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A UNICITY THEOREM FOR MEROMORPHIC MAPPINGS INTO COMPACTIFIED LOCALLY SYMMETRIC SPACES

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Introduction

The classical theorem of Nevanlinna states that non-constant holomorphic mappings $f, g: \boldsymbol{C} \to \boldsymbol{P}_1(\boldsymbol{C})$ satisfying $f^{-1}(a_i)=g^{-1}(a_i)$ with multiplicities for distinct five points $a_1, \cdots a_5 \in \boldsymbol{P}_1(\boldsymbol{C})$ are identical ([11]). The unicity theorems of this type for holomorphic (or meromorphic) mappings were studied by several authors (cf., *e.g.*, [4], [5], [6] and [14]). For instance, in [6], H. Fujimoto studied meromorphic mappings $f: \boldsymbol{C}^n \to \boldsymbol{P}_m(\boldsymbol{C})$, using Borel's theorem and obtained many interesting results. On the other hand, S. Drouilhet [5] proved a unicity theorem of another type for meromorphic mappings $f: M \to V$, where M is a smooth affine variety and V is a smooth projective variety with $\dim V \leqq \dim M$. He used the second main theorem for meromorphic mappings due to Shiffman [15]. In this paper, we prove some unicity theorems for meromorphic mappings of a finite analytic covering space over $\boldsymbol{C}^n$ into a smooth toroidal compactification of a locally symmetric space, by making use of a second main theorem proved in [1].

Let $\mathcal{D}$ be a bounded symmetric domain in $\boldsymbol{C}^m$ and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ a neat arithmetic group. Let γ be a positive rational number such that the holomorphic sectional curvature of the Bergman metric on $\mathcal{D}$ is bounded by $-\gamma$ from above. We denote by $\overline{\Gamma \backslash \mathcal{D}}$ a smooth toroidal compactification of $\Gamma \backslash \mathcal{D}$ such that $D = \overline{\Gamma \backslash \mathcal{D}} - \Gamma \backslash \mathcal{D}$ is a hypersurface with only normal crossings. Let $\iota: \overline{\Gamma \backslash \mathcal{D}} \to \boldsymbol{P}_N(\boldsymbol{C})$ be a non-constant holomorphic mapping and $[H] \to \boldsymbol{P}_N(\boldsymbol{C})$ the hyperplane bundle over $\boldsymbol{P}_N(\boldsymbol{C})$. Let $\pi: X \to \boldsymbol{C}^n$ be a finite analytic covering with ramification divisor R. Then we have the following unicity theorem for meromorphic mappings $f: X \to \overline{\Gamma \backslash \mathcal{D}}$ in the case $1 \leqq n < m$ (see Theorem 2.1 in § 2):

Let $f, g: X \to \overline{\Gamma \backslash \mathcal{D}}$ be meromorphic mappings of maximal rank such that $f^{-1}(D)=g^{-1}(D)=E$ and $f=g$ on E. Assume that

$$L = K(\overline{\Gamma \backslash \mathcal{D}}) \otimes [D] \otimes \frac{2}{\gamma} \iota^*[H]^{-1}$$

is big and $|\nu L \otimes [D]^{-1}|$ has no base point in $\Gamma \backslash \mathcal{D}$ for $\nu \gg 0$. We also assume that

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$$\liminf_{r\to+\infty} \frac{2N(r, R)}{T_f(r, L)+T_g(r, L)}<\gamma .$$

Then $\iota\circ f=\iota\circ g$ *on* X.

Moreover, in the case $1\leqq m\leqq n$, we have a similar result (see Theorem 2.4 in § 2). In § 3, we consider meromorphic mappings of X into a compact Riemann surface and give some unicity theorems which imply the classical unicity theorem in the case $X=\boldsymbol{C}$.

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1. Preliminaries

(a) Line bundles. Let M be a compact complex manifold of dimension m and let $L\to M$ be a holomorphic line bundle over M. We denote by νL the ν-th tensor power of L for a positive integer ν and by L^{-1} the dual bundle of L. Let $\Gamma(M, L)$ be the space of all holomorphic sections of $L\to M$ and $|L|=\boldsymbol{P}(\Gamma(M, L))$ the complete linear system of L.

DEFINITION 1.1. A line bundle $L\to M$ is said to be big if

$$\dim \Gamma(M, \nu L)\geqq C\nu^m$$

for all sufficiently large integers ν and some $C\in\boldsymbol{R}^+=\{r\in\boldsymbol{R};\, r>0\}$.

Let $\Phi_L: M\to\boldsymbol{P}_N(\boldsymbol{C})$ ($N=\dim\Gamma(M, L)-1$) be the meromorphic mapping associated with $|L|$; i.e., $\Phi_L=(\xi_0:\cdots:\xi_N)$ for a basis $\{\xi_0, \cdots, \xi_N\}$ of $\Gamma(M, L)$. It is well known that L is big if and only if $\dim \Phi_{\nu L}(M)=m$ for some positive integer ν. Therefore, if L is big, we can take a system of generators $\{\varphi_1, \cdots, \varphi_l\}$ of the function field $\boldsymbol{C}(M)$ of M such that each φ_i belongs to the quotient field of $\Gamma(M, \nu L)$ for some $\nu\in\boldsymbol{Z}^+$.

