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SUFFICIENT CONDITIONS FOR UNIMODALITY OF NON-SYMMETRIC LÉVY PROCESSES

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1. Introduction and results.

A measure μ on R^1 is said to be *unimodal* with mode a if $\mu(dx)=c\delta_a(dx)+f(x)dx$, where $c\geqq 0$, δ_a is the delta measure at a, $f(x)$ is non-decreasing for $x<a$ and non-increasing for $x>a$. A probability measure μ on R^1 is said to be *strongly unimodal* if, for every unimodal probability measure η, the convolution $\mu*\eta$ is unimodal. Let $\{X_t\}$ $(t\geqq 0)$ be a Lévy process on R^1 (that is a process with stationary independent increments starting at the origin). The process $\{X_t\}$ is said to be of class L if the distribution μ_t of X_t is of class L for every $t>0$ (equivalently, for some $t>0$). A necessary and sufficient condition for an infinitely divisible distribution μ with Lévy measure ν to be of class L is that $|x|\nu(dx)$ is unimodal with mode 0. The process $\{X_t\}$ is said to be unimodal if the distribution μ_t is unimodal for every $t>0$. Medgyessy [1] and Wolfe [13] show that symmetric Lévy processes are unimodal if and only if their Lévy measures are unimodal with mode 0. Yamazato [14] proves that every process of class L is unimodal. Watanabe [8] shows that there exist unimodal non-symmetric Lévy processes that are not of class L. Also, Watanabe [10] gives a necessary and sufficient condition for unimodality of one-sided Lévy processes by using zeros of some polynomials. However it has not been successful to find a necessary and sufficient condition in terms of their Lévy measures. Other results on the unimodality of Lévy processes are obtained by Sato [2, 3], Sato-Yamazato [4], Steutel-van Harn [6], Watanabe [9, 11], Wolfe [12], and Yamazato [15]. The purpose of this paper is to improve the previous paper [8] and to give sufficient conditions for unimodality of non-symmetric Lévy processes that are not of class L, in terms of their Lévy measures. To describe our results, we need to introduce some notations.

From now on, let n be a positive integer,

$$0=b_0<a_1<b_1<a_2<b_2<\cdots<a_n<b_n<a_{n+1}\leqq\infty,$$

and let $k(x)$ be a function on $(0, \infty)$ such that $k(0+)<\infty$, $k(x)>0$ on $(0, a_{n+1})$, $k(x)=0$ on $[a_{n+1}, \infty)$, $k(x)$ is non-increasing on $[b_m, a_{m+1}]$ $(0\leqq m\leqq n)$, non-

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decreasing and concave on (a_m, b_m) $(1 \leqq m \leqq n)$, and $\int_0^\infty (1+x)^{-1} k(x) dx < \infty$. Let $E = \bigcup_{m=1}^{n} (a_m, b_m)$. Let

$$d_m = k(a_m -) - k(a_m +) > 0, \tag{1.1}$$

$$e_m = k(b_m -) - k(b_m +) > 0,$$

$$h_m = k(b_m -) - k(a_m +) > 0,$$

$$\delta_m = k^*(a_m +) < \infty,$$

for $1 \leqq m \leqq n$, and

$$K(x) = \int_0^x (k(0+) - k(u)) u^{-1} du < \infty$$

for $0 < x < \infty$, where $k^*(x)$ is the Radon-Nikodym derivative of $k(x)$ on E.

Our results are as follows.

THEOREM 1.1. *Let $\{X_t\}$ be a one-sided Lévy process without drift such that*

$$E \exp(-z X_t) = \exp(t\psi(z)), \tag{1.2}$$

$$\psi(z) = \int_0^\infty (e^{-zx} - 1) x^{-1} k(x) dx$$

for $z \geqq 0$. Then $\{X_t\}$ is unimodal if the following additional conditions are satisfied:

(H.1) $k(0+) \leqq 2$.

(H.2) $k(x) = g(x) + h(x)$, *where $g(x)$ and $h(x)$ are nonnegative on $(0, \infty)$, $g(0+) \geqq 1$, the set $\{x : g(x) > 0\}$ is an interval, $\log g(x)$ is concave on this interval, and $h(x)$ is non-increasing on $(0, \infty)$.*

(H.3) *There exists a real number α snch that $0 < \alpha < a_1$, and, for every m $(1 \leqq m \leqq n)$,*

(H.3.a) $\alpha(e_m - h_m) \geqq e_m(b_m - a_m)$,

(H.3.b) $(k(\alpha -) - K(\alpha)) d_m > k(\alpha -) h_m$,

(H.3.c) $(k(\alpha -) - k(b_m -)) d_m \geqq \delta_m (k(\alpha -) a_m + \alpha k(b_m -))$.

Remark 1.1. Assume that

(H.4) $\log k(x)$ is concave on (β, a_{n+1}) for some β satisfying $b_n \leqq \beta \leqq a_{n+1}$.

In this case, we can define a function $g(x)$ on $(0, \infty)$ such that $g(x)$ is absolutely continuous on $(0, a_{n+1})$ and

$$g^*(x) = \sup_{x \leqq y \in E} k^*(y), \qquad \text{for } 0 < x < b_n,$$

if $\beta < a_{n+1}$,

$$g(x) = k(\beta +), \qquad \text{for } b_n \leqq x \leqq \beta,$$

$$=k(x), \qquad \text{for} \quad \beta<x<\infty,$$

if $\beta=a_{n+1}$,

$$g(x)=k(\beta-) \qquad \text{for} \quad b_n\leqq x<\beta,$$

$$g(x)=0 \qquad \text{for} \quad x\geqq\beta,$$

where $k^*(y)$ and $g^*(x)$ are the Radon-Nikodym derivatives of $k(y)$ and $g(x)$, respectively. Then (H.2) is satisfied for this $g(x)$ if $g(0+)\geqq 1$.

Remark 1.2. It should be noted that $g(x)$ in (H.2) is positive and non-decreasing on $(0, b_n)$. Hence $k(x)\geqq g(x)\geqq g(0+)\geqq 1$ for $0<x<b_n$ and $d_m\leqq k(0+)-g(0+)\leqq 1$ for $1\leqq m\leqq n$ by (H.1) and (H.2). Also we note that, for every m $(1\leqq m\leqq n)$, $\alpha>b_m-a_m$, $e_m>h_m$, and $d_m>h_m$ by (H.3.a) and (H.3.b).

Combining the above theorem with Corollary 1.2 of Watanabe [11], we obtain the following result.

