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THE SECOND VARIATION OF THE DIRICHLET ENERGY ON CONTACT MANIFOLDS

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1. Introduction

S. S. Chern and R. S. Hamilton in a paper of 1985 [5] studied a kind of Dirichlet energy in terms of the torsion $\tau(\tau=\mathcal{L}_\xi g)$ of a 3-dimensional compact contact manifold and a problem analogous to the Yamabe problem. They raised the question of determining all 3-dimensional contact manifolds with $\tau=0$ (i.e. K-contact). In a long paper of 1989 [8] S. Tanno studied the Dirichlet energy and gauge transformations of contact manifolds. D. E. Blair [2] obtained the critical point condition of $I(g)=\int_M Ric(\xi)dV_g$ over $\mathcal{M}(\eta)$ (the space of all the associated metrics), and proved that the regularity of the characteristic vector field ξ and the critical point condition force the metric to be K-contact. Since $Ric(\xi)=2n-1/4|\tau|^2$, the study of $I(g)$ is the same as the study of the Dirichlet energy. In this paper we investigate the second variation and prove the following result.

THEOREM 2. *Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy $L(g)=\int_M|\tau|^2dV_g$, i.e. $\nabla_\xi L_\xi g=2(\mathcal{L}_\xi g)\phi$, then along any path $g_{ij}(t)=g_{ir}[\delta^r_j+tH^r_j+t^2K^r_j+O(t^3)]$ in $\mathcal{M}(\eta)$*

$$\frac{d^2L}{dt^2}(0)=2\int_M|\mathcal{L}_\xi H^i_j|^2dV_g\geqq 0\,,$$

and $L(g)$ has minimum at each critical metric.

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2. Contact manifolds

A C^∞ manifold M^{2n+1} is said to be a *contact manifold* if it carries a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere. Given a contact form η it is well

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known that there exists a unique vector field ξ on M satisfying $d\eta(\xi, X)=0$ and $\eta(\xi)=1$; ξ is called the *characteristic vector field* of the contact structure. A Riemannian metric g is said to be an *associated metric* if there exists a tensor field ϕ of type (1, 1) such that $d\eta(X, Y)=g(X, \phi Y)$, $\phi^2=-I+\eta\otimes\xi$ and $\eta(X)=g(X, \xi)$. We call (ϕ, ξ, η, g) a *contact metric structure*. Such ϕ and g can be constructed by the polarization of $d\eta$ and they are not unique (see [4]). All associated metrics have the same volume element, namely $dV=(1/2^n n!)\eta\wedge d\eta^n$.

Let $\tau=\mathcal{L}_\xi g$ be the torsion and let $h=(1/2)\mathcal{L}_\xi\phi$. We have

$$\tau_{ij}=-2\phi_{ir}h_j^r$$
$$h_r^i\phi_j^r+\phi_r^i h_j^r=0$$
$$\nabla_i\eta_j=\phi_{ij}-\phi_{ir}h_j^r$$

where $\phi_{ij}=g_{ir}\phi_j^r$, and

$$Ric(\xi)=2n-|h|^2=2n-\frac{1}{4}|\tau|^2.$$

We call the contact metric structure with $\tau=h=0$ (or ξ is Killing) *K-contact*. For general reference see [1], [7] and [9].

3. The space of all associated metrics and the Dirichlet energy

The space of all Riemannian metrics of M^{2n+1} with fixed volume, denoted by $\mathcal{M}_1$, is a symmetric Hilbert manifold; geodesics in $\mathcal{M}_1$ are of the form ge^{Ht} (here H is a type (1, 1) tensor field; see [6]). The space of all the associated metrics $\mathcal{M}(\eta)$ is a totally geodesic submanifold of $\mathcal{M}_1$. See [4] for details about $\mathcal{M}(\eta)$. Let $g(t)$ be any curve in $\mathcal{M}(\eta)$ with $g(0)=g$. Then the structure tensors $(\phi(t), \xi, \eta, g(t))$ corresponding to $g(t)$ satisfy the following:

$$g_{ir}(t)\xi^r=\eta_i$$
$$2g_{ir}(t)\phi_j^r(t)=2\phi_{ij}=\nabla_i\eta_j-\nabla_j\eta_i$$
$$\phi_r^i(t)\phi_j^r(t)=-\delta_j^i+\xi^i\eta_j.$$

Now we put

$$g_{ij}(t)=g_{ir}[\delta_j^r+tH_j^r+t^2K_j^r+O(t^3)]$$
$$\phi_j^i(t)=\phi_j^i+tS_j^i+t^2T_j^i+O(t^3).$$

