INNER RADII OF TEICHMÜLLER SPACES OF FINITELY GENERATED FUCHSIAN GROUPS

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1. Introduction

Let $\Gamma$ be a Fuchsian group keeping the lower half plane $L$ invariant. The Teichmüller space $T(\Gamma)$ of $\Gamma$ is a bounded domain of the Banach space $B(L, \Gamma)$ of bounded quadratic differentials for $\Gamma$. The inner radius $i(\Gamma)$ of $T(\Gamma)$ is the radius of the maximal ball in $B(L, \Gamma)$ centered at the origin which is included in $T(\Gamma)$. If $T(\Gamma)$ is not a single point, then by a theorem of Ahlfors-Weill [3] it holds that $i(\Gamma) \geq 2$. In particular, if $\Gamma$ is finitely generated of the first kind and if $T(\Gamma)$ is not a single point, then the strict inequality $i(\Gamma) > 2$ holds (cf. [10]). Denote by $I(\Gamma) = \inf \{i(WGW^{-1})\}$, where the infimum is taken over for all quasiconformal automorphisms $W$ of the upper half plane compatible with $\Gamma$. Recently T. Nakanishi [10] proved the following.

THEOREM 1 (T. Nakanishi). Let $\Gamma$ be a finitely generated Fuchsian group of the first kind such that $T(\Gamma)$ is not a single point. Then $I(\Gamma)$ is equal to 2.

The purpose of this note is to prove the following generalization to Theorem 1.

THEOREM 2. Let $\Gamma$ be a finitely generated Fuchsian group such that $T(\Gamma)$ is not a single point. Then $I(\Gamma)$ is equal to 2.

The proof of Theorem 2 is immediate from Theorem 1 and the following.

THEOREM 3. Let $\Gamma$ be a finitely generated Fuchsian group of the second kind. Then $i(\Gamma)$ is equal to 2.

A careful reading of the proof of Theorem 3 shows the readers an alternative proof of Theorem 1, though we omit it. Our proof of Theorem 3 depends on results on B-groups [1], [4] and Koebe groups [9].

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2. Preliminaries

2.1. Let $PSL(2, C)$ be the group of all conformal automorphisms of the extended complex plane $C \cup \{\infty\}$. Denote by $PSL(2, R)$ the subgroup of $PSL(2, C)$ which consists of all conformal automorphisms of the upper half plane $U=\{z; \text{Im} z>0\}$. A Fuchsian group is a discrete subgroup of $PSL(2, R)$. A Fuchsian group is of the first kind (resp. the second kind) if it acts discontinuously at no point (resp. some point) of the real axis.

2.2. We define a hyperbolic metric $\rho_{U}(z)|dz|$ in $U$ as $(2\text{Im} z)^{-1}|dz|$. Let $f$ be a holomorphic function of $U$ onto a domain $D\subset \mathbb{C}$ with more than two boundary points. Then the hyperbolic metric $\rho_{D}(z)|dz|$ is defined by $\rho_{D}(f(z))|f'(z)|=\rho_{U}(z)$. Assume moreover that $D$ is a connected and simply connected domain of $\mathbb{C}$. Then $(4X(z))^{-1}\leq \rho_{\mathbb{C}}(z)$, where $X(z)$ is the Euclidean distance between a point $z$ of $D$ and the boundary of $D$. In particular, if $D=\{z; |\text{Im} z|<\pi/2\}$, then $1/(2\pi)\leq \rho_{D}(z)$. If $D,\subset D_{a}$, then by Schwarz's lemma we see that $\rho_{D_{1}}(z)\leq \rho_{D_{2}}(z)$ [5; p. 45].

2.3. A holomorphic function $\phi(z)$ in the lower half plane $L=\{z; \text{Im} z<0\}$ is a bounded quadratic differential for a Fuchsian group $\Gamma$ if

$$\|\phi\| = \sup_{z\in L} \rho_{L}(z)^{-1}|\phi(z)| < \infty$$

and

$$\phi(\gamma(z))\gamma'(z)\phi(z)$$

for all $\gamma \in \Gamma$ and all $z \in L$.

The space $B(L, \Gamma)$ of all bounded quadratic differentials for $\Gamma$ can be regarded as a Banach space with the norm $\|\|$ defined above.

