THOM'S CONJECTURE ON SINGULARITIES
OF GRADIENT VECTOR FIELDS

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1. Introduction.

In [3], R. Thom gave the following conjecture.

CONJECTURE. *Let $f(x)$ be a germ of real analytic function at the origin $0 \in R^n$ and let $X = \text{grad} f(x)$ be the gradient vector field of $f(x)$ with respect to the ordinary Riemannian metric on R^n. If an integral curve $g(t)$ of X tends to the origin $0 \in R^n$, then there exists a unique tangential direction $\lim_{t \to +\infty} g(t)/|g(t)|$.*

Thom proved the case where $f(x)$ is a homogeneous polynomial and for the general case he gave an outline of a proof. In this paper, we give a partial answer to the above problem. The essential idea of our proof is the same as Thom's one (see [3]).

Let $f(x): (R^n, 0) \to (R, 0)$ be a germ of analytic function. And we express $f(x)$ in the form

$$f(x) = P_k(x) + P_{k+1}(x) + \cdots + P_m(x) + \cdots$$

where $P_m(x)$ is a homogeneous polynomial of degree m.
We define the *cone spectrum* $Sp(P_m)$ as follows:

$$Sp(P_m) = \left\{ x = (x_1, \cdots, x_n) \in R^n ; \ x_i \frac{\partial P_m}{\partial x_j} = x_j \frac{\partial P_m}{\partial x_i} \ \ i, j = 1, \cdots, n \right\}.$$

Obviously, $Sp(P_m)$ is a cone algebraic set and it contains $0 \in R^n$.
In this paper, we prove the following theorem.

THEOREM. *Let $f(x) = P_k(x) + P_{k+1}(x) + \cdots$ be a real analytic function germ at $0 \in R^n$. If $\dim Sp(P_k) \leq 1$, then any integral curve of $\text{grad} f(x)$ which tends to $0 \in R^n$ has a unique tangential direction at the origin.*

Remark. We see later on that the condition $\dim Sp(P_k) \leq 1$ is equivalent to that the restricted function $P_k|_{S^{n-1}}$ of $P_k(x)$ to the unit sphere S^{n-1} has only isolated singularities. Thus, the above condition is a generic property on the initial term of $f(x)$.

COROLLARY. *In the two dimensional case, Thom's conjecture holds.*

2. Lojasiewicz's Theorem and Blowing up of vector field.

The proof of our theorem is based on two important theorems. One is Lojasiewicz's theorem on analytic gradient vector fields and the other is Takens's blowing up construction of singularities of vector fields.

Now, for its importance we start with recalling them.

LEMMA 2.1. *Let $f(x)$ be a real analytic function defined on a neighbourhood U of $a \in R^n$ and $f(a)=0$. Then, there exists $0 < \theta < 1$ such that*

$$|\operatorname{grad} f(x)| \geq |f(x)|^\theta$$

in some neibourhood of $a \in R^n$.

The proof can be found in Lojasiewicz [1] pp 92.

THEOREM 2.2. (*Lojasiewicz*) *Let $f(x)$ be a real analytic function defined on a neibourhood U of R^n and let A denote the set $f^{-1}(0) \cap U$. If an integral curve $g(t)$ of $\operatorname{grad} f(x)$ tends toward A, then $g(t)$ tends to a unique point of A.*

Proof. Let $g(t)=(g_1(t), g_2(t), \cdots, g_n(t))$ denote the integral curve of $\operatorname{grad} f(x)$ with $g(0)=x$, $x \in U$ i.e.

$$\frac{dg}{dt}(t) = \left(\frac{\partial f}{\partial x_1}(g(t)), \cdots, \frac{\partial f}{\partial x_n}(g(t)) \right) \quad \text{and} \quad g(0)=x.$$

Now, easily we have

$$\frac{d}{dt} f(g(t)) = \frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial t}$$

$$= |\operatorname{grad} f(g(t))|^2 \geq 0. \tag{1}$$

On the other hand. if $g(t)$ tends to a point $a \in A$, then from Lemma 2.1, there exists $0 < \theta < 1$ such that

$$|\operatorname{grad} f(x)| \geq |f(x)|^\theta$$

in some neibourhood of $a \in U$.