Let $\mathrm{Pic}(M)=H^1(M, \mathcal{O}^*)$ be the Picard group over M. An element of $\mathrm{Pic}(M)\otimes\boldsymbol{Q}$ is called a $\boldsymbol{Q}$-line bundle over M. A $\boldsymbol{Q}$-line bundle L is said to be big if a line bundle $\nu L\in\mathrm{Pic}(M)$ is big for some positive integer ν.

For a non-zero holomorphic section σ of $L\to M$, we denote by (σ) the effective divisor of zeros of σ. Then we have $|L|=\{(\sigma);\, \sigma\in\Gamma(M, L)-\{0\}\}$. When $|L|\neq\emptyset$, we say that $p\in M$ is a base point for $|L|$ if p is contained in the support of every $D\in|L|$. We denoted by $Bs|L|$ the set of all base points for $|L|$.

(b) First main theorem. Let $\pi: X\to\boldsymbol{C}^n$ be a finite analytic covering: that is, X is a normal complex space and π is a proper surjective holomorphic mapping with finite fibre. A finite analytic covering $\pi: X\to\boldsymbol{C}^n$ is said to be algebraic covering if X is biholomorphic to an affine variety and π is a rational mapping. We denote by k the sheet number of $\pi: X\to\boldsymbol{C}^n$. Let

$z=(z_1, \cdots, z_n)$ be the natural complex coordinate system in C^n and set

$$\|z\|^2=\sum_{\nu=1}^{n} z_\nu \bar{z}_\nu, \qquad B(r)=\{z\in C^n ; \|z\|<r\},$$

$$X(r)=\pi^{-1}(B(r)), \qquad d^c=\frac{\sqrt{-1}}{4\pi}(\bar{\partial}-\partial),$$

$$\alpha=\pi^* dd^c\|z\|^2 .$$

For a (1, 1)-current φ of order 0 on X, we set

$$n(r, \varphi)=r^{2-2n}\langle\varphi\wedge\alpha^{n-1}, \chi_{X(r)}\rangle$$

and

$$N(r, \varphi)=\int_1^r \frac{n(t, \varphi)}{kt} dt ,$$

where $\chi_{X(r)}$ denotes the characteristic function of $X(r)$.

Let M be a compact complex manifold. Let $L\to M$ be a holomorphic line bundle over M with a hermitian fibre metric h and ω its Chern form. For a meromorphic mapping $f : X\to M$, we set

$$T_f(r, L)=N(r, f^*\omega)$$

and call it the characteristic function of f with respect to L. We note that $T_f(r, L)$ is independent of the choice of a metric h up to an $O(1)$-term. If L is ample, it is clear that $T_f(r, L)\to\infty$ as $r\to+\infty$. Even if L is big, $T_f(r, L)$ also has this property. In fact, we can show the following proposition:

PROPOSITION 1.2. *Let* $f : X\to M$ *be a non-constant meromorphic mapping. Assume that* L *is big and* $f(X)\not\subset Bs|\mu L|$ *for some* $\mu\in Z^+$. *Then there exists a positive constant* C *such that*

$$C \log r\leqq T_f(r, L)+O(1) .$$

In paticular, $T_f(r, L)\to+\infty$ *as* $r\to+\infty$.

Proof. Let $\Phi_{\nu L} : M\to P_N(C)$ be the meromorphic mapping associated with $|\nu L|$ and $W=\Phi_{\nu L}(M)$. Since L is big, $\dim W=m$ for some $\nu\in Z^+$. We may assume that $f(X)\not\subset Bs|\nu L|$. Let $F=\Phi_{\nu L}\circ f$ and $[H]\to P_N(C)$ the hyperplane bundle. Then $F: X\to P_N(C)$ is a non-constant meromorphic mapping and

$$T_f(r, L)=T_F(r, [H])+O(1) .$$

Let $\{\varphi_1, \cdots, \varphi_l\}$ be a system of generators of $C(W)$ such that $F^*\varphi_j$ are well defined for $j=1, \cdots, l$. It is well known that

$$T(r, F^*\varphi_j)\leqq T_F(r, [H])+O(1)$$

for $j=1, \cdots, l$, where $T(r, F^*\varphi_j)$ denotes the characteristic function of a meromorphic mapping $F^*\varphi_j : X \rightarrow \boldsymbol{P}_1(\boldsymbol{C})$ with respect to the point bundle over $\boldsymbol{P}_1(\boldsymbol{C})$. Since at least one of $F^*\varphi_j$ is non-constant, we have

$$C \log r \leqq T_F(r, [H]) + O(1)$$
$$= T_f(r, L) + O(1)$$

for some $C \in \boldsymbol{R}^+$. This completes the proof.