Then from the above conditions we have

$$H_{ir}\xi^r=K_{ir}\xi^r=S_r^i\xi^r=T_r^i\xi^r=0$$
$$H_{ij}+H_{rs}\phi_i^r\phi_j^s=0, \qquad \textit{hence } H_i^i=0$$
$$S_j^i=\phi_r^i H_j^r, \qquad S_r^i S_j^r=H_r^i H_j^r$$
$$T_j^i=\phi_r^i K_j^r$$

$$K_{ij}+K_{rs}\phi_i^r\phi_j^s=H_{ir}H_j^r$$
$$2K_r^r=H^{rs}H_{rs}$$

where $H_{ij}=g_{ir}H_j^r$, etc., and the inverse of $g(t)$ is given by ([8])

$$g^{ij}(t)=g^{ij}-tH^{ij}+t^2(H_r^iH^{rj}-K^{ij})+O(t^3).$$

The critical point condition of the Dirichlet energy $L(g)=\int_M |\tau|^2 dV_g$ is given by the following theorem, see [2], [5] and [8] for proof.

THEOREM 1. *Let M^{2n+1} be a compact contact manifold. An associated metric $g\in\mathcal{M}(\eta)$ is critical with respect to the Dirichlet energy if and only if*

$$\nabla_\xi\tau=2\tau\phi.$$

Remarks. Chern and Hamilton studied this over the set of all the CR-structures. Strongly pseudo-convex CR-manifolds are contact manifolds satisfying an integrability condition i.e. $Q=0$; in dimension 3 $Q=0$ trivially (see [8]).

4. Proof of Theorem 2

THEOREM 2. *Let M^{2n+1} be a compact contact manifold. If g is a critical metric of the Dirichlet energy, i.e. $\nabla_\xi\tau=2\tau\phi$, then along any path $g(t)$ in $\mathcal{M}(\eta)$ with $g(0)=g$*

$$\frac{d^2L}{dt^2}(0)=2\int_M |\mathcal{L}_\xi H_j^i|^2 dV_g\geqq 0,$$

and $L(g)$ has minimum at each critical metric.

Proof. Let $g_{ij}(t)=g_{ij}+tH_{ij}+t^2K_{ij}+O(t^3)$ be any curve in $\mathcal{M}(\eta)$ with $g(0)=g$ critical. Then for the curvature tensor we have

$$\begin{aligned}R_{ijk}{}^h(t)=R_{ijk}{}^h&+\frac{t}{2}(\nabla_iD_{jk}{}^h-\nabla_jD_{ik}{}^h)\\&+\frac{t^2}{2}\Big[\nabla_i(E_{jk}{}^h-H_r^hD_{jk}{}^r)-\nabla_j(E_{ik}{}^h-H_r^hD_{ik}{}^r)\\&+\frac{1}{2}(D_{ir}{}^hD_{jk}{}^r-D_{jr}{}^hD_{ik}{}^r)\Big]+O(t^3)\end{aligned}$$

where $D_{jk}{}^i=\nabla_jH_k^i+\nabla_kH_j^i-\nabla^iH_{jk}$, $E_{jk}{}^i=\nabla_jK_k^i+\nabla_kK_j^i-\nabla^iK_{jk}$. Therefore we have

$$\begin{aligned}R_{jk}(t)=R_{jk}&+\frac{t}{2}(\nabla_r\nabla_jH_k^r+\nabla_r\nabla_kH_j^r-\nabla^r\nabla_rH_{jk})\\&+\frac{t^2}{4}[2(\nabla_r\nabla_jK_k^r+\nabla_r\nabla_kK_j^r-\nabla^r\nabla_rK_{jk}-\nabla_j\nabla_kK_r^r)\end{aligned}$$

$$-2H^{rs}(\nabla_s\nabla_j H_{rk}+\nabla_s\nabla_k H_{rj}-\nabla_s\nabla_r H_{jk}-\nabla_j\nabla_k H_{rs})$$
$$-2\nabla_s H^{sr}(\nabla_j H_{rk}+\nabla_k H_{rj}-\nabla_r H_{jk})$$
$$+\nabla_j H^{rs}\nabla_k H_{rs}-2\nabla_r H^s_j\nabla_s H^r_k+2\nabla_r H^s_j\nabla^r H_{sk}]+O(t^3).$$

See [8] for some details. Let $I(g)=\int_M Ric(\xi)dV_g$. For any associated metric we have $Ric(\xi)=2n-(1/4)|\tau|^2$, hence $I(g)=2n\, vol(M)-(1/4)L(g)$. Now we assume

$$I_1=\int_M \xi^j\xi^l(\nabla_r\nabla_l K^r_j+\nabla_r\nabla_j K^r_l-\nabla_r\nabla^r K_{jl}-\nabla_l\nabla_j K^r_r)dV_g$$

$$I_2=\int_M \xi^j\xi^l\Big[-H^{rs}(\nabla_r\nabla_l H_{sj}+\nabla_r\nabla_j H_{sl}-\nabla_r\nabla_s H_{jl}-\nabla_l\nabla_j H_{rs})$$
$$-\nabla_s H^{sr}(\nabla_l H_{rj}+\nabla_j H_{rl}-\nabla_r H_{jl})+\frac{1}{2}\nabla_l H^{rs}\nabla_j H_{rs}$$
$$+\nabla_r H_{sj}\nabla^r H^s_l-\nabla_r H_{sj}\nabla^s H^r_l\Big]dV_g.$$