Then, the length of integral curve $g(t)$ from $t=0$ to $t=m$ is estimated as follows:

$$\int_0^m |\operatorname{grad} f(g(t))| \, dt = \int_0^m \frac{(d/dt)f(g(t))}{|\operatorname{grad} f(g(t))|} \, dt$$

$$\leq \int_0^m \frac{(d/dt)f(g(t))}{|f(g(t))|^\theta} \, dt$$

$$= \frac{-1}{1-\theta} \int_0^m \frac{d}{dt} [(-f(g(t))^{1-\theta}] dt$$

$$= \frac{1}{1-\theta} [(-f(x))^{1-\theta} - (-f(g(m))^{1-\theta}]$$

$$< \frac{1}{1-\theta} (-f(x))^{1-\theta} < \infty .$$

Here, from (1) we note that $f(g(t))$ is an increasing function and $\lim_{t\to+\infty} f(g(t)) = f(a) = 0$, thus $f(g(t)) < 0$. If the ω-limit set of $g(t)$ contains two or more points, then the length of integral curve $g(t)$ must be ∞. This contradicts the above estimation. □

From Theorem 2.2, we can easily obtain the following.

PROPOSITION 2.3. *Let M^n be a real analytic Riemannian manifold of dimension n and $f : M^n \to R$ be a real analytic function on M^n. Then, every integral curve $g(t)$ of $\mathrm{grad}\, f(x)$ has unique α-limit and ω-limit points. Moreover, the points $\lim_{t\to+\infty} g(t)$ and $\lim_{t\to-\infty} g(t)$ are singular points of $f(x)$.*

Next, we recall the blowing-up construction of vector field. For more details see Takens [2].

THEOREM 2.4. (*Takens [2]*) *Let X be a C^∞-vector field on R^n with $X(0) = 0$. Let $\Phi : S^{n-1} \times R \to R^n$ be a C^∞-mapping defined by $\Phi(\bar{x}_1, \cdots, \bar{x}_n, r) = (r\bar{x}_1, \cdots, r\bar{x}_n)$ where $(\bar{x}_1, \cdots, \bar{x}_n)$ with $\sum_{i=1}^n \bar{x}_i^2 = 1$ is the coordinate system of S^{n-1}. Then there exists a C^∞-vector field $\tilde{X}$ such that $\Phi_*(\tilde{X}) = X$.*

For the purpose of our proof, we repeat shortly the outline of Takens's proof.

Proof. From a direct calculation we have

$$\left(\sum_{i=1}^n x_i^2\right) X = \langle R, X \rangle R + 2 \sum_{i,j=1}^n \langle V_{ij}, X \rangle V_{ij} \tag{2}$$

where R, V_{ij} are the vector fields on R^n given by

$$R = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}, \qquad V_{ij} = \frac{1}{2}\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right)$$

and $\langle\ ,\ \rangle$ denotes the inner product of R^n.

We define the vector fields $\tilde{R}$ and $\tilde{V}_{ij}$ on $S^{n-1} \times R$ by

$$\tilde{R} = r \frac{\partial}{\partial r}, \qquad \tilde{V}_{ij} = \frac{1}{2}\left(\bar{x}_i \frac{\partial}{\partial \bar{x}_j} - \bar{x}_j \frac{\partial}{\partial \bar{x}_i}\right).$$

Then, we have

$$\Phi_*(\tilde{R}) = R \quad \text{and} \quad \Phi_*(\tilde{V}_{ij}) = V_{ij}$$

Now, it is clear that the vector field

$$\tilde{X} = \frac{1}{r^2}\Big[(\langle R, X\rangle \circ \Phi)\hat{R} + 2\sum_{i,j=1}^{n}(\langle V_{ij}, X\rangle \circ \Phi)\tilde{V}_{ij}\Big]$$

satisfies the required condition $\Phi_*(\tilde{X}) = X$. □

Remark. If the $(k-1)$-jet of X at the origin equals 0, in other words the degree of initial term of X is k, then we set

$$\bar{X} = \frac{1}{r^{k+1}}\Big[(\langle R, X\rangle \circ \Phi)\tilde{R} + 2\sum_{i,j=1}^{n}(\langle V_{ij}, X\rangle \circ \Phi)\tilde{V}_{ij}\Big].$$

Then $\bar{X}$ is also C^∞-vector field on $S^{n-1} \times R$ and the integral curves of $\tilde{X}$ and $\bar{X}$ coincide as sets. Thus the ω-limit sets of $\tilde{X}$ and $\bar{X}$ coincide. We say that $\bar{X}$ is the *blowing-up* of X at $0 \in R^n$.