The following proposition is obtained by a direct calculation and the definition of characteristic function (cf. [7]).

PROPOSITION 1.3. (a) *Let M_1 and M_2 be compact complex manifolds with holomorphic line bundles L_1, L_2, respectively. Let $\pi_i : M_1 \times M_2 \rightarrow M_i$ $(i=1, 2)$ be the natural projections. Suppose $f : X \rightarrow M_1$ and $g : X \rightarrow M_2$ are meromorphic mappings. Set $\varphi = (f, g) : X \rightarrow M_1 \times M_2$. Then*

$$T_\varphi(r, \pi_1^* L_1 \otimes \pi_2^* L_2) = T_f(r, L_1) + T_g(r, L_2) + O(1). \tag{1.1}$$

(b) *Let M be a compact complex manifold and $L_i \rightarrow M$ $(i=1, 2)$ holomarphic line bundles over M. Then, for a meromorphic mapping $f : X \rightarrow M$,*

$$T_f(r, L_1 \otimes L_2) = T_f(r, L_1) + T_f(r, L_2) + O(1). \tag{1.2}$$

Let $f : X \rightarrow M$ be a meromorphic mapping and let $D \in |L|$ such that $f(X) \not\subset \operatorname{Supp} D$, where $\operatorname{Supp} D$ denotes the support of D. Set

$$N_f(r, D) = N(r, f^*D)$$

and

$$\bar{N}_f(r, D) = N(r, \operatorname{Supp} f^*D).$$

Now, we can state the First Main Theorem for meromorphic mappings in the following form.

THEOREM 1.4. *Let $L \rightarrow M$ be a holomorphic line bundle over M and $f : X \rightarrow M$ be a meromorphic mapping. Then*

$$N_f(r, D) \leqq T_f(r, L) + O(1) \tag{1.3}$$

for $D \in |L|$ with $f(X) \not\subset \operatorname{Supp} D$.

For a proof, see Stoll [16].

(c) Inequality of second main theorem type. Let $\mathscr{D}$ be a bounded symmetric domain in $\boldsymbol{C}^m$ and h the Bergman metric on $\mathscr{D}$ normalized in such a way that the Ricci tensor of h is equal to $-h$. It is well known that the holomorphic sectional curvature of h does not exceed $-\gamma$ for some rational number γ with $1/m \leqq \gamma \leqq 1$ (see e.g. [3], p. 219).

Let $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ be a neat arithmetic group ([2], p. 219). Since Γ is torsion-free, the quotient space $V=\Gamma \backslash \mathcal{D}$ is a smooth quasi-projective variety, called a locally symmetric variety. We denote by $\bar{V}$ a smooth toroidal compactification of V such that $D=\bar{V}-V$ is a hypersurface with only normal crossings. Note that, in general, $\bar{V}$ is an only smooth Moishezon variety. We denote by $K(\bar{V})$ the canonical bundle over $\bar{V}$ and by $[D]$ the associated line bundle to D. Set

$$K(\bar{V}, D)=K(\bar{V})\otimes[D].$$

It is well known that $K(\bar{V}, D)$ is big (see e.g., [10]). We also note that the complete linear system $|\nu K(\bar{V})+(\nu-1)[D]|$ has no base point in V for a sufficiently large integer ν.

Let $\pi: X \to \boldsymbol{C}^n$ be a finite analytic covering with the ramification divisor R. For a meromorphic mapping $f: X \to \bar{V}$, we denote by $I(f)$ the indeterminancy locus of f. Define

$$\operatorname{rank} f=\max\{(\operatorname{rank} df(z)\,;\, z\in X-(S(X)\cup I(f))\},$$

where $S(X)$ is the singular locus of X. We denote by $\mathrm{Mer}^*(X, \bar{V})$ the set of all meromorphic mappings $f: X\to\bar{V}$ with maximal rank (i.e. $\operatorname{rank} f=\min\{m, n\}$) such that $f(X)\cap V\neq\emptyset$. Let $A(r)$ and $B(r)$ be real functions defined on $[1, +\infty)$. We write

$$A(r)\leqq B(r) \;\|_E,$$

if $E\subset[1, +\infty)$ is a Borel subset with finite measure and if $A(r)\leqq B(r)$ for $r\in[1, +\infty)-E$. We set $\log^+ s=\log\max\{1, s\}$ for $s\in\boldsymbol{R}$.

The following inequality of second main theorem type will play an essential role in the next section.

THEOREM 1.5. *Let $f, g\in\mathrm{Mer}^*(X, \bar{V})$ and let $0<\varepsilon<1$ be fixed. Then*

$$\gamma T_f(r, K(\bar{V}, D))\leqq \bar{N}_f(r, D)+N(r, R)+S_f(r, \varepsilon) \tag{1.4}$$

in the case of $1\leqq n<m$, and

$$T_f(r, K(\bar{V}, D))\leqq \bar{N}_f(r, D)+N(r, R)+S_f(r, \varepsilon) \tag{1.5}$$

in the case of $1\leqq m\leqq n$, where

$$S_f(r, \varepsilon)=O(\log^+ T_f(r, [D]))+n(2n-1)\varepsilon\log r \;\|_{E(\varepsilon)}. \tag{1.6}$$

For the proof, see [1].

