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DUAL CONVERGENCE THEOREMS FOR THE INFINITE PRODUCTS OF RESOLVENTS IN BANACH SPACES

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1. Introduction

Let E be a Banach space, $A\subset E\times E$ an m-accretive operator, and J_r the resolvent of A. Given a sequence $\{r_n\}_{n=0}^{\infty}$ of positive reals and $x_0\in E$, we define an iterative scheme by

$$x_{n+1}=J_{r_n}x_n\,,\qquad n=0, 1, 2, \cdots. \tag{1}$$

We shall consider this scheme in particular under the assumption that

$$\sum_{n=0}^{\infty} r_n=\infty\,. \tag{2}$$

The convergence of (1) in Hilbert spaces has been studied by Rockafellar [17], Brézis and Lions [2], and Pazy [11]. Bruck and Reich [4] and Reich [14] have obtained several results in uniformly convex Banach spaces. Bruck and Passty [3] have established the convergence of weighted averages $y_n=\sum_{i=0}^{n} r_i x_i/\sum_{i=0}^{n} r_i$ in the same Banach space.

The purpose of this paper is to study convergence theorems for iterative scheme (1) in Banach spaces. In Section 3. we prove a dual convergence theorem (Theorem 1) for (1) in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and then apply this result to study the problem of weak convergence. We also use Theorem 1 to show a result in a Hilbert space, which is closely related to the results of Brézis and Lions [2], and Pazy [11]. In Section 4, we present additional results. Furthermore, using the method of the proof of Theorem 1, we give a related result on the asymptotic behavior of a certain nonlinear evolution equation.

2. Preliminaries

Let E be a real Banach space and let I denote the identity operator. Re-

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call that a subset $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $\|x_1-x_2\| \leqq \|x_1-x_2+r(y_1-y_2)\|$ for all $[x_i, y_i] \in A$, $i=1, 2$, and $r>0$. If A is accretive, for each positive r, the resolvent $J_r: R(I+rA) \to D(A)$ and the Yosida approximation $A_r: R(I+rA) \to R(A)$ are defined by $J_r=(I+rA)^{-1}$ and $A_r=(I-J_r)/r$, respectively. We know that $A_r x \in AJ_r x$ for every $x \in R(I+rA)$ and $\|A_r x\| \leqq |Ax|$ for every $x \in D(A) \cap R(I+rA)$, where $|Ax|=\inf\{\|y\|: y \in Ax\}$; see [1]. We also know that $A^{-1}0=F(J_r)$ for each $r>0$, where $F(J_r)$ is the set of fixed points of J_r. We say that A is m-accretive if A is accretive and $R(I+rA)=E$ for each $r>0$. We denote the closure of a subset D of E by $cl(D)$ and its distance from a point x in E by $d(x, D)$. We also define $|D|=d(0, D)$.

Recall that a Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$$

exists for each x and y in $U=\{x \in E: \|x\|=1\}$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. The norm is said to be Fréchet differentiable if for each $x \in U$, this limit is attained uniformly for $y \in U$. Finally, the norm is said to be uniformly Fréchet differentiable if the limit is attained uniformly for $[x, y] \in U \times U$. In this case, E is said to be uniformly smooth. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The reverse is false.

The duality mapping from E into the family of nonempty subsets of its dual E^* is defined by

$$J(x)=\{x^* \in E^*: (x, x^*)=\|x\|^2=\|x^*\|^2\}.$$

It is single valued if and only if E is smooth. If E is smooth, the duality mapping J is said to be weakly sequentially continuous at 0 if $\{J(x_n)\}$ converges to 0 in the sense of the weak-star topology of E^*, as $\{x_n\}$ converges weakly to 0 in E. We also know that an operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A_i)$ and each $y_i \in Ax_i$, $i=1, 2$, there exists $j \in J(x_1-x_2)$ such that $(y_i-y_2, j) \geqq 0$.

A Banach limit LIM is a bounded linear functional on l^∞ such that

$$\inf t_n \leqq \mathrm{LIM}\, t_n \leqq \sup t_n$$

and $\mathrm{LIM}\, t_n=\mathrm{LIM}\, t_{n+1}$ for all $\{t_n\}$ in l^∞. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z)=\mathrm{LIM}\, \|x_n-z\|^2$$

for each $z \in E$. The following lemma was proved in [7, 18].