COROLLARY 1.1. *Let $k^{(1)}(x)$ and $k^{(2)}(x)$ be two functions satisfying the conditions for $k(x)$ and let $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ be independent unimodal one-sided Lévy processes in Theorem* 1.1 *associated with $k^{(1)}(x)$ and $k^{(2)}(x)$, respectively. Let a, b, σ be nonnegative, $\gamma\in R^1$, and $\{B(t)\}$ be a Brownian motion independent of $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$. Then the process $X_t=X_{at}^{(1)}-X_{bt}^{(2)}+\sigma B(t)+\gamma t$ is unimodal.*

In order to prove Theorem 1.1, we show in Section 2 integro-differential equations and fundamental inequalities satisfied by the density function of the process. In Section 3 we prove Theorem 1.1.

2. Preliminaries.

In this section, let $\{X_t\}$ be a one-sided Lévy process satisfying (1.2) with $k(x)$ described in the paragraph preceding Theorem 1.1.

Assume that

(S.1) $k(x)$ is a step function on $[b_m, a_{m+1}]$ for $0\leqq m\leqq n$.

(S.2) $b_m-a_m\leqq\alpha<a_1$, $e_m>h_m$, and $d_m>h_m$ for $1\leqq m\leqq n$.

Remark 2.1. The condition (S.2) implies that $k(u)\geqq k(a_m-)$ for $0<u<a_m$ for $1\leqq m\leqq n$.

Let $\{c_j: 0\leqq j\leqq N+1\}$ be the union of $\{b_j: 0\leqq j\leqq n\}$, $\{a_j: 1\leqq j\leqq n+1\}$ and the set of jumping points of $k(x)$ in (b_m, a_{m+1}) for $0\leqq m\leqq n$. They are numbered so that

$$0=c_0<c_1<c_2<\cdots<c_{N-1}<c_N=a_{n+1}\leqq c_{N+1}=\infty.$$

Let $c_{j(m)}=a_m$ and $c_{j(m)+1}=b_m$ for $1\leqq m\leqq n$. Define

$$J_1=\{j: 0\leqq j\leqq N \text{ and } j\neq j(m) \text{ for } 1\leqq m\leqq n\}$$

and

$$J_2=\{j: 1\leqq j\leqq N \text{ and } j\neq j(m),\ j(m)+1 \text{ for } 1\leqq m\leqq n\}.$$

Define $p_j=k(c_j-)-k(c_j+)$ for $j\in J_2$, understanding $p_N=0$ if $c_N=\infty$ and $p_N>0$ if $c_N<\infty$. Define $p_j=0$ for $j\in J_2^c$. The distribution μ_t of X_t is absolutely continuous for every $t>0$ by Tucker [7], since $\int_0^1 u^{-1}k(u)du=\infty$. Let $f(x)$ be the density function of μ_t. We do not write dependence of $f(x)$ on t explicitely.

LEMMA 2.1. *For $t>0$, we have*

$$\begin{aligned}(2.1)\qquad xf(x)&=t\int_0^x f(x-u)k(u)du\\ &=t\lambda F(x)-t\sum_{j=1}^N p_jF(x-c_j)\\ &\quad+t\sum_{m=1}^n\left\{-d_mF(x-a_m)-e_mF(x-b_m)+\int_{E_m}F(x-u)k^*(u)du\right\}\end{aligned}$$

for $x\neq 0$, where $\lambda=k(0+)$, $F(x)=\int_{-\infty}^x f(u)du$, and $E_m=(a_m, b_m)$ for $1\leqq m\leqq n$.

Proof. The identity (2.1) follows from Steutel [5] and integration by parts.

Remark 2.2. We see from (2.1) that $f(x)=0$ for $x<0$, $f(x)>0$ for $x>0$, and $f(x)$ is continuous for $x>0$.

LEMMA 2.2. *For $t>0$, we have*

$$\begin{aligned}(2.2)\qquad xf'(x)&=(\lambda t-1)f(x)-t\sum_{j=1}^N p_jf(x-c_j)\\ &\quad+t\sum_{m=1}^n\left\{-d_mf(x-a_m)-e_mf(x-b_m)+\int_{E_m}f(x-u)k^*(u)du\right\}\end{aligned}$$

for $x\neq c_j$ $(0\leqq j\leqq N)$.

Proof. Since $f(x)$ is continuous for $x>0$, $F(x-c_j)$ is differentiable for $x\neq c_j$. Because $f(x)$ is integrable on $(-\infty, \infty)$ and $k^*(u)\leqq\delta_m<\infty$ on E_m, $\int_{E_m}f(x-u)k^*(u)du$ is continuous in x. We get

$$\begin{aligned}(2.3)\qquad \frac{d}{dx}\int_{E_m}F(x-u)k^*(u)du&=\lim_{h\to 0}h^{-1}\int_x^{x+h}dy\int_{E_m}f(y-u)k^*(u)du\\ &=\int_{E_m}f(x-u)k^*(u)du\end{aligned}$$

for $-\infty<x<\infty$ for $1\leqq m\leqq n$. Hence, differentiating (2.1), we obtain (2.2).

LEMMA 2.3. *For $t>0$, we have*

$$f(x)=Cx^{\lambda t-1}, \tag{2.4}$$

$$f'(x)=C(\lambda t-1)x^{\lambda t-2}, \tag{2.5}$$

and

$$f''(x)=C(\lambda t-1)(\lambda t-2)x^{\lambda t-3} \tag{2.6}$$

for $0<x<c_1$, where C is a positive constant which depends on t.

Proof. The identity (2.2) means

$$f'(x)=(\lambda t-1)f(x) \tag{2.7}$$

for $0<x<c_1$. Hence we obtain (2.4), which implies (2.5) and (2.6).

Remark 2.3. We find from (2.2), (2.4), and Remark 2.2 that, for $t>0$, $f'(x)$ is continuous except at $x=c_j$ $(0\leqq j\leqq N)$. Moreover, for $t>\lambda^{-1}$, $f(x)$ is continuous on $(-\infty, \infty)$ and $f'(x)$ is continuous on $(0, \infty)$.

LEMMA 2.4. *For $t>\lambda^{-1}$, we have*

$$\begin{aligned} xf''(x)=&(\lambda t-2)f'(x)-t\sum_{j=1}^{N} p_j f'(x-c_j) \\ &+t\sum_{m=1}^{n}\left\{-d_m f'(x-a_m)-e_m f'(x-b_m)+\int_{E_m} f'(x-u)k^*(u)du\right\} \end{aligned} \tag{2.8}$$

for $x\neq c_j$ $(0\leqq j\leqq N)$.

Proof. Since $f'(x)$ is integrable on $(0, M)$ for any $M>0$, $\int_{E_m} f(x-u)k^*(u)du$ is differentiable on $(-\infty, \infty)$ for every m $(1\leqq m\leqq n)$ as in (2.3). Hence, differentiating (2.2), we get (2.8).