3. The proof of theorem.

Now, let $f(x) = P_k + P_{k+1} + \cdots : (R^n, 0) \to (R, 0)$ be a real analytic function germ. Then the blowing-up vector field $\bar{X}$ of grad $f(x)$ is given by:

$$\bar{X} = \frac{1}{r^k}\Big[\langle R, \operatorname{grad} f(x)\rangle \circ \Phi)\hat{R} + 2\sum_{i,j=1}^{n}(\langle V_{ij}, \operatorname{grad} f(x)\rangle \circ \Phi)\tilde{V}_{ij}\Big].$$

Then the restriction of vector field $\bar{X}$ to $S^{n-1} \times \{0\}$ is given by

$$2\sum_{i,j=1}^{n} \langle V_{ij}, \operatorname{grad} P_k\rangle \tilde{V}_{ij}. \tag{3}$$

We denote the above vector field (3) by $\bar{X}_0$.

LEMMA 3.1. *Let* $f(x) = P_k + P_{k+1} + \cdots$ *and* $\bar{X}$ *and* $\bar{X}_0$ *be as above. Then*

$$\bar{X}_0 = \operatorname{grad}(P_k|_{S^{n-1}})$$

where S^{n-1} *has the ordinary Riemannian metric.*

Remark. The coordinate systems of the both sides of above equation are different, but there will be no confusion.

Proof. Let TS^{n-1} denote the tangent space of S^{n-1}. From (2) we easily see that the TS^{n-1} component of $(\operatorname{grad} P_k)|_{S^{n-1}}$ is given by (3).

Let $\partial/\partial t$ denote the unit vector on R. Then for any vector $v \in TS^{n-1}$, we have

$$\langle \bar{X}_0, v\rangle \frac{\partial}{\partial t} = \langle TS^{n-1} \text{ component of } (\operatorname{grad} P_k)|_{S^{n-1}}, v\rangle \frac{\partial}{\partial t}$$

$$= \langle \operatorname{grad}(P_k|_{S^{n-1}}), v\rangle \frac{\partial}{\partial t}$$

$$=(P_k)_* v$$

$$=(P_k|_{S^{n-1}})_* v \,.$$

Therefore, we have

$$\bar{X}_0 = \mathrm{grad}\,(P_k|_{S^{n-1}}) \,. \quad \square$$

Proof of Theorem. Let $g(t)$ be an integral curve of $\mathrm{grad}\,f(x)$ such that $\lim_{t\to+\infty} g(t)=0$ and let L denote the ω-limit set of $\lim_{t\to+\infty} g(t)/|g(t)|$. Let $\tilde{g}(t)$ be the integral curve of the blowing-up vector field $\bar{X}$ with $\tilde{g}(0)=\Phi^{-1}(g(0))$. Then we easily see that

$$L \times \{0\} = \text{the } \omega-\text{limit set of } \tilde{g}(t) \text{ and } L \times \{0\} \subset S^{n-1} \times \{0\} \,.$$

From the elementary general theory of dynamical systems, we easily see that $L \times \{0\}$ is a connected closed set and an invariant set by the flow of $\bar{X}_0$.
Now, at the point $x \in Sp(P_k) \cap S^{n-1}$ the following holds:

$$x_i \frac{\partial P_k}{\partial x_j} = x_j \frac{\partial P_k}{\partial x_i} \qquad \text{for any } i, j = 1, 2, \cdots, n \,.$$

Thus, the position vector $\vec{x}$ and $\mathrm{grad}\,P_k(x)$ are parallel i.e. $\mathrm{grad}\,P_k(x)$ has no TS^{n-1} components. Therefore x is a singular point of $P_k|_{S^{n-1}}$ and the condition $\dim Sp(P_k) \leq 1$ means that the singularities of $P_k|_{S^{n-1}}$ (or equivalently the singularities of $\bar{X}_0 = \mathrm{grad}\,(P_k|_{S^{n-1}})$ are finite and isolated. Since $L \times \{0\}$ is a connected set, it is enough to prove that $L \times \{0\}$ does not contain regular points of $\bar{X}_0$. Then $L \times \{0\}$ is a one point set and L will give the tangential direction $\lim_{t\to+\infty} g(t)/|g(t)|$.
Now, we suppose that $L \times \{0\}$ contains a regular point p_1 of $\bar{X}_0$. Let $\tilde{g}_{p_1}(t)$ be the integral curve of $\bar{X}_0$ with $\tilde{g}_{p_1}(0)=p_1$. Since $L \times \{0\}$ is invariant by the flow of $\bar{X}_0$, we have