LEMMA 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit and $u\in E$. Then*

$$\mathrm{LIM}\,\|x_n-u\|^2=\inf_{z\in E}\mathrm{LIM}\,\|x_n-z\|^2$$

if and only if

$$\mathrm{LIM}\,(z,\ J(x_n-u))=0$$

for all $z\in E$, where J is the duality mapping of E into E^*.

3. Convergence theorems

We begin this section by recalling the following definition. A sequence $\{t_n\}$ in l^∞ is said to be almost convergent if all of its Banach limits agree. Lorentz's characterization of almost convergent sequence $\{t_n\}$ is that $\lim_{n\to\infty}\left(\sum_{i=1}^{n} t_{i+k}\right)/n$ exists uniformly in $k\geqq 0$ [10]. We also say that a sequence $\{x_n\}$ in a Banach space E is weakly almost convergent to $z\in E$ if the weak $\lim_{n\to\infty}\left(\sum_{i=1}^{n} x_{i+k}\right)/n=z$ uniformly in $k\geqq 0$.

In [9], we proved the following result on the asymptotic behavior of infinite products of resolvents, which is crucial in the proof of Theorem 1.

LEMMA 2. *Let E be a Banach space and $A\subset E\times E$ an m-accretive. Suppose that $\{r_n\}$ are positive numbers with $\sum_{i=0}^{\infty} r_i=\infty$. If $\{x_n\}$ is defined by* (1), *then for all $k\geqq 1$,*

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|/r_n=\lim_{n\to\infty}\|x_n-x_{n+k}\|/\sum_{i=n}^{n+k-1} r_i$$

$$=\lim_{n\to\infty}\|x_n\|/\sum_{i=0}^{n-1} r_i=d(0,\ R(A)).$$

Now, we establish a dual convergence theorem for infinite products of resolvents.

THEOREM 1. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A\subset E\times E$ be m-accretive and $0\in R(A)$. Suppose that $\{r_n\}$ are positive numbers with $\sum_{i=0}^{\infty} r_i=\infty$. If $\{x_n\}$ is defined by* (1), *then there exists a point v in $A^{-1}0$ such that $\{J(x_n-v)\}$ is weakly almost convergent to* 0.

Proof. Since $0\in R(A)$, $\{x_n\}$ is bounded and $d(0,\ R(A))=0$. Then, $\lim_{n\to\infty} A_{r_n}x_n=0$ by Lemma 2. So, for $r>0$, $\lim_{n\to\infty}\|x_n-J_r x_n\|=0$. In fact, we know that

$$\left\|\frac{x_n-J_r x_n}{r}\right\|=\|A_r x_n\|\leqq|Ax_n|=|AJ_{r_{n-1}}x_{n-1}|\leqq\|A_{r_{n-1}}x_{n-1}\|.$$

Let LIM be a Banach limit and define a real valued function ϕ on E by

$$\phi(z)=\text{LIM}\,\|x_n-z\|^2$$

for each $z\in E$. Then, ϕ is a continuous convex function and $\phi(z)\to\infty$ as $\|z\|\to\infty$. Since E is reflexive, ϕ attains its infimum over E. Let

$$K=\{u\in E:\phi(u)=\inf\{\phi(z):z\in E\}\}.$$

Then it is easy to verify that K is nonempty, bounded, closed, and convex. Furthermore K is invariant under J_r for $r>0$. In fact, since $\lim_{n\to\infty}\|x_n-J_r x_n\|=0$, we have, for each $u\in K$,

$$\begin{aligned}\phi(J_r u)&=\text{LIM}\,\|x_n-J_r u\|^2\\&=\text{LIM}\,\|J_r x_n-J_r u\|^2\\&\leqq\text{LIM}\,\|x_n-u\|^2=\phi(u).\end{aligned}$$