Remark 1.6. The assumption for Γ to be neat is used only to ensure a good compactification $\overline{\Gamma\backslash\mathcal{D}}$ of $\Gamma\backslash\mathcal{D}$. Thus Theorem 1.5 also remains valid in the case where $\Gamma\subset\mathrm{Aut}(\mathcal{D})$ is a torsion-free discrete subgroup such that $\Gamma\backslash\mathcal{D}$

has a good compactification $\overline{\Gamma\backslash\mathcal{D}}$. In the special case of $m=1$, $\mathcal{D}$ is the upper half plane $\boldsymbol{H}$ and (1.5) remains valid for arbitary Fuchsian group Γ of first kind without torsion. In fact, in the case $m=1$, h is the Poincaré metric in $\boldsymbol{H}$ and $\Gamma\backslash\boldsymbol{H}$ is a finite Riemann surface $\bar{S}=S-\{a_i\}_{i=1}^{d}$, where $\bar{S}$ denotes a compact Riemann surface of genus g_0 and $a_1, \cdots, a_d$ are distinct d points in $\bar{S}$ with $2g_0-2+d>0$. Thus h defines a singular hermitian metric which is good on $\bar{S}$ (see [10], p. 242), so S always has a good compactification $\bar{S}$.

2. Unicity theorems for $f\in\mathbf{Mer}^*(X, \bar{V})$

In this section, we will prove some unicity theorems for meromorphic mappings of a finite analytic covering $\pi: X\to\boldsymbol{C}^n$ into a smooth troidal compactification $\bar{V}$ of a locally symmetric variety V.

We keep the same notation as in § 1, (c). Let $[H]\to\boldsymbol{P}_N(\boldsymbol{C})$ be the hyperplane bundle over $\boldsymbol{P}_N(\boldsymbol{C})$ and $\iota: \bar{V}\to\boldsymbol{P}_N(\boldsymbol{C})$ a non-constant holomorphic mapping.

Now we can state our main result in the case of $1\leqq n<m$ as follows.

THEOREM 2.1. *Let* $1\leqq n<m$ *and* $f, g\in\mathrm{Mer}^*(X, \bar{V})$. *Set*

$$L=K(\bar{V}, D)\otimes\frac{2}{\gamma}\iota^*[H]^{-1}.$$

Assume that

(a) $f^{-1}(D)=g^{-1}(D)\neq\emptyset$ *as a point set (say* E*)*

(b) $f=g$ *on* $E-(I(f)\cup I(g))$

(c) *the* $\boldsymbol{Q}$*-line bundle* L *is big and* $|\nu L\otimes[D]^{-1}|$ *has no base point in* V *for some* $\nu\in\boldsymbol{Z}^+$ *with* $\nu L\in\mathrm{Pic}(\bar{V})$

(d) $\displaystyle\liminf_{r\to+\infty}\frac{2N(r, R)}{T_f(r, L)+T_g(r, L)}<\gamma.$

Then $\iota\circ f=\iota\circ g$ *on* X.

Remark 2.2. (i) In the case $X=\boldsymbol{C}^n$, we always have $E\neq\emptyset$. In fact, since the holomorphic sectional curvature of h is bounded from above by $-\gamma$, V is complete hyperbolic. Thus f is holomorphic on $\boldsymbol{C}^n-E$ and $E\neq\emptyset$ (see [9], p. 90).

(ii) We note that $N(r, R)=O(\log r)$ if and only if $\pi: X\to\boldsymbol{C}^n$ is an algebraic covering ([12], p. 274). Therefore the assumption (*d*) is satisfied if at least one of f, g is transcendental and if $\pi: X\to\boldsymbol{C}^n$ is an algebraic covering.

Proof of Theorem 2.1. Set $M=\boldsymbol{P}_N(\boldsymbol{C})\times\boldsymbol{P}_N(\boldsymbol{C})$. Denote by Δ the diagonal of M. We define a meromorphic mapping $\varphi: X\to M$ by $\varphi=(\iota\circ f, \iota\circ g)$. For the proof of Theorem 2.1, it suffices to show that the image of X by φ is contained in Δ. Assume the contrary. Let $\pi_i: M\to\boldsymbol{P}_N(\boldsymbol{C})$ $(i=1, 2)$ be the natural projections. Set

$$L_0=\pi_1^*[H]\otimes\pi_2^*[H],$$

LEMMA 2.3. *There exists a holomorphic section* σ *of* $L_0 \to M$ *such that* $\Delta \subset \mathrm{Supp}(\sigma)$ *and* $\varphi^*\sigma \neq 0$.