LEMMA 2.5. *Let $t>0$ and $x\neq c_j$ $(0\leqq j\leqq N)$. Suppose that $f(u)$ is non-increasing for $0<u<x$.*

(i) *If $x\in E^c=\left(\bigcup_{m=1}^{n} E_m\right)^c$, then*

$$xf'(x)\leqq(k(x-)t-1)f(x). \tag{2.9}$$

(ii) *If $x\in E_q$ for some q $(1\leqq q\leqq n)$, then*

$$xf'(x)<f(x)\{tk(a_q+)-1+\delta_q(b_q-ta_q k(a_q+))k(a_q+)^{-1}\}. \tag{2.10}$$

Proof. We have

$$f(x-c_j) \geqq f(x) \tag{2.11}$$

for $c_j < x$ and

$$\sum_{b_m < x} \int_{E_m} f(x-u)k^*(u)du \leqq \sum_{b_m < x} f(x-b_m)h_m . \tag{2.12}$$

Hence, if $x \in E^c$, then we obtain from (S.2) and (2.2) that

$$\begin{aligned} xf'(x) &\leqq \{\lambda t - 1 - t \sum_{c_j < x} p_j - t \sum_{a_m < x} d_m\} f(x) \\ &\quad - t \sum_{b_m < x} (e_m - h_m) f(x-b_m) \\ &\leqq (k(x-)t-1)f(x), \end{aligned} \tag{2.13}$$

noting that

$$\lambda - \sum_{c_j < x} p_j - \sum_{a_m < x} d_m - \sum_{b_m < x} (e_m - h_m) = k(x-) .$$

Similarly we get

$$xf'(x) \leqq (k(a_q+)t-1)f(x) + t\int_{E_q} f(x-u)k^*(u)du \tag{2.14}$$

for $x \in E_q$. We have, for $x \in E_q$,

$$\int_{E_q} f(x-u)k^*(u)du \leqq \delta_q \int_0^{x-a_q} f(u)du . \tag{2.15}$$

We obtain from (2.1) and Remark 2.1 that

$$\begin{aligned} xf(x) &= t\int_0^x f(x-u)k(u)du \\ &\geqq t\,k(a_q+)\int_0^x f(u)du \\ &\geqq t\,k(a_q+)\int_0^{x-a_q} f(u)du + tk(a_q+)a_q f(x) \end{aligned} \tag{2.16}$$

for $x \in E_q$, which implies that

$$(b_q - ta_q k(a_q+))f(x) > tk(a_q+)\int_0^{x-a_q} f(u)du . \tag{2.17}$$

Hence (2.10) follows from (2.14), (2.15), and (2.17). Thus we have proved Lemma 2.5.

LEMMA 2.6. *Let $\lambda t > 1$. Suppose that $f(u)$ is non-decreasing for $0 < u < x$ and non-increasing for $x \leqq u \leqq y$.*

(i) *If $c_Q \leqq x < c_{Q+1}$ for some Q $(0 \leqq Q \leqq N)$, then*

$$xf'(x) \geqq (k(c_Q+)t-1)f(x). \tag{2.18}$$

(ii) *If* $a_q \leqq x < b_q$ *for some* q $(1 \leqq q \leqq n)$ *and* $y < a_q + \alpha$, *then*

$$y f'(y) \leqq (k((y-x)+)t-1) f(y) - t(k((y-x)+) - k(b_q-)) f(y-a_q). \tag{2.19}$$

Proof of (i). We have

$$f(x-c_j) \leqq f(x) \tag{2.20}$$

for $c_j \leqq x$ and

$$\sum_{b_m \leqq x} \int_{E_m} f(x-u) k^*(u) du \geqq \sum_{b_m \leqq x} f(x-b_m) h_m. \tag{2.21}$$

Hence we obtain from (S.2) and (2.2) that

$$\begin{aligned} x f'(x) &\geqq \Big(\lambda t - 1 - t \sum_{c_j \leqq x} p_j - t \sum_{a_m \leqq x} d_m\Big) f(x) - t \sum_{b_m \leqq x} (e_m - h_m) f(x-b_m) \\ &\geqq (k(c_Q+)t-1) f(x), \end{aligned} \tag{2.22}$$

noting that $\lambda - \sum_{c_j \leqq x} p_j - \sum_{a_m \leqq x} d_m - \sum_{b_m \leqq x} (e_m - h_m) = k(c_Q+)$. Thus we have proved (i).

Proof of (ii). Let R be such that $y - c_{R+1} < x \leqq y - c_R$ with $0 \leqq R < j(1)$. As in the proof of (i), we get

$$f(y-c_j) \geqq f(y) \tag{2.23}$$

for $0 \leqq j \leqq R$ and

$$f(y-c_j) \geqq f(y-a_q) \tag{2.24}$$

for $R+1 \leqq j \leqq j(q)$. We have

$$\int_{E_m} f(y-u) k^*(u) du \leqq f(y-a_m) h_m \tag{2.25}$$

for $1 \leqq m \leqq n$, noting that $y - a_m < a_q \leqq x$ for $1 \leqq m \leqq n$. Hence we obtain from (S.2) and (2.2) that

$$\begin{aligned} y f'(y) &\leqq \Big\{t\Big(\lambda - \sum_{j=1}^{R} p_j\Big) - 1\Big\} f(y) - t\Big(\sum_{j=R+1}^{j(q)} p_j + \sum_{m=1}^{q-1} e_m\Big) f(y-a_q) \\ &\quad - t \sum_{m=1}^{n} (d_m - h_m) f(y-a_m) \\ &\leqq (k((y-x)+)t-1) f(y) - t(k((y-x)+) - k(b_q-)) f(y-a_q), \end{aligned} \tag{2.26}$$

noting that $\sum_{j=R+1}^{j(q)} p_j + \sum_{m=1}^{q-1} e_m + \sum_{m=1}^{q} (d_m - h_m) = k((y-x)+) - k(b_q-)$ and $\lambda - \sum_{j=1}^{R} p_j = k((y-x)+)$. Thus we have proved Lemma 2.6.

When $1 < \lambda t < 2$, define

$$S=\{x: x>0 \text{ and } f \text{ attains local maximum at } x\}, \tag{2.27}$$

$$T=\{x: x>0,\ x\neq c_j\ (0\leqq j\leqq N), \text{ and } f''(x)\geqq 0\}, \tag{2.28}$$

and

$$\inf S=s_0 \text{ and } \inf T=y. \tag{2.29}$$

Obviously the set S is non-empty. The set T is also non-empty, because the support of μ_t is unbounded. We find from (2.5) and (2.6) that

$$s_0>c_1 \text{ and } y\geqq c_1. \tag{2.30}$$

LEMMA 2.7. *Let* $1<\lambda t<2$ *and, in addition to* (S.1) *and* (S.2), *suppose* (H.3.c). *Then we have*

$$y\geqq s_0. \tag{2.31}$$

Proof. Suppose that $y<s_0$. We shall consider three possible cases and show that absurdity occurs in each case.