Then for $I(g)$ we have

$$\frac{d^2 I}{dt^2}(0)=I_1+I_2.$$

Using Green's Theorem, the critical point condition and the facts that

$$H^i_r H^r_s h^s_i=\nabla_\xi H^r_s H^s_i h^i_j\phi^j_r=0$$
$$\nabla^r\xi^i\nabla_i\xi^s=-g^{rs}+\xi^r\xi^s+h^r_j h^{js}$$
$$\nabla^i\xi^r\nabla_i\xi^s=g^{rs}-\xi^r\xi^s-2h^{rs}+h^r_j h^{js}$$

we compute as follows

$$\int_M \xi^j\xi^l\nabla_r\nabla_l K^r_j dV_g=\int_M(\xi^l\nabla_l\nabla^r\xi^s+\nabla_l\xi^s\nabla^r\xi^l)K_{rs}dV_g$$

$$\int_M \xi^j\xi^l\nabla_r\nabla^r K_{jl}dV_g=2\int_M\nabla_r\xi^j\nabla^r\xi^l K_{jl}dV_g$$

$$\int_M \xi^j\xi^l\nabla_l\nabla_j K^r_r dV_g=0$$

and hence

$$I_1=2\int_M(\xi^l\nabla_l\nabla^r\xi^s+\nabla_l\xi^s\nabla^r\xi^l-\nabla_i\xi^s\nabla^i\xi^r)K_{rs}dV_g$$
$$=-4\int_M K^r_r dV_g$$
$$=-2\int_M |H|^2 dV_g.$$

Now consider I_2

$$\int_M \xi^j\xi^l H^{rs}\nabla_r\nabla_l H_{sj}dV_g=\int_M [\nabla_l\nabla_r\xi^j\xi^l H^{rs}H_{sj}+\nabla_r\xi^j\xi^l\nabla_l H^{rs}H_{sj} +\nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj}-\xi^j\xi^l\nabla_r H^{rs}\nabla_l H_{sj}]dV_g$$

$$\int_M \xi^j\xi^l H^{rs}\nabla_r\nabla_s H_{jl}dV_g=\int_M [2\nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl}-\xi^j\xi^l\nabla_r H^{rs}\nabla_s H_{jl}]dV_g$$

$$\int_M \xi^j\xi^l H^{rs}\nabla_l\nabla_j H_{rs}dV_g=-\int_M |\nabla_\xi H|^2 dV_g$$

$$\int_M \xi^j\xi^l\nabla_r H_{sj}\nabla^r H^s_l dV_g=\int_M \nabla_r\xi^j\nabla^r\xi^l H_{sj}H^s_l dV_g$$

$$\int_M \xi^j\xi^l\nabla_r H_{sj}\nabla^s H^r_l dV_g=\int_M \nabla_r\xi^j\nabla^s\xi^l H_{sj}H^r_l dV_g .$$

Therefore

$$I_2=\int_M\Big[-2\nabla_l\nabla_r\xi^j\xi^l H^{rs}H_{sj}-2\nabla_r\xi^j\xi^l\nabla_l H^{rs}H_{sj} -2\nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj}+2\nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl} +\nabla_r\xi^j\nabla^r\xi^l H_{js}H^s_l-\nabla_r\xi^j\nabla^s\xi^l H_{sj}H^r_l -\frac{1}{2}|\nabla_\xi H|^2\Big]dV_g$$

but

$$\int_M \xi^l\nabla_l\nabla_r\xi^j H^{rs}H_{sj}dV_g=0$$

$$\int_M \nabla_l\xi^j\nabla_r\xi^l H^{rs}H_{sj}dV_g=\int_M(-|H|^2+|hH|^2)dV_g$$

$$\int_M \nabla_r\xi^j\nabla_s\xi^l H^{rs}H_{jl}dV_g=\int_M(-|H|^2-tr(hH)^2)dV_g$$

$$\int_M \nabla_r\xi^j\nabla^r\xi^l H_{sj}H^s_l dV_g=\int_M(|H|^2+|hH|^2)dV_g$$

$$\int_M \nabla_r\xi^j\nabla^s\xi^l H_{sj}H^r_l dV_g=\int_M(|H|^2-tr(hH)^2)dV_g$$

and hence

$$I_2=\int_M\Big[2(\phi^j_r+\phi_{ri}h^{ij})\nabla_\xi H^r_s H^s_j-tr(hH)^2 -|hH|^2-\frac{1}{2}|\nabla_\xi H|^2\Big]dV_g .$$