Proof of Lemma 2.3. Fix a homogeneous coordinate system $((\zeta_0 : \cdots : \zeta_N), (\xi_0 : \cdots : \xi_N))$ on M. Let $\{a_{kl}; 0 \leqq k < l \leqq N\}$ be a set of complex numbers such that at least one of them is not zero and

$$R(\zeta; \xi) = \sum_{0 \leqq k < l \leqq N} a_{kl}(\zeta_k \xi_l - \zeta_l \xi_k).$$

Then the bihomogeneous polynomial $R(\zeta; \xi)$ naturally determines a holomorphic section σ of $L_0 \to M$. It is clear that $\Delta \subset \mathrm{Supp}(\sigma)$. Assume that $\varphi^*\sigma = 0$ for any choice of $\{a_{kl}\}$. Write $\iota \circ f = (f_0 : \cdots : f_N)$ and $\iota \circ g = (g_0 : \cdots : g_N)$. Then $\varphi^*\sigma = 0$ implies

$$\sum_{0 \leqq k < l \leqq N} a_{kl}(f_k g_l - f_l g_k) = 0$$

on X. It follows that, for all $0 \leqq k < l \leqq N$,

$$f_k g_l - f_l g_k = 0.$$

Then we have $\iota \circ f = \iota \circ g$. This is absurd. Thus $\varphi^*\sigma \neq 0$ for some choice of $\{a_{kl}\}$. This completes the proof.

Let $\sigma \in \Gamma(M, L_0)$ be as in Lemma 2.3. Then, by (1.1) and (1.3),

$$N_\varphi(r, (\sigma)) \leqq T_f(r, \iota^*[H]) + T_g(r, \iota^*[H]) + O(1). \tag{2.1}$$

On the other hand, by the assumptions (a) and (b), we have $\varphi(E) \subset \Delta$. It is easy to see that

$$\bar{N}_f(r, D) \leqq N_\varphi(r, (\sigma)) \quad \text{and} \quad \bar{N}_g(r, D) \leqq N_\varphi(r, (\sigma)). \tag{2.2}$$

By (2.1) and (2.2), we have

$$\bar{N}_f(r, D) + \bar{N}_g(r, D) \leqq 2(T_f(r, \iota^*[H]) + T_g(r, \iota^*[H])) + O(1). \tag{2.3}$$

Applying (1.4) to (2.3), we obtain

$$\begin{aligned} &\gamma(T_f(r, K(\bar{V}, D)) + T_g(r, K(\bar{V}, D))) \\ &\leqq 2(T_f(r, \iota^*[H]) + T_g(r, \iota^*[H]) + N(r, R)) + S_f(r, \varepsilon) + S_g(r, \varepsilon). \end{aligned}$$

Thus we have

$$\gamma(T_f(r, L) + T_g(r, L)) \leqq S_f(r, \varepsilon) + S_g(r, \varepsilon) + 2N(r, R). \tag{2.4}$$

By the assumption (c), $|\nu L \otimes [D]^{-1}|$ has no base point in V for some $\nu \in \boldsymbol{Z}^+$ with $\nu L \in \mathrm{Pic}(\bar{V})$. Let $\tau \in \Gamma(\bar{V}, \nu L \otimes [D]^{-1})$ with $f^*\tau \neq 0$. By Theorem 1.4, we have

$$\begin{aligned} N_f(r, (\tau)) &\leqq T_f(r, \nu L \otimes [D]^{-1}) + O(1) \\ &\leqq \nu T_f(r, L) - T_f(r, [D]) + O(1). \end{aligned}$$

Hence

$$T_f(r, [D]) \leqq \nu T_f(r, L) + O(1).$$

In the same way, we also obtain

$$T_g(r, [D]) \leqq \nu T_g(r, L) + O(1).$$

On the other hand, by Proposition 1.2, there exists a constant $C>0$ such that

$$C \log r \leqq T_f(r, L) + O(1)$$

and

$$C \log r \leqq T_g(r, L) + O(1).$$

Dividing (2.4) by $T_f(r, L)+T_g(r, L)$ and letting $r \to +\infty$, we have $0<\gamma<C'\varepsilon+\gamma'$ for some non-negative constants C' and $\gamma'<\gamma$. Note that C' is independent of ε. Letting $\varepsilon \to 0$, we obtain a contradiction, $\gamma \leqq \gamma'$. This completes the proof of Theorem 2.1.

In the case $n \geqq m$, using (1.5), we can show the following unicity theorem in the same way.

THEOREM 2.4. *Let* $1 \leqq m \leqq n$ *and* $f, g \in \mathrm{Mer}^*(X, \bar{V})$. *Set*

$$L = K(\bar{V}, D) \otimes 2\iota^*[H]^{-1}.$$

Assume that

(a) $f^{-1}(D) = g^{-1}(D) \neq \emptyset$ *as a point set (say* E*)*
(b) $f = g$ *on* $E - (I(f) \cup I(g))$
(c) *the line bundle* $L \to \bar{V}$ *is big*
(d) $\liminf_{r \to +\infty} \dfrac{2N(r, R)}{T_f(r, L)+T_g(r, L)} < 1.$

Then $\iota \circ f = \iota \circ g$ *on* X.