Case 1. $y<s_0$ and $y=c_Q$ $(1\leqq Q\leqq N)$. There exists a sequence y_k such that $y<y_k<s_0$, $f''(y_k)\geqq 0$, and $y_k\to y$ as $k\to\infty$. Since $f'(y_k-u)\geqq 0$ for $0<u<y_k$, we get

$$\int_{E_m} f'(y_k-u)k^*(u)du\leqq \delta_m(f(y_k-a_m)-f(y_k-b_m)) \tag{2.32}$$

for $1\leqq m\leqq n$. Since, as $k\to\infty$, $f'(y_k)\to f'(y)\geqq 0$,

$$f'(y_k-c_j)\longrightarrow f'(y-c_j)>0 \quad \text{for } 1\leqq j\leqq Q-1,\ f'(y_k-c_Q)\to f'(0+)=\infty,$$

$$f'(y_k-c_j)\longrightarrow f'(y-c_j)=0 \quad \text{for } j\geqq Q+1, \text{ and } f(y_k-c_j)\to f(y-c_j)$$

for $0\leqq j\leqq N$, we obtain from (2.8) and (2.32) that

$$\begin{aligned} 0\leqq y_k f''(y_k)&\leqq(\lambda t-2)f'(y_k)-t\sum_{j=1}^{N} p_j f'(y_k-c_j)\\ &+t\sum_{m=1}^{n}\{-d_m f'(y_k-a_m)-e_m f'(y_k-b_m)+\delta_m(f(y_k-a_m)-f(y_k-b_m))\}\\ &\longrightarrow -\infty \end{aligned} \tag{2.33}$$

as $k\to\infty$. This is a contradiction.

Case 2. $y<s_0$ and $c_Q<y<c_{Q+1}$ with $Q\in J_1$. Since $f'(y)\geqq 0$, $f'(y-c_j)>0$ for $1\leqq j\leqq Q$, and $f'(y-u)<f'(y-b_m)$ for $u\in E_m$ $(j(m)<Q)$, we have by (2.8)

$$\begin{aligned} 0\leqq yf''(y)&\leqq(\lambda t-2)f'(y)-t\sum_{j=1}^{Q} p_j f'(y-c_j)\\ &+t\sum_{j(m)<Q}\{-d_m f'(y-a_m)-(e_m-h_m)f'(y-b_m)\}<0. \end{aligned} \tag{2.34}$$

This is a contradiction.

Case 3. $y<s_0$ and $a_q<y<b_q$ $(1\leqq q\leqq n)$. We shall first prove that

$$d_q f'(y-a_q)>\delta_q f(y-a_q). \tag{2.35}$$

We get by (i) of Lemma 2.6 that

$$\begin{aligned}\alpha f'(y-a_q)&\geqq(y-a_q)f'(y-a_q)\\&\geqq(k((y-a_q)+)t-1)f(y-a_q)\\&\geqq(k(\alpha-)t-1)f(y-a_q),\end{aligned} \tag{2.36}$$

noting that $y-a_q<\alpha<a_1$ by (S.2). We obtain from (ii) of Lemma 2.6 with $y=x$ and from $f(y)\leqq f(y-a_q)+f'(y-a_q)a_q$ that

$$\begin{aligned}0&\leqq yf'(y)\leqq(\lambda t-1)f(y)-t(\lambda-k(b_q-))f(y-a_q)\\&\leqq(tk(b_q-)-1)f(y-a_q)+(\lambda t-1)f'(y-a_q)a_q,\end{aligned} \tag{2.37}$$

and hence

$$(-tk(b_q-)+1)f(y-a_q)<f'(y-a_q)a_q, \tag{2.38}$$

noting that $\lambda t-1<1$. The identity (2.36) shows that (2.35) holds for $t>T_1$ and the identity (2.38) implies that (2.35) holds for $t\leqq T_2$, where

$$T_1=d_q^{-1}k(\alpha-)^{-1}(\delta_q\alpha+d_q)$$

and

$$T_2=d_q^{-1}k(b_q-)^{-1}(d_q-\delta_q a_q).$$

Hence it is sufficient for (2.35) that $T_2\geqq T_1$, which follows from (H.3.c). In fact, we have by (H.3.c)

$$\begin{aligned}&T_2-T_1\\&=d_q^{-1}k(b_q-)^{-1}k(\alpha-)^{-1}\{(k(\alpha-)-k(b_q-))d_q-\delta_q(a_qk(\alpha-)+\alpha k(b_q-))\}\\&\geqq 0.\end{aligned} \tag{2.39}$$

Since $\int_{E_q}f'(y-u)k^*(u)du\leqq\delta_q f(y-a_q)$ and $\int_{E_m}f'(y-u)k^*(u)du<h_m f'(y-b_m)$ for $1\leqq m\leqq q-1$, we obtain from (2.8) and (2.35) that

$$\begin{aligned}0&\leqq yf''(y)\leqq(\lambda t-2)f'(y)-t\sum_{j=1}^{j(q)}p_j f'(y-c_j)\\&\quad+t\sum_{m=1}^{q-1}\{-d_m f'(y-a_m)-(e_m-h_m)f'(y-b_m)\}\\&\quad-t\{d_q f'(y-a_q)-\delta_q f(y-a_q)\}<0.\end{aligned} \tag{2.40}$$

This is a contradiction. Thus we have proved Lemma 2.7.

3. Proof of Theorem 1.1.

Let $\{X_t\}$ be a one-sided Lévy process satisfying (1.2). Let μ_t be the distribution of X_t.

PROPOSITION 3.1. *Let* $\lambda t \leqq 1$. *Suppose* (S.2) *and that, for every* m $(1 \leqq m \leqq n)$,
(S.3.a) $k(a_m+) \geqq \delta_m a_m$,
(S.3.b) $k(a_m+)(\lambda - k(a_m+)) \geqq \delta_m(\lambda b_m - k(a_m+)a_m)$.
Then μ_t *is unimodal with mode* 0.

Proof. We divide the proof into two steps.

First step. Suppose (S.1) and continue to use the notation in Section 2. The identity (2.5) implies that $f'(x) \leqq 0$ for $0 < x < c_1$. We shall prove that $f'(x) \leqq 0$ for $0 < x \neq c_j$ $(1 \leqq j \leqq N)$. Suppose that $f'(x_0) > 0$ for some $x_0 \neq c_j$ $(1 \leqq j \leqq N)$. Let

$$x_1 = \inf\{x : x > 0,\ x \neq c_j\ (1 \leqq j \leqq N), \text{ and } f'(x) > 0\}.$$

Then we note that $c_1 \leqq x_1 < x_0$, $f'(x_1+) \geqq 0$, and $f'(x) \leqq 0$ for $0 < x < x_1$ except at $x = c_j$ $(1 \leqq j \leqq N)$. We shall consider three possible cases and show that absurdity occurs in each case.