Since $\phi_{ri}h^{ij}\nabla_\xi H^r_s H^s_j=0$, we have

$$\frac{d^2I}{dt^2}(0)=I_1+I_2$$

$$=\int_M\Big[-2|H|^2+2\phi^j_r\nabla_\xi H^r_s H^s_j-\frac{1}{2}|\nabla_\xi H|^2$$

$$-tr(hH)^2-|hH|^2\Big]dV_g$$

$$=\int_M\Big[-\frac{1}{2}|2H-\phi\nabla_\xi H|^2-tr(hH)^2-|hH|^2\Big]dV_g .$$

Now note that

$$|\mathcal{L}_\xi H^i_j|^2=|\nabla_\xi H-2H\phi|^2+|Hh+hH|^2$$
$$=|\nabla_\xi H-2H\phi|^2+2tr(hH)^2+2|hH|^2$$

therefore

$$\frac{d^2L}{dt^2}(0)=(-4)\frac{d^2I}{dt^2}(0)=2\int_M|\mathcal{L}_\xi H^i_j|^2dV_g\geqq 0 .$$

We show in the next proposition that $|\tau(t)|^2$ is constant along any geodesic $g(t)=ge^{Ht}$ with $\mathcal{L}_\xi H^i_j=0$, hence, $L(g)$ is constant along all such geodesics. $\mathcal{M}(\eta)$ is geodesically complete [4], therefore $L(g)$ has minimum at each critical metric.

Q. E. D.

PROPOSITION. *$\tau^i_j(t)=\tau^i_j(0)$ along any geodesic $g(t)=ge^{Ht}$ with $\mathcal{L}_\xi H^i_j=0$. In particular, $|\tau(t)|^2$ is constant along such geodesics.*

Proof. Let $D^{(n)i}_{jk}=\nabla_j(H^n)^i_k+\nabla_k(H^n)^i_j-\nabla^i(H^n)_{jk}$. If $\mathcal{L}_\xi H^i_j=0$, we have $\nabla_\xi H=2H\phi$ and $hH=-Hh$, and hence

$$D^{(n)i}_{jk}\xi^k=\nabla_\xi(H^n)^i_j+(H^n)^i_k\phi^k_j+\phi^i_k(H^n)^k_j-(H^n)^i_r h^r_k\phi^k_j-\phi^i_r h^r_k(H^n)^k_j$$
$$=2(H^n)^i_r\phi^r_j$$

for any n. Thus along ge^{Ht} with $\mathcal{L}_\xi H^i_j=0$,

$$\nabla^{(t)}_j\xi^i=\nabla_j\xi^i+\frac{t}{2}D_{jk}{}^i\xi^k+\frac{t^2}{2}\Big(\frac{1}{2}D^{(2)i}_{jk}\xi^k-H^i_rD_{jk}{}^i\xi^k\Big)+\cdots$$
$$+\frac{t^n}{2}\Big[\frac{1}{n!}D^{(n)i}_{jk}+\frac{1}{(n-1)!}(-1)H^i_rD^{(n-1)r}_{jk}+\frac{1}{(n-2)!2!}(H^2)^i_rD^{(n-2)r}_{jk}+\cdots$$
$$+\frac{1}{(n-l)!l!}(-1)^l(H^l)^i_rD^{(n-l)r}_{jk}+\cdots+\frac{1}{(n-1)!}(-1)^{n-1}(H^{n-1})^i_rD_{jk}{}^r\Big]\xi^k$$
$$+\cdots$$

and therefore

$$-\phi^i_j(t)+\frac{1}{2}\tau^i_j(t)=-\phi^i_j+\frac{1}{2}\tau^i_j-t\phi^i_rH^r_j-\cdots-\frac{t^n}{n!}\phi^i_r(H^n)^r_j-\cdots .$$

Note that $\phi(t)=\phi e^{Ht}$; therefore we have

$$\tau^i_j(t)=\tau^i_j(0)$$

along ge^{Ht} with $\mathcal{L}_\xi H^i_j=0$. Q. E. D.

Example 1. Any K-contact manifold, since $\tau=0$, $L(g)$ has minimum trivially.

Example 2. The tangent sphere bundle of a compact Riemannian manifold of constant curvature (-1), i.e. $T_1M(-1)$ (see [3]). In this case the standard associated metric is a critical point of $L(g)$, but τ is not 0. In fact, non-trivial examples must be irregular (see [2]). Theorem 2 says that $L(g)$ has local minimum at the standard metric. It seems that it is also a global minimum, or in other words, one can not deform the metric to have $\tau=0$.

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