2.4. An element $\gamma$ of $\Gamma$ is primitive if $j^{n}=\gamma$ has no solution in $\Gamma$ for $n \neq \pm 1$. The following lemma is well known but the author has never seen what is stated in this form.

**Lemma 1.** Let $\Gamma$ be a Fuchsian group keeping the upper half plane invariant which contains a primitive parabolic element $p(z) = z + 1$. Then for each $\phi \in B(L, \Gamma)$ it holds that

$$\sup_{|\text{Im} z| < 1} \rho_{L}(z)^{-1}|\phi(z)| = \sup_{|\text{Im} z| < 1} \rho_{L}(z)^{-1}|\phi(z)|.$$ 

**Proof.** Recall that $\phi(z)$ has a Fourier expansion $\sum_{n=-1}^{\infty} \exp(-2\pi inz)$ [5; p. 111]. Note that

$$4y^{4}|\phi(z)| = 4y^{4}\exp(2\pi y)|\sum_{n=1}^{\infty} \exp(-2\pi i(n-1)z)|,$$

where $y=\text{Im} z$. Then by the principle of the maximal absolute value and $d(y^{4}\exp 2\pi y)/dy \geq 0$ for $y \leq -1/\pi$, we have the desired conclusion. \qed

2.5. Let $Q(\Gamma')$ be the set of all conformal homeomorphisms $f$ of $L$ admitting quasiconformal extensions $\hat{f}$ to the extended complex plane which are
compatible with $\Gamma$, that is, $f\Gamma f^{-1}\subset PSL(2, C)$. For each $f\in Q(\Gamma)$, its Schwarzian derivative $[f]=(f''/f')-(f''/f')/2$ belongs to $B(L, \Gamma)$. The Teichmüller space $T(\Gamma)$ of $\Gamma$ is the image of $Q(\Gamma)$ under the mapping $f\rightarrow [f]$. The inner radius $i(\Gamma)$ of $T(\Gamma)$ is $\inf_{\phi\in L, n-\tau cn}\phi l$. If $g\in PSL(2, \mathbb{C})$, then $[s, / sJ=(l\Omega°gi)gi'$ and $||[s, / sJ||=||\phi||$. In particular, if $g\in PSL(2, \mathbb{R})$, then $f\circ g\circ f^{-1}=Q(g\Gamma g^{-1})$ and $i(g\Gamma g^{-1})=i(\Gamma)$.

2.6. A component of the region of discontinuity of a Kleinian group $G$ is called a component of $G$. An invariant component of $G$ is a component of $G$ which is invariant under $G$. A Kleinian group $G$ is a B-group if $G$ has exactly one simply connected invariant component. An Euclidean disc (including a half plane) $D$ is a horodisc of a primitive parabolic element $g$ of $G$ if $j(D)=D$ for each $f\in\langle g\rangle$, the cyclic group generated by $g$ and $j(D)\cap D=\emptyset$ for each $f\in G-\langle g\rangle$. A B-group $G$ is regular if for each primitive parabolic element $g$ of $G$ there exist two mutually disjoint horodiscs of $g$ (Abikoff [1]). A regular B-group is a Koebe group if each noninvariant component of $G$ is an Euclidean disc. Note that our definition of a Koebe group is stronger than Maskit’s original one [9].

3. Proof of theorem 3

3.1. Let $\Gamma$ be a finitely generated Fuchsian group of the second kind such that $L/\Gamma$ is a compact Riemann surface with finitely many points and $m\geq 1$ discs removed. Then classical is the existence of a hyperbolically convex fundamental region $\omega$ for $\Gamma$ in $L$ satisfying the following: There exist $2m$ sides $S_1, \ldots, S_{2m}$ of $\omega$ consisting of hyperbolic half lines and primitive hyperbolic elements $a_1, \ldots, a_m$ of $\Gamma$ such that $a_1(S_1)=S_{k+1}$ and such that a component of $R\cup\{\infty\}$ minus the fixed points of $a_k$ is included in the region of discontinuity of $\Gamma$, $k=1, \ldots, m$.

Let $E_k$ be the geodesic included in $\omega$ tangent to $S_k$ and $S_{k+1}$, $k=2, \ldots, m$. Let $H_k, H_k'$ and $E_{1, n}$ be geodesics included in $\