Case 1. $c_Q < x_1 < c_{Q+1}$ with $1 \leqq Q \in J_1$. We get by (i) of Lemma 2.5 that

$$0 \leqq x_1 f'(x_1) \leqq (k(x_1)t - 1) f(x_1) < 0, \tag{3.1}$$

noting that $k(x_1)t - 1 < \lambda t - 1 \leqq 0$. This is a contradiction.

Case 2. $a_q < x_1 < b_q$ with $1 \leqq q \leqq n$. We find from (S.3.a), (S.3.b), and from $t \leqq \lambda^{-1}$ that

$$\begin{aligned} & tk(a_q+) - 1 + \delta_q(b_q - t a_q k(a_q+))k(a_q+)^{-1} \\ & \leqq \lambda^{-1} k(a_q+)^{-1}\{k(a_q+)(k(a_q+) - \lambda) + \delta_q(\lambda b_q - a_q k(a_q+))\} \leqq 0. \end{aligned} \tag{3.2}$$

We obtain from (3.2) and (ii) of Lemma 2.5 that

$$\begin{aligned} 0 \leqq x_1 f'(x_1) < f(x_1)\{tk(a_q+) - 1 + \delta_q(b_q - t a_q k(a_q+))k(a_q+)^{-1}\} \\ \leqq 0. \end{aligned} \tag{3.3}$$

This is a contradiction.

Case 3. $x_1 = c_Q$ with $1 \leqq Q \leqq N$. There exists a sequence y_k such that $x_1 < y_k$, $f'(y_k) > 0$, and $y_k - x_1$ as $k \to \infty$. Hence we can show that contradiction occurs in this case by argument similar to Case 1. In fact let $y_k - x_1 < c_1$. Then we have

$$f(y_k - c_j) \geqq f(x_1) \tag{3.4}$$

for $c_j < y_k$ $(j \geqq 1)$ and

(3.5) $$\sum_{b_m < y_k} \int_{E_m} f(y_k - u) k^*(u) du \leqq \sum_{b_m < y_k} f(y_k - b_m) h_m .$$

Hence we get, as in the proof of Lemma 2.5, that

(3.6) $$0 < y_k f'(y_k) \leqq (\lambda t - 1) f(y_k) - t(\lambda - k(y_k -)) f(x_1)$$

if $1 \leqq Q \in J_1$, and

(3.7) $$0 < y_k f'(y_k) \leqq (\lambda t - 1) f(y_k) - t(\lambda - k(a_q +)) f(x_1)$$
$$+ t \int_{E_q} f(y_k - u) k^*(u) du$$

if $c_Q = a_q$ with $1 \leqq q \leqq n$. Letting $k \to \infty$ in (3.6) and (3.7), we have

(3.8) $$0 \leqq (k(x_1 +) t - 1) f(x_1) < 0 .$$

This is a contradiction.

Second step. We can find a sequence of Lévy processes $\{X_t^{(n)}\}$ such that each $\{X_t^{(n)}\}$ satisfies (S.1), (S.2), (S.3.a), and (S.3.b) and the distribution $\mu_t^{(n)}$ converges weakly to μ_t as $n \to \infty$ for every $t > 0$. Hence μ_t is unimodal with mode 0. The proof of Proposition 3.1 is complete.

PROPOSITION 3.2. *Let* $1 < \lambda t < 2$. *Suppose* (H.1), (H.2), *and* (H.3). *Then* μ_t *is unimodal.*

Proof. We first assume (S.1) and continue to use the notation in Section 2. As in the second step in the proof of Proposition 3.1, we can prove general case. Let us suppose that $\mu_t(dx) = f(x) dx$ is not unimodal for some t $(1 < \lambda t < 2)$. Then the set S defined in (2.27) contains at least two points and there are two possible cases:

Case A. s_0 is an isolated point of S.

Case B. s_0 is a limit point of S.

In Case A, let $s_1 = s_0$ and

$$s_2 = \inf\{x : x > s_1 \text{ and } f \text{ attains local minimum at } x\}.$$

Then $c_1 < s_1 < s_2$ by (2.30) and we have

(A.1) $f'(s_1) = f'(s_2) = 0$,

and

(A.2) $f(x)$ is strictly increasing for $0 < x < s_1$ and strictly decreasing for $s_1 < x < s_2$.

In Case B, we can choose, for any $\varepsilon > 0$, s_1 and s_2 such that $c_1 < s_0 < s_1 < s_2 < s_0 + \varepsilon$, $f(s_2) \leqq f(s_1)$, and $f(x)$ attains local maximum at $x = s_1$ and local minimum at $x = s_2$. Hence we get

(B.1) $f'(s_1)=f'(s_2)=0$.

We shall prove that

(a) Existence of s_1 and s_2 leads to a contradiction.

This will imply the unimodality of μ_t.

Consider Case A and let Q and R be such that

(3.9) $c_Q \leqq s_1 < c_{Q+1}$ and $s_2 - c_{R+1} \leqq s_1 < s_2 - c_R$ with $1 \leqq Q \leqq N$ and $0 \leqq R \leqq N$.

There are two possible cases: $Q<R$ (Case 1) and $Q \geqq R$ (Case 2). We shall prove that absurdity occurs in each case. Define $I(s_i)=s_i f'(s_i)$ for $i=1, 2$.

Case 1. $Q<R$. We obtain from (2.2) and (A.1) that

$$0=I(s_2)=\sum_{k=1}^{4} I_k(s_2), \tag{3.10}$$

where

$$I_1(s_2)=(\lambda t-1)f(s_2)-t\sum_{j=1}^{Q} p_j f(s_2-c_j)-t\sum_{j(m)\leqq Q} d_m f(s_2-a_m)$$
$$+t\sum_{j(m)<Q}\left\{-e_m f(s_2-b_m)+\int_{E_m} f(s_2-u)k^*(u)du\right\}, \tag{3.11}$$

$$I_2(s_2)=-t\sum_{j=Q+1}^{R} p_j f(s_2-c_j)-t\sum_{Q<j(m)<R} d_m f(s_2-a_m)$$
$$+t\sum_{Q\leqq j(m)<R}^{R}\left\{-e_m f(s_2-b_m)+\int_{E_m} f(s_2-u)k^*(u)du\right\}, \tag{3.12}$$

$$I_3(s_2)=-td_r f(s_2-a_r)-te_r f(s_2-b_r)+t\int_{E_r} f(s_2-u)k^*(u)du \tag{3.13}$$

if $R=j(r)$ $(1\leqq r\leqq n)$ $(I_3(s_2)=0$ if $R\in J_1)$, and

$$I_4(s_2)=-t\sum_{j=R+1}^{N} p_j f(s_2-c_j)$$
$$+t\sum_{R<j(m)}\left\{-d_m f(s_2-a_m)-e_m f(s_2-b_m)+\int_{E_m} f(s_2-u)k^*(u)du\right\}. \tag{3.14}$$