We also observe that K contains a fixed point v of J_r. To see this, let $w\in A^{-1}0$ and define

$$K'=\{u\in K:\|u-w\|=d(w,K)\}.$$

Then, since E is strictly convex, K' is a singleton. Let $K'=\{v\}$. Then $\|J_r v-w\|=\|J_r v-J_r w\|\leqq\|v-w\|$, and so $J_r v=v$. On the other hand, since $\{\|x_n-w\|\}$ is nonincreasing for any $w\in A^{-1}0$, it converges. Then, $\phi(w)$ is independent of Banach limits. Thus we may assume that v minimizes ϕ for any Banach limit LIM. If follows from Lemma 1 that

$$\text{LIM}\,(z,J(x_n-v))=0$$

for all $z\in E$ and any LIM. Thus $\{(z,J(x_n-v))\}$ is almost convergent to 0. In other words, $\{J(x_n-v)\}$ is weakly almost convergent to 0.

Applying Theorem 1, we obtain the following result.

THEOREM 2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A\subset E\times E$ be m-accretive and $0\in R(A)$. Assume that $J^{-1}:E^*\to E$ is weakly sequentially continuous at 0. Let $\{r_n\}$ be positive numbers with $\sum_{i=0}^{\infty}r_i=\infty$. If $\{x_n\}$ is defined by (1), and if $x_n-x_{n+1}\to 0$ as $n\to\infty$, then there exists a point $v\in A^{-1}0$ such that $\{x_n\}$ converges weakly to v.*

Proof. By Theorem 1, there exists a point $v\in A^{-1}0$ such that $\{J(x_n-v)\}$ is weakly almost convergent to 0. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subset of E from the strong topology of E to the weak-star topology of E^*. Thus, since $\{x_n\}$ is bounded and $x_n - x_{n+1} \to 0$, $\{J(x_n - v) - J(x_{n+1} - v)\}$ converges weakly to 0. However this is a Tauberian condition for almost convergence, so $\{J(x_n - v)\}$ converges weakly to 0. Since J^{-1} is weakly sequentially continuous at 0, $\{x_n\}$ converges weakly to v.

Remark 1. The conclusion of Theorem 2 has been known for a uniformly convex Banach space with a Fréchet differentiable norm or with a duality mapping that is weakly sequentially continuous at 0 (cf. [4], [14]). The weak convergence of the sequence $\left\{\sum_{i=0}^{n} r_i x_i / \sum_{i=0}^{n} r_i\right\}$ in uniformly convex Banach space with a Fréchet differentiable norm was shown by Bruck and Pussty [3]. Theorem 2 also implies that sequence $\left\{\sum_{i=0}^{n} r_i x_i / \sum_{i=0}^{n} r_i\right\}$ converges weakly to a point of $A^{-1}0$.

As a consequence of Theorem 1, we also have the following.

COROLLARY 1. *Let H be a Hilbert space, $A \subset H \times H$ a maximal monotone operator and $0 \in R(A)$. Suppose that $\{r_n\}$ are positive numbers with $\sum_{i=0}^{\infty} r_i = \infty$. If $\{x_n\}$ is defined by* (1), *then $\{x_n\}$ is weakly almost convergent to a point v of $A^{-1}0$, which is the asymptotic center of $\{x_n\}$.*

Proof. In a Hilbert space, the duality mapping J is just the identity mapping. Thus, by Theorem 1, $\{x_n\}$ is weakly almost convergent to a point v of $A^{-1}0$. It is also clear that v is the asymptotic center of $\{x_n\}$.

COROLLARY 2. *Let H be a Hilbert space, $A \subset H \times H$ a maximal monotone operator and $0 \in R(A)$. Suppose that $\{r_n\}$ are positive numbers with $\sum_{i=0}^{\infty} r_i = \infty$. If $\{x_n\}$ is defined by the iteration* (1), *then $\{x_n\}$ converges weakly to a point of $A^{-1}0$ if and only if $\{x_n - x_{n+1}\}$ converges weakly to* 0.

Proof. Weak $\lim_{n \to \infty} (x_n - x_{n+1}) = 0$ is a Tauberian condition for almost convergence. Hence, by Corollary 1, $\{x_n\}$ converges weakly to a point of $A^{-1}0$. The reverse is obvious.

Remark 2. In [2], Brézis and Lions showed that $\{x_n\}$ converges weakly to a point of $A^{-1}0$ provided $A = \partial\phi$ is the subdifferential of a lower-semicontinuous proper convex function ϕ on H, or A is demipositive, or $\sum_{i=0}^{\infty} r_i^2 = \infty$ (cf. [11]). In this sense, Corollaries 1 and 2 are new results in Hilbert space.