Note that Theorem 2.4 is obtained by S. Drouilhet in the case where X is a smooth affine variety and target spaces are smooth projective varieties.

3. Meromorphic mappings into compact Riemann surfaces

In this section, we consider the case of $m=1$ and deduce some unicity theorems for non-constant meromorphic mappings of X into a compact Riemann surface (cf. [5]). In the case $m=1$, $\mathcal{D}$ is the upper half plane $\boldsymbol{H}$ and Γ is a finitely generated Fuchsian group of first kind which has no elliptic element. The quotient space $\Gamma \backslash \boldsymbol{H}$ is a finite Riemann surface $S = \bar{S} - \{a_i\}_{i=1}^{d}$, where $\bar{S}$ is a compact Riemann surface of genus g_0 and $a_1, \cdots, a_d$ are distinct d points in $\bar{S}$ with $2g_0-2+d>0$. For a non-constant meromorphic mapping $f: X \to \bar{S}$, we denote by $T_f(r)$ the characteristic function of f with respect to the point bundle over $\bar{S}$. We can identify $H^2(\bar{S}, \boldsymbol{Z})$ with $\boldsymbol{Z}$ (cf. [8]). Then we have $c_1([D])=d$, so

$$T_f(r, [D])=dT_f(r)+O(1).$$

Set

$$\mu(f, R)=\liminf_{r\to+\infty}\frac{N(r, R)}{T_f(r)}$$

and

$$\Theta_f(D)=1-\limsup_{r\to+\infty}\frac{\bar{N}_f(r, D)}{T_f(r, [D])}.$$

Then, from Theorem 1.5, we obtain the following defect relation which will be used later.

PROPOSITION 3.1. *Let $f: X\to\bar{S}$ be a non-constant meromorphic mapping. Then*

$$\sum_{i=1}^{d}\Theta_f(a_i)\leqq 2-2g_0+\mu(f, R). \tag{3.1}$$

Remark 3.2. If there exists a non-constant meromorphic mapping $f: X\to\bar{S}$ for which (3.1) is valid in its proper sense; i.e., $\mu(f, R)<+\infty$, we have

$$2g_0-2\leqq\mu(f, R)\leqq 2g_0-2+d.$$

In the case $g_0\geqq 2$, the existence of such a mapping is a delicate matter. It is an interesting problem to determine the case where there exist non-constant meromorphic mappings of X into $\bar{S}$ for which (3.1) remains valid in its proper sense. In the case $n=1$, the existence of non-constant holomorphic mappings is discussed from this view point in [13].

First we show a unicity theorem for meromorphic mappings of X into $\boldsymbol{P}_1(\boldsymbol{C})$ which yields Nevanlinna's unicity theorem in the case $X=\boldsymbol{C}$. We denote by k the sheet number of $\pi: X\to\boldsymbol{C}^n$.

THEOREM 3.3. *Let $f, g: X\to\boldsymbol{P}_1(\boldsymbol{C})$ be non-constant meromorphic mappings. Assume that $f^{-1}(a_i)=g^{-1}(a_i)$ $(i=1, \cdots, 2k+3)$ for distinct $2k+3$ points $a_1, \cdots, a_{2k+3}$ in $\boldsymbol{P}_1(\boldsymbol{C})$. Then $f=g$ on X.*

For the proof of Theorem 3.3, we need the following lemma.

LEMMA 3.4. *Let $f: X\to\boldsymbol{P}_1(\boldsymbol{C})$ be a non-constant meromorphic mapping. Then*

$$N(r, R)\leqq 2(k-1)T_f(r)+O(1). \tag{3.2}$$

In paticular,

$$\mu(f, R)\leqq 2(k-1).$$

For the proof, see [12].

Proof of Theorem 3.3. Set $D=\{a_i\}_{i=1}^{2k+3}$ and $S=\boldsymbol{P}_1(\boldsymbol{C})-D$. Then there exists a finitely generated Fuchsian group Γ of first kind without torsion such that $S=\Gamma\backslash\boldsymbol{H}$. In Theorem 2.4, we let $\boldsymbol{P}_N(\boldsymbol{C})=\boldsymbol{P}_1(\boldsymbol{C})$ and $\iota:\boldsymbol{P}_1(\boldsymbol{C})\to\boldsymbol{P}_1(\boldsymbol{C})$ the identity. Since $[D]=(2k+3)[H]$ and $K(\boldsymbol{P}_1(\boldsymbol{C}))=2[H]^{-1}$, we obtain

$$L=K(\boldsymbol{P}_1(\boldsymbol{C}),\ D)\otimes 2\iota^*[H]^{-1}=(2k-1)[H]>0\,.$$

By Lemma 3.4, we have

$$2N(r,\ R)\leqq(2k-2)\{T_f(r)+T_g(r)\}+O(1)\,,$$

so that

$$\liminf_{r\to+\infty}\frac{2N(r,\ R)}{T_f(r,\ L)+T_g(r,\ L)}\leqq\frac{2k-2}{2k-1}<1\,.$$

Note that $f^{-1}(D)=g^{-1}(D)\neq\emptyset$ by Proposition 3.1. Therefore we infer the desired conclusion from Theorem 2.4.