We shall prove that

$$0=I(s_1)\geqq(k(c_Q+)t-1)f(s_1) \tag{3.15}$$

and

$$0=I(s_2)<(k(c_Q+)t-1)f(s_2), \tag{3.16}$$

which will lead to a contradiction. The inequality (3.15) follows from (2.18) in Lemma 2.6. By argument similar to (2.9), we get

$$(3.17)\qquad I_1(s_2)<(k(c_Q+)t-1)f(s_2).$$

Since $f(s_2-u)\leqq f(s_2-b_m)$ for $u\in E_m$ $(j(m)<R)$, we have

$$(3.18)\qquad I_2(s_2)\leqq -t\sum_{j=Q+1}^{R} p_j f(s_2-c_j)-t\sum_{Q<j(m)<R} d_m f(s_2-a_m)$$

$$-t\sum_{Q\leqq j(m)<R}(e_m-h_m)f(s_2-b_m)\leqq 0.$$

We shall show that, if $R=j(r)$ $(1\leqq r\leqq n)$, then

$$(3.19)\qquad I_3(s_2)\leqq 0.$$

There are two cases.

(i) Suppose that $s_2-a_r\geqq\alpha$. Then we get by (H.3.a) that

$$(3.20)\qquad \begin{aligned} s_1h_r-e_r(s_2-b_r)&<(s_2-a_r)h_r-e_r(s_2-b_r)\\ &=(s_2-a_r)(h_r-e_r)+e_r(b_r-a_r)\\ &\leqq\alpha(h_r-e_r)+e_r(b_r-a_r)\leqq 0\end{aligned}$$

and $s_2-b_r>0$. We have $f(s_1)/s_1\leqq f(s_2-b_r)/(s_2-b_r)$ by Lemma 2.7. Hence

$$(3.21)\qquad \int_{E_r} f(s_2-u)k^*(u)du-e_rf(s_2-b_r)<f(s_1)h_r-e_rf(s_2-b_r)$$

$$\leqq\frac{f(s_2-b_r)}{s_2-b_r}\{s_1h_r-e_r(s_2-b_r)\}\leqq 0$$

by (3.20). Therefore, (3.19) follows from (3.13) in this case.

(ii) Suppose that $s_2-a_r<\alpha$. Then we note that $s_1<s_2-a_r<\alpha<a_1<s_2$. Hence we have by (A.2)

$$(3.22)\qquad f(s_2-a_r)>f(\alpha).$$

It follows from (2.2) that

$$(3.23)\qquad \begin{aligned} xf'(x)&\geqq(\lambda t-1)f(x)+tf(s_1)(k(x-)-\lambda)\\ &>tf(s_1)(k(x-)-\lambda)\end{aligned}$$

for $s_1<x<\alpha$. Hence we have

$$(3.24)\qquad f(s_1)-f(\alpha)<tf(s_1)\int_0^{\alpha}x^{-1}(\lambda-k(x-))dx,$$

equivalently,

$$(3.25)\qquad f(\alpha)>(1-tK(\alpha))f(s_1).$$

We obtain from (3.22), (3.25), and (H.3.b) that

$$\int_{E_r} f(s_2-u)k^*(u)du-d_r f(s_2-a_r)<f(s_1)h_r-d_r f(\alpha)$$
$$<f(s_1)\{h_r-(1-k(\alpha-)^{-1}K(\alpha))d_r\}\leqq 0, \tag{3.26}$$

noting that $t\leqq k(c_Q+)^{-1}\leqq k(\alpha)^{-1}$ from (3.15) and from $c_Q<\alpha<a_1$. Therefore, (3.19) follows from (3.13) and (3.26) in this case. Thus we have proved (3.19). Since $f(s_2-u)\leqq f(s_2-a_m)$ for $u\in E_m$ ($j(m)>R$), we find

$$\sum_{R<j(m)}\left\{\int_{E_m} f(s_2-u)k^*(u)du-d_m f(s_2-a_m)\right\}$$
$$\leqq \sum_{R<j(m)}(h_m-d_m)f(s_2-a_m)\leqq 0, \tag{3.27}$$

which shows

$$I_4(s_2)\leqq 0. \tag{3.28}$$

Hence (3.16) follows from (3.17), (3.18), (3.19), and (3.28). Thus the proof of the assertion (a) in Case 1 is complete.

Case 2. $R\leqq Q$. We obtain from (2.2) and (A.1) that, for $i=1, 2$.

$$0=I(s_i)=\sum_{k=1}^{5} I_k(s_i), \tag{3.29}$$

where

$$I_1(s_i)=(\lambda t-1)f(s_i)-t\sum_{j=1}^{R} p_j f(s_i-c_j)-t\sum_{j(m)\leqq R} d_m f(s_i-a_m)$$
$$+t\sum_{j(m)<R}\left\{-e_m f(s_i-b_m)+\int_{E_m} f(s_i-u)k^*(u)du\right\}, \tag{3.30}$$

$$I_2(s_i)=t\int_{E_r} f(s_i-u)k^*(u)du-te_r f(s_i-b_r) \tag{3.31}$$

if $R=j(r)$ $(1\leqq r\leqq n)$ $(I_2(s_i)=0$ if $R\in J_1)$,

$$I_3(s_i)=-t\sum_{j=R+1}^{Q} p_j f(s_i-c_j)$$
$$+t\sum_{R<j(m)<Q}\left\{-d_m f(s_i-a_m)-e_m f(s_i-b_m)+\int_{E_m} f(s_i-u)k^*(u)du\right\}, \tag{3.32}$$

$$I_4(s_i)=t\int_{E_q} f(s_i-u)k^*(u)du-td_q f(s_i-a_q)-te_q f(s_i-b_q) \tag{3.33}$$

if $R<Q=j(q)$ $(1\leqq q\leqq n)$ $(I_4(s_i)=0$ if $Q\in J_1$ or $R=Q)$,

$$I_5(s_i)=-t\sum_{j=Q+1}^{N} p_j f(s_i-c_j)$$
$$+t\sum_{j(m)>Q}\left\{-d_m f(s_i-a_m)-e_m f(s_i-b_m)+\int_{E_m} f(s_i-u)k^*(u)du\right\}. \tag{3.34}$$