4. Additional results

In this section, we obtain some results using the theorems of the previous section.

In the iteration scheme (1), let $r>0$ and $r_n=r$ for all $n=0, 1, \cdots$. Then for each $x\in E$, $x_{n+1}=J_r{}^n x$. By Theorem 1, we obtain the following result.

THEOREM 3. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A\subset E\times E$ be m-accretive and $r>0$. If $0\in R(A)$, then there exists a point v of $A^{-1}0$ such that $\{J(J_r{}^n x-v)\}$ is weakly almost convergent to 0 for each $x\in E$.*

Remark 3. Theorem 3 has been known in case of uniformly smooth Banach spaces which involve the fixed point property for nonexpansive mappings (cf. [5, 15]). However, our result does not require the property.

As a consequence of Theorem 2, we obtain the following result, which is known under the assumption that E is a uniformly convex Banach space with a Fréchet differentiable norm (cf. [6, p. 53], [16]) or with a duality mapping that is weakly sequentially continuous at 0 (cf. [4]).

COROLLARY 3. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A\subset E\times E$ be m-accretive, $r>0$ and $0\in R(A)$. If $J^{-1}: E^*\to E$ is weakly sequentially continuous at 0, then for each $x\in E$, $\{J_r{}^n x\}$ converges weakly to a point of $A^{-1}0$.*

Finally, by the method of the proof of Theorem 1, we study the convergence of the solutions of an evolution equation.

Let $A\subset E\times E$ be accretive operator, $g:[0, \infty)\to[0, \infty)$ a nonincreasing function of class C^1 such that $\lim_{t\to\infty} g(t)=0$ and $\int_0^\infty g(r)dr=\infty$, $x\in E$, $x_0\in D(A)$, and consider the following initial value problem:

$$\begin{cases} \dfrac{du(t)}{dt}+Au(t)+g(t)u(t)\ni g(t)x, & 0\leqq t<\infty \\ u(0)=x_0 \end{cases} \tag{3}$$

Several results which are related to this equation can be found in [8, 12, 13]. The following result is proved without using the fixed point property for nonexpansive mappings (cf. [8, Theorem 12]).

THEOREM 4. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and $A\subset E\times E$ be an accretive operator that satisfies $R(I+rA)\supset cl(D(A))$ for all $r>0$. Assume that $cl(D(A))$ is convex,*

$\lim_{t\to\infty} g'(t)/g^2(t)=0$, $0\in R(A)$, and $x\in cl(D(A))$. Let $u:[0, \infty)\to E$ be a limit solution of (4). *Then the strong $\lim_{t\to\infty} u(t)$ exists and belongs to $A^{-1}0$.*

Proof. Let $x_n=u(t_n)$ with $t_n\to\infty$. Since $0\in R(A)$, the sequence $\{x_n\}$ is bounded. Since we may assume that u is a strong solution of (4), $\lim_{n\to\infty}\|x_n-J_r x_n\| \leqq \lim_{n\to\infty} r|Ax_n|=0$, where J_r is the resolvent of A (cf. [13]). Let LIM be a Banach limit and define a real valued, continuous and convex function ϕ on $cl(D(A))$ by $\phi(z)=\text{LIM}\|x_n-z\|^2$.

Let K be the set of minimizers of ϕ over $cl(D(A))$ as in the proof of Theorem 1. Then, by the argument used in the proof, K contains a fixed point of J_r. Since $v\in A^{-1}0$, by the proof of [8, Proposition 11], in which was used the condition on g, $\limsup_{n\to\infty}(x_n-x, J(x_n-v))\leqq 0$. On the other hand, since $v\in K$, we can also show that $\text{LIM}(x-v, J(x_n-v))\leqq 0$. Thus $\text{LIM}\|x_n-v\|^2\leqq 0$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to v. If $\{u(s_n)\}$ converges to w, then we have $(v-x, J(v-w))\leqq 0$ and $(w-x, J(w-v))\leqq 0$. Therefore we have $v=w$ and hence the result follows.

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