The following theorem is another type extension of Nevanlinna's theorem:

THEOREM 3.5. *Let f, $g: X\to\boldsymbol{P}_1(\boldsymbol{C})$ be non-constant meromorphic mappings. Assume that*

(a) $f^{-1}(a_i)=g^{-1}(a_i)$ $(i=1, \cdots, 5)$ *for distinct five points* $a_1, \cdots, a_5$ *in* $\boldsymbol{P}_1(\boldsymbol{C})$

(b) $\min\{\mu(f,\ R),\ \mu(g,\ R)\}<1/2$.

Then $f=g$ on X.

Proof of Theorem 3.5. Set $D=\{a_1, \cdots, a_5\}$. Let Γ and $\iota:\boldsymbol{P}_1(\boldsymbol{C})\to\boldsymbol{P}_1(\boldsymbol{C})$ be as in the proof of Theorem 3.3. Then we have

$$L=K(\boldsymbol{P}_1(\boldsymbol{C}),\ D)\otimes 2\iota^*[H]^{-1}=[H]>0\,.$$

Hence

$$\begin{aligned}&\liminf_{r\to+\infty}\frac{2N(r,\ R)}{T_f(r,\ L)+T_g(r,\ L)}\\ &=\liminf_{r\to+\infty}\frac{2N(r,\ R)}{T_f(r)+T_g(r)}\\ &\leqq 2\min\{\mu(f,\ R),\ \mu(g,\ R)\}<1\,.\end{aligned}$$

The remainder of the proof is the same as in that of Theorem 3.3.

In the case $X=\boldsymbol{C}$, we have Nevanlinna's original unicity theorem. Note that the number five in the case $X=\boldsymbol{C}$ is sharp. In fact, R. Nevanlinna has given an example to show the number five is sharp:

Example. Let $f(z)=e^{-z}$ and $g(z)=e^{z}$. Put $a_1=0$, $a_2=1$, $a_3=-1$ and $a_4=\infty$. Then $f^{-1}(a_i)=g^{-1}(a_i)$ for $i=1, \cdots, 4$ but $f\neq g$.

Let us consider the case of $g_0\geqq 1$. In [14], E. M. Schmid proved a unicity theorem for holomorphic mappings of an open Riemann surface into a smooth elliptic curve T with some conditions. In the case $X=\boldsymbol{C}$, Schmid's unicity theorem is stated as follows:

Let $f, g: \boldsymbol{C}\to T$ be non-constant holomorphic mappings such that $f^{-1}(a_i)=g^{-1}(a_i)$ for distinct five point $a_1, \cdots, a_5$ in T. Then $f=g$ on $\boldsymbol{C}$.

We will give unicity theorems for meromorphic mappings of X into $\bar{S}$ with $g_0\geqq 1$. We can deduce the above result from these theorems. Set

$$l_0=\min\{l\in\boldsymbol{Z}^+;\ \text{there exists a non-constant holomorphic mapping } \varphi:\bar{S}\to\boldsymbol{P}_1(\boldsymbol{C}) \text{ of } \deg\varphi=l\}.$$

THEOREM 3.6. *Let $\bar{S}$ be a compact Riemann surface with genus $g_0\geqq 1$. Set*

$$d=2\{(2g_0+1)(k-1)+l_0\}+1.$$

Let $f, g: X\to\bar{S}$ be non-constant meromorphic mappings. Assume that $f^{-1}(a_i)=g^{-1}(a_i)$ $(i=1, \cdots, d)$ for distinct d points $a_1, \cdots, a_d$ in $\bar{S}$. Then $f=g$ on X.

Remark 3.7. By Riemann-Roch's theorem, we give an upper bound of l_0:

$$l_0\leqq g_0+1. \tag{3.3}$$

Hence we have

$$d\leqq(2g_0+1)(2k-1)+2.$$

In particular, if $\bar{S}$ is a smooth elliptic curve T, we have $l_0=2$ and $d=6k-1$.