We shall prove that

(3.35) $$I_1(s_1) \geqq (k(c_R+)t-1)f(s_1),$$

(3.36) $$I_1(s_2) \leqq (k(c_R+)t-1)f(s_2)$$

(equality holds if and only if $R=0$),

(3.37) $$I_2(s_2)<0 \quad \text{if} \quad R=j(r)\ (1\leqq r\leqq n),$$

(3.38) $$I_2(s_2)<I_2(s_1) \quad \text{if} \quad R=j(r)\ (1\leqq r\leqq n),$$

(3.39) $$I_3(s_2)\leqq 0,$$

(3.40) $$I_3(s_2)\leqq I_3(s_1),$$

(3.41) $$I_4(s_2)<0 \quad \text{if} \quad R<Q=j(q)\ (1\leqq q\leqq n),$$

(3.42) $$I_4(s_2)\leqq I_4(s_1) \quad \text{if} \quad R<Q=j(q)\ (1\leqq q\leqq n),$$

(3.43) $$I_5(s_1)=0,$$

and

(3.44) $$I_5(s_2)\leqq 0.$$

These inequalities lead to a contradiction. In fact, we obtain from (3.36), (3.37), (3.39), (3.41), and (3.44) that

(3.45) $$0=I(s_2)\leqq I_1(s_2)<(tk(c_R+)-1)f(s_2)$$

if $R\geqq 1$. Hence we have

(3.46) $$tk(c_R+)-1>0.$$

Note that this holds even for $R=0$. We find from (A.2), (3.35), (3.36), (3.38), (3.40), (3.42), (3.43), (3.44), and (3.46) that

(3.47) $$\begin{aligned} 0=I(s_2) &\leqq (tk(c_R+)-1)f(s_2)+I_2(s_2)+I_3(s_2)+I_4(s_2) \\ &<(tk(c_R+)-1)f(s_1)+I_2(s_1)+I_3(s_1)+I_4(s_1) \\ &\leqq I(s_1)=0. \end{aligned}$$

This is a contradiction.

Now we prove the inequalities (3.35)-(3.44). We can prove (3.35) as in (2.18) of Lemma 2.6 and (3.36) as in (2.9) of Lemma 2.5. Also the proof of (2.9) shows that equality in (3.36) holds if and only if $R=0$, because $f(x)$ is strictly decreasing for $s_1<x<s_2$ by (A.2). We can prove (3.37) as in (3.21). To prove (3.38), we consider two possible cases.

(i) Suppose that $Q>R=j(r)$ $(1\leqq r\leqq n)$. Lemma 2.7 implies that

$$f(s_2-u)-f(s_1-u)<f(s_2-b_r)-f(s_1-b_r)$$

for $s_2-s_1<u<b_r$ and

$$f(s_2-u)-f(s_1-u)\leqq f(s_1)-f(s_1-u)\leqq f(s_2-b_r)-f(s_1-b_r)$$

for $a_r\leqq u\leqq s_2-s_1$. Hence, noting $e_r>h_r$ we have that

$$I_2(s_2)-I_2(s_1)<0, \tag{3.48}$$

which means (3.38).

(ii) Suppose that $Q=R=j(r)$ $(1\leqq r\leqq n)$. Then we note that $I_2(s_1)=t\int_{E_r}f(s_1-u)k^*(u)du\geqq 0$. Therefore, (3.38) follows from (3.37). Thus we have proved (3.38).

Since $f(s_2-u)\leqq f(s_2-a_m)$ for $u\in E_m$ $(R<j(m)<Q)$, we get

$$I_3(s_2)\leqq -t\sum_{j=R+1}^{Q}p_jf(s_2-c_j) \tag{3.49}$$
$$-t\sum_{R<j(m)<Q}\{(d_m-h_m)f(s_2-a_m)+e_mf(s_2-b_m)\}\leqq 0,$$

which implies (3.39). Since

$$f(s_2-u)-f(s_1-u)<f(s_2-b_m)-f(s_1-b_m)$$

for $u\in E_m$ $(R<j(m)<Q)$ by Lemma 2.7, we have

$$\int_{E_m}f(s_2-u)k^*(u)du-e_mf(s_2-b_m) \tag{3.50}$$
$$<\int_{E_m}f(s_1-u)k^*(u)du-e_mf(s_1-b_m)$$

for $R<j(m)<Q$ as in (3.48). Hence (3.40) follows from $f(s_2-c_j)>f(s_1-c_j)$ for $R+1\leqq j\leqq Q$. If $R<Q=j(q)$ $(1\leqq q\leqq n)$, then

$$I_4(s_2)\leqq t(h_q-d_q)f(s_2-a_q)-te_qf(s_2-b_q)<0, \tag{3.51}$$

because $f(s_2-u)\leqq f(s_2-a_q)$ for $u\in E_q$. Thus we have proved (3.41). To prove (3.42), we consider two possible cases.

(i) Suppose that $s_2-a_q\geqq\alpha$. We get as in (3.21) that

$$\int_{E_q}f(s_2-u)k^*(u)du-e_qf(s_2-b_q)<f(s_2-a_q)h_q-e_qf(s_2-b_q) \tag{3.52}$$
$$\leqq\frac{f(s_2-b_q)}{s_2-b_q}\{(s_2-a_q)h_q-(s_2-b_q)e_q\}\leqq\frac{f(s_2-b_q)}{s_2-b_q}\{\alpha(h_q-e_q)+(b_q-a_q)e_q\}$$
$$\leqq 0.$$

Hence (3.42) follows from $f(s_1-b_q)=0$ and from $f(s_2-a_q)>f(s_1-a_q)$.

(ii) Suppose that $\alpha > s_2 - a_q$. Define

$$G=\int_{E_q} f(s_2-u)k^*(u)du-\int_{E_q} f(s_1-u)k^*(u)du\,. \tag{3.53}$$

We shall show that

$$G \leqq \delta_q(s_2-s_1)f(s_2-a_q)\,. \tag{3.54}$$

We can write G as

$$G=\int_{D_1} f(u)k^*(s_2-u)du-\int_{D_2} f(u)k^*(s_1-u)du\,, \tag{3.55}$$

where $D_1=(s_2-b_q,\ s_2-a_q)$ and $D_2=(0,\ s_1-a_q)$. If $s_2-b_q \leqq s_1-a_q$, then, by concavity of $k(u)$ on E_q, $k^*(s_2-u) \leqq k^*(s_1-u)$ for $s_2-b_q<u<s_1-a_q$, and hence, by (3.55),

$$G \leqq \int_{D_3} f(u)k^*(s_2-u)du \leqq \delta_q(s_2-s_1)f(s_2-a_q)\,, \tag{3.56}$$

where $D_3=(s_1-a_q,\ s_2-a_q)$. If $s_2-b_q>s_1-a_q$, then by (3.55),

$$\begin{aligned} G &\leqq \int_{D_1} f(u)k^*(s_2-u)du \leqq \delta_q(b_q-a_q)f(s_2-a_q) \\ &< \delta_q(s_2-s_1)f(s_2-a_q)\,. \end{aligned} \tag{3.57}$$

