

where $\sum_{r=1}^{q+1}n_r=n$. With respect to the standard orthonormal basis of $\boldsymbol{R}_{q+1}^{n+q+1}$ a class of space-like submanifolds

$$H^{n_1}(c_1)\times\cdots\times H^{n_{q+1}}(c_{q+1}) \tag{5.13}$$

of $\boldsymbol{R}_{q+1}^{n+q+1}$ is defined as the Pythagorean product

$$\begin{aligned}&H^{n_1}(c_1)\times\cdots\times H^{n_{q+1}}(c_{q+1})\\&\quad=\left\{(x_1,\cdots,x_{q+1})\in\boldsymbol{R}_{q+1}^{n+q+1}=\boldsymbol{R}_1^{n_1+1}\times\cdots\times\boldsymbol{R}_1^{n_{q+1}+1}:\ |x_r|^2=-\frac{1}{c_r}>0\right\},\end{aligned}$$

where $r=1,\cdots,q+1$. The mean curvature vector $\boldsymbol{h}'$ of M in $H_q^{n+q}(c')$ is given by

$$\boldsymbol{h}=-\frac{1}{n}\sum_{r=1}^{q+1}(n_rc_rx_r)-c'x$$

at $x=(x_1,\cdots,x_{q+1})\in M$, which is parallel in the normal bundle of M in $H_q^{n+q}(c')$. So the mean curvature H' and the squared norm S' of the second fundamental form of M in $H_q^{n+q}(c')$ are given by

$$n^2H'^2=n^2c'-\sum_{r=1}^{q+1}n_r{}^2c_r,\qquad S'=nc'-\sum_{r=1}^{q+1}n_rc_r\,. \tag{5.14}$$

For the mean curvature vector $\boldsymbol{h}'$ of M in $H_q^{n+q}(c')$ the mean curvature vector $\boldsymbol{h}$ of M in $H_p^{n+p}(c)$ is given by $\boldsymbol{h}=\boldsymbol{h}'+\boldsymbol{h}''$, where $\boldsymbol{h}''$ is the mean curvature vector of $H_q^{n+q}(c')$ in $H_p^{n+p}(c)$. Consequently, by using (5.14) the mean curvature H and the squared norm S of M in $H_p^{n+p}(c)$ are given by

$$n^2H^2=n^2c'-\sum_{r=1}^{q+1}n_r{}^2c_r+(p-q)^2(c-c'),$$

$$S=nc-\sum_{r=1}^{q+1} n_r c_r+(p-q)(c-c'),$$

from which it follows that we have

$$\text{(5.15)}\qquad S-\{npH^2-n(p-1)c\}$$
$$=\frac{1}{n}\sum_{r=1}^{q+1} n_r(pn_r-n)c_r+\left\{(p-q)+pn-\frac{p}{n}(p-q)^2\right\}(c-c').$$

Suppose that $p\leqq 3$. Then we see $Ric(M)<0$ by (5.1). Since M is congruent to the product manifold (5.13) and it is 3-dimensional, the negative definiteness of the Ricci curvature means that M is totally umbilic, a contradiction. Accordingly, we obtain $p\geqq 4$. On the other hand, q must be less than 3, because of $n=3$. In order to check whether or not these situations occur, it is divided into three cases: $q=0$, 1 and 2.

First we consider the case $q=0$. Then M is totally umbilic, a contradiction.

Next we consider the case $q=1$. If $p\geqq 5$, then the first term of the right hand side in (5.15) is negative and the second one is of non-positive. This also leads a contradiction. So we have $p=4$ and c_1 and c_2 are determined by constant curvatures c and c', because of $1/c_1+1/c_2=1/c'$.

The case $q=2$. If $p\geqq 6$, then the first term of the right hand side in (5.15) is negative and the second one is of non-positive. Accordingly this case can not occur. So we have $p=4$ or $p=5$.

This completes the proof. ■

Remark 5.1. A product manifold $H^1(c_1)\times H^1(c_2)\times H^1(c_3)$ is a canonical space-like submanifold with parallel mean curvature vector of $H_3^6(c)$ and it satisfies $Ric(M)=0$ and $S=9H^2-6c$. This means that the estimate of the Ricci curvature is best possible.

Remark 5.2. In the case of $p=1$, two conditions in (5.1) are equivalent with each other.

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