Proof of Theorem 3.6. Let $D=\{a_1, \cdots, a_d\}$ and $S=\bar{S}-D$. Take a non-constant meromorphic function $\varphi:\bar{S}\to\boldsymbol{P}_1(\boldsymbol{C})$ with $\deg\varphi=l_0$. In Theorem 2.4, let $\boldsymbol{P}_N(\boldsymbol{C})=\boldsymbol{P}_1(\boldsymbol{C})$ and $\iota=\varphi$. Since $c_1(\iota^*[H])$ is the number of zeros of any holomorphic section of $\iota^*[H]\to\bar{S}$, it is easy to see that $c_1(\iota^*[H])=l_0$. Thus we have

$$\begin{aligned} c_1(L)&=c_1(K(\bar{S}, D)\otimes 2\iota^*[H]^{-1}) \\ &=2g_0-2+d-2l_0 \\ &=(2g_0+1)(2k-1)-2. \end{aligned} \tag{3.4}$$

On the other hand, by Riemann-Roch's theorem, $(2g_0+1)[p]$ $(p\in\bar{S})$ is very ample. It follows easily from Lemma 3.4 that

$$N(r, R)\leqq(2g_0+1)(2k-2)T_f(r)+O(1) \tag{3.5}$$

and

$$N(r, R)\leqq(2g_0+1)(2k-2)T_g(r)+O(1).$$

Hence by (3.4) and (3.5),

$$\liminf_{r\to+\infty} \frac{2N(r, R)}{T_f(r, L)+T_g(r, L)}=\frac{(2g_0+1)(2k-2)}{(2g_0+1)(2k-1)-2}<1.$$

Hence $\iota\circ f=\iota\circ g$ by Theorem 2.4. Let $B=\{e_1, \cdots, e_t\}$ be the branch locus of $\iota=\varphi$. By Riemann-Hurwitz' formula, it is easy to see that $t\leqq 2(g_0+l_0-1)$. By (3.5) and Proposition 3.1, we have

$$\sum_{i=1}^{d} \Theta_f(a_i)\leqq(2g_0+1)(2k-3)+3.$$

Since

$$d-\{(2g_0+1)(2k-3)+3\}=2(g_0+l_0)-1>t,$$

there exists at least one a_i, say a_1, which is not contained in B and $\Theta_f(a_1)=\Theta_g(a_1)<1$. Let $q\in f^{-1}(a_1)=g^{-1}(a_1)$. Since φ is a one-to-one mapping on a neighbourhood W of $f(q)$, $f(z)=g(z)$ for all $z\in f^{-1}(W)\cap g^{-1}(W)$. Thus we have $f=g$ on X. This completes the proof.

We can also prove a unicity theorem of another type which is proved by S. Drouilhet [5] in the case where $g_0=1$ and X is a smooth affine algebraic variety:

THEOREM 3.8. *Let $\bar{S}$ be a compact Riemann surface with genus $g_0\geqq 1$. Set*

$$d=2(g_0+l_0)-1.$$

Let $f, g: X\to\bar{S}$ be non-constant meromorphic mappings. Assume that
(a) *$f^{-1}(a_i)=g^{-1}(a_i)$ for distinct d points $a_1, \cdots, a_d$ in $\bar{S}$*
(c) $\min\{\mu(f, R), \mu(g, R)\}<(1/2)(4g_0-3)$.
Then $f=g$ on X.

Proof. Let D, Γ and ι be as in the proof of Theorem 3.6. Then we have

$$\begin{aligned} c_1(L)&=c_1(K(\bar{S}, D)\otimes 2\iota^*[H]^{-1}) \\ &=2g_0-2+d-2l_0 \\ &=4g_0-3. \end{aligned}$$

Thus

$$\begin{aligned} &\liminf_{r\to+\infty} \frac{2N(r, R)}{T_f(r, L)+T_g(r, L)} \\ &=\frac{2}{4g_0-3}\liminf_{r\to+\infty}\frac{N(r, R)}{T_f(r)+T_g(r)} \\ &\leqq\frac{2}{4g_0-3}\min\{\mu(f, R), \mu(g, R)\}<1. \end{aligned}$$

Hence $\iota\circ f=\iota\circ g$ by Theorem 2.4. By the assumption (b), we have

$$\Theta_f(a_i)=\Theta_g(a_i)<\frac{1}{2}$$

for $i=1, \cdots, d$. The remainder of the proof is the same as in that of Theorem 3.7.

In the case $g_0=1$, we can take $d=5$. The following example which is due to E. M. Schmid [14] shows the number five is sharp in the case where $\bar{S}$ is a smooth elliptic curve:

Example. Let $\Lambda=\boldsymbol{Z}+\tau\boldsymbol{Z}$ ($\mathrm{Im}\,\tau>0$) be a lattice in $\boldsymbol{C}$ such that $T=\Lambda\diagdown\boldsymbol{C}$ and $\pi:\boldsymbol{C}\to T$ the natural projection. We define transcendental holomorphic mappings $f, g:\boldsymbol{C}\to T$ by $f(z)=\pi(z)$ and by $g(\varepsilon)=\pi(-z)$. Put $a_1=\pi(0)$, $a_2=\pi(1/2)$, $a_3=\pi(\tau/2)$ and $a_4=\pi(1+\tau)/2$. Then $f^{-1}(a_i)=g^{-1}(a_i)$ ($i=1, \cdots, 4$) but $f\neq g$.

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