Thus we have proved (3.54). We can show $d_q f'(s_2-a_q) > \delta_q f(s_2-a_q)$ as in (2.35) of the proof of Lemma 2.7, using (ii) of Lemma 2.6 with $s_1=x$ and $s_2=y$. Hence we obtain from (3.54) and Lemma 2.7 that

$$\begin{aligned} I_4(s_1)-I_4(s_2) &\geqq t d_q(f(s_2-a_q)-f(s_1-a_q))-tG \\ &> t(s_2-s_1)(d_q f'(s_2-a_q)-\delta_q f(s_2-a_q))>0\,, \end{aligned} \tag{3.58}$$

which means (3.42). The proof of (3.42) is complete.

The inequality (3.44) can be proved as in (3.28). This finishes the proof of (3.35)-(3.44). Thus the assertion (a) is established in Case A.

In Case B, we can prove the assertion (a) more simply. In fact, we can find s_1 and s_2 such that $0<s_2-s_0<c_1$ and $c_Q \leqq s_0<s_1<s_2<c_{Q+1}$ with $1 \leqq Q \leqq N$. For $i=1, 2$, we obtain from (2.2) and (B.1) that

$$0=s_i f'(s_i)=I_1(s_i)+I_2(s_i)\,, \tag{3.59}$$

where

$$\begin{aligned} I_1(s_i) &= (\lambda t-1)f(s_1)-t\sum_{j=1}^{Q} p_j f(s_i-c_j) \\ &\quad + t\sum_{j(m)<Q}\left\{-d_m f(s_i-a_m)-e_m f(s_i-b_m)+\int_{E_m} f(s_i-u)k^*(u)du\right\}, \end{aligned} \tag{3.60}$$

and

$$I_2(s_\iota)=t\int_{E_q} f(s_\iota-u)k^*(u)du-td_qf(s_i-a_q) \tag{3.61}$$

if $Q=j(q)$ $(1\leqq q\leqq n)$ $(I_2(s_\iota)=0$ if $Q\in J_1)$. Since

$$0<f(s_2-u)-f(s_1-u)\leqq f(s_2-b_m)-f(s_1-b_m)$$

for $u\in E_m$ $(j(m)<Q)$ by Lemma 2.7, we have

$$\int_{E_m} f(s_2-u)k^*(u)du-e_mf(s_2-b_m) \tag{3.62}$$

$$\leqq\int_{E_m} f(s_1-u)k^*(u)du-e_mf(s_1-b_m)$$

for $j(m)<Q$ as in (3.48). Since $f(s_1)\geqq f(s_2)$ and $f(s_2-c_j)>f(s_1-c_j)$ for $1\leqq j\leqq Q$, we get by (3.62) that

$$I_1(s_1)>I_1(s_2). \tag{3.63}$$

If $Q=j(q)$ $(1\leqq q\leqq n)$, then, choosing s_1 and s_2 sufficiently close to s_0, we have

$$I_2(s_1)\geqq I_2(s_2) \tag{3.64}$$

as in (3.42). Hence we obtain from (3.63) and (3.64) that

$$0=I_1(s_1)+I_2(s_1)>I_1(s_2)+I_2(s_2)=0. \tag{3.65}$$

This is a contradiction. Thus the assertion (a) is true in Case B and the proof of Proposition 3.2 is complete.

Proof of Theorem 1.1. We divide the proof into three cases.

Case (I). $0<t\leqq\lambda^{-1}$. We find from Proposition 3.1 that μ_t is unimodal with mode 0, since (S.3.a) and (S.3.b) follow from (H.1), (H.2), and (H.3). In fact, we note by Remark 1.2 that $k(a_m+)\geqq 1\geqq d_m$ for $1\leqq m\leqq n$. Hence, (S.3.a) is true by (H.3.c). We obtain from (S.3.a) and from $\lambda^{-1}\leqq k(\alpha-)^{-1}$ that

$$\begin{aligned} &k(a_m+)(\lambda-k(a_m+))-\delta_m(\lambda b_m-k(a_m+)a_m) \\ &=\lambda\{\lambda^{-1}k(a_m+)(\delta_m a_m-k(a_m+))+k(a_m+)-\delta_m b_m\} \\ &\geqq\lambda\, k(\alpha-)^{-1}\{k(a_m+)(k(\alpha-)-k(a_m+))-\delta_m(k(\alpha-)b_m-k(a_m+)a_m)\} \end{aligned} \tag{3.66}$$

for $1\leqq m\leqq n$. We note by (H.1) and Remark 1.2 that, for $1\leqq m\leqq n$, $2\geqq k(\alpha-)\geqq k(a_m+)\geqq 1\geqq d_m$ and hence

$$k(a_m+)(k(\alpha-)-k(a_m+))\geqq d_m(k(\alpha-)-k(b_m-)) \tag{3.67}$$

and

(3.68) $$k(\alpha-)a_m+\alpha k(b_m-)-(k(\alpha-)b_m-k(a_m+)a_m)$$
$$\geqq\alpha(-k(\alpha-)+2k(a_m+))\geqq 0.$$

Now (S.3.b) follows from (3.66), (3.67), (3.68), and (H.3.c).

Case (II). $\lambda^{-1}<t<1$. Proposition 3.2 shows the unimodality of μ_t in this case, because $1\leqq 2\lambda^{-1}$ by (H.1).

Case (III). $t\geqq 1$. Let $\phi_1(x)=h(x)$ and $\phi_2(x)=g(x)$. For $i=1, 2$ define one-sided Lévy processes $\{X_t^{(i)}\}$ with the distribution $\mu_t^{(i)}$ such that

$$E\exp(-zX_t^{(i)})=\exp(t\psi_i(z)),$$

$$\psi_i(z)=\int_0^\infty(e^{-zx}-1)x^{-1}\phi_i(x)dx.$$

Then we find from (H.2) that the process $\{X_t^{(1)}\}$ is of class L and hence unimodal by Wolfe [12], and that $\mu_t^{(2)}$ is strongly unimodal for $t\geqq 1$ by Yamazato's theorem [15]. Hence $\mu_t=\mu_t^{(1)}*\mu_t^{(2)}$ is unimodal for $t\geqq 1$. The proof of Theorem 1.1 is complete.

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