$$\tilde{g}_{p_1}((-\infty, +\infty)) \subset L \times \{0\} \,.$$

From Proposition 2.3 there exist unique points $\lim_{t\to-\infty} \tilde{g}_{p_1}(t)$ and $\lim_{t\to+\infty} \tilde{g}_{p_1}(t)$, we set $q_1 = \tilde{g}_{p_1}(-\infty)$ and $q_2 = \tilde{g}_{p_1}(+\infty)$. Then q_1, q_2 are singular points of $P_k|_{S^{n-1}}$. Since $L \times \{0\}$ is a closed set, we have $q_1, q_2 \in L \times \{0\}$.
Now, we take small upper half n-disks $D_{q_1}^+$ and $D_{q_2}^+$ centered q_1 and q_2 on $S^{n-1} \times R^+$ such that $D_{q_1}^+ \cap S^{n-1} \times \{0\}$ and $D_{q_2}^+ \cap S^{n-1} \times \{0\}$ contain no singular points of $\bar{X}_0$ except q_1, q_2, here R^+ denotes the set of non-negative real numbers. Next we take a flow box W_1 of $\bar{X}$ on $S^{n-1} \times R^+$ which contains $\tilde{g}_{p_1}(t)$ and joins $D_{q_1}^+$ to $D_{q_2}^+$ (see Fig. 1.).
Since $p_1 \in L \times \{0\}$, there exist $t_1 < t_2 < \cdots < t_i < \cdots$ such that

$$\lim_{i\to+\infty} t_i = +\infty, \quad \lim_{i\to+\infty} \tilde{g}(t_i) = p_1 \text{ and } \tilde{g}(t_i) \in W_1 \text{ for any } i = 1, 2, \cdots \,.$$

Let u_i be the point of the boundary $\partial D_{q_2}^+$ at which the integral curve $\tilde{g}(t)$ started from $\tilde{g}(t_i)$ leaves $D_{q_2}^+$ for the first time. Since $\partial D_{q_2}^+$ is compact, the set $\{u_i\}$ has

the accumulation points. Let p_2 be one of them. Then $p_2 \in S^{n-1} \times \{0\}$ and $p_2 \in L \times \{0\}$, because if $p_2 \notin S^{n-1} \times \{0\}$ then the integral curve $g(t)$ arrives at two different points 0 and $\Phi(p_2)$ but this contradicts Lojasiewicz's theorem.
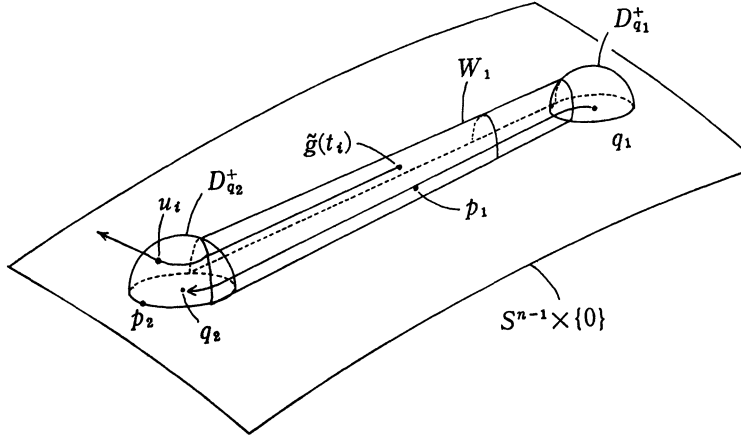

Fig. 1.

Obviously

$$p_2 \in L \times \{0\}, \quad \bar{X}_0(p_2) \neq 0 \quad \text{and} \quad \tilde{g}_{p_2}((-\infty, +\infty)) \subset L \times \{0\}.$$

We set $q = \lim_{t \to -\infty} \tilde{g}_{p_2}(t)$ and $q_3 = \lim_{t \to +\infty} \tilde{g}_{p_2}(t)$, then we see that $q = q_2$. Because if $q \neq q_2$, we take the upper small half n-disks D_q^+, $D_{q_3}^+$ and the flow box W_2 in the same way as $D_{q_1}^+$, $D_{q_2}^+$ and W_1 with $D_q^+ \cap D_{q_2}^+ = \emptyset$, $p_2 \in W_2$ and $W_1 \cap W_2 = \emptyset$ (see Fig. 2.). Since $u_i \in W_2$, the integral curve $\tilde{g}(t)$ which starts from $\tilde{g}(t_i)$ must pass through a point of $\partial D_{q_2}^+$ different from u_i and must go into D_q^+ and W_2 before it arrives at u_i. But this contradicts the definition of u_i.

Now, we have the arc $\overbrace{q_1 q_2 q_3}$ which consists of integral curves $\tilde{g}_{p_1}(t)$ and $\tilde{g}_{p_2}(t)$ of $\bar{X}_0$. If $\overbrace{q_1 q_2 q_3}$ does not contain a loop, then we repeat the above argument for $D_{q_3}^+$ and costruct the arc $\overbrace{q_1 q_2 q_3 q_4}$ and so on. Since the singular points set of $\bar{X}_0$ is finite, by the above finite constructions we have the arc $\overbrace{q_1 q_2 \cdots q_s}$ which contain a loop γ. However, $\bar{X}_0$ is the gradient vector field of $P_k |_{S^{n-1}}$, thus the function $P_k |_{S^{n-1}}$ increases along the integral curve of $\bar{X}_0$. This contradics the existance of loop γ. This completes the proof of Theorem. $\square$

LEMMA 3.2. *Let $P_k(x)$ be a non-zero homogeneous polynomiol of degree k on R^n. If $\dim S_p(P_k) = n$, then k is even and $P_k(x) = \alpha r^k$ where $\alpha \in R$ and $r = \sqrt{x_1^2 + \cdots + x_n^2}$.*

Proof. In the proof of theorem we see that $S_p(P_k) \cap S^{n-1}$ is the singular point set of $P_k |_{S^{n-1}}$. Since $\dim S_p(P_k) = n$, we have $\dim S_p(P_k) \cap S^{n-1} = n-1$ *i.e.* $P_k |_{S^{n-1}}$ is a constant function. If k is odd then $P_k(-x) = -P_k(x)$. Thus $P_k |_{S^{n-1}} \equiv 0$ but this contradicts that $P_k(x)$ is non-zero. Hence k is even. We set $\alpha \equiv$
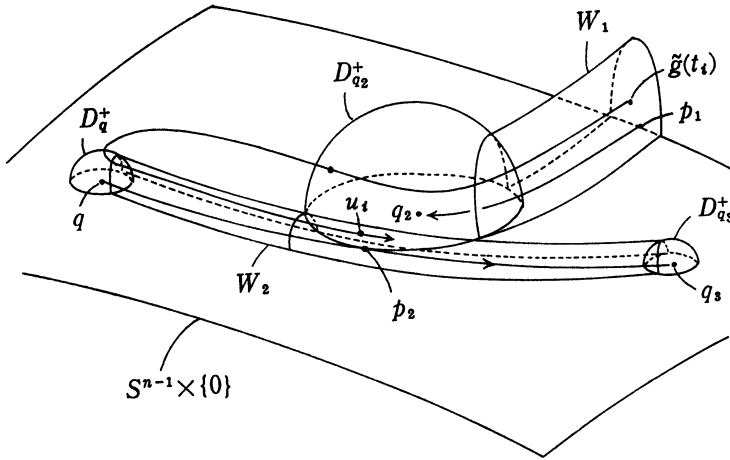
Fig. 2.

$P_k|_{S^{n-1}}$. Then we have that for any $x \in R^n$

$$P_k(x)=P_k\Big(|x|\frac{x}{|x|}\Big)=|x|^k P_k\Big(\frac{x}{|x|}\Big)=\alpha r^k \quad \square$$

PROPOSITION 3.3.   Let $f(x)=P_k(x)+P_{k+1}(x)+\cdots$ be a real analytic function germ at $0 \in R^n$. If $\dim S_p(P_k)=n$, then any integral curve $g(t)$ of $\operatorname{grad} f(x)$ which tends to $0 \in R^n$ has a unique tangential direction at the origin.

Proof.   Let $\bar{X}$ and $\bar{X}_0$ be as above.   Then, $\bar{X}_0$ is given by (3).   From Lemma 3.2 we see that $\operatorname{grad} P_k=k\alpha r^{k-2}\sum_{i=1}^{n} x_i \partial/\partial x_i$ and we have $\bar{X}_0 \equiv 0$.   Set $\bar{\bar{X}}=(1/r)\bar{X}$.   Then $\bar{\bar{X}}$ is also a C^∞ vector field on $S^{n-1}\times R$ and $\bar{\bar{X}}$ no singular points on $S^{n-1}\times\{0\}$.   For $x=g(0)$ the integral curve of $\bar{\bar{X}}$ started from $\varPhi^{-1}(x)$ meets $S^{n-1}\times\{0\}$ at a unique point of $S^{n-1}\times\{0\}$.   This point gives the tangential direction $\lim_{t \to +\infty} g(t)/|g(t)|$. $\square$

From Theorem and Proposition 3.3 we obtain Corollary.

REFERENCES

[1]   S. LOJASIEWICZ,   Ensembles semi-analytiques, I.H.E.S. Lecture Notes (1965).
[2]   F. TAKENS,   Singularities of vector fields, Publ. Math. I.H.E.S. 43 (1973), 47–100.
[3]   R. THOM,   Gradients des fonctions analytiques, preprint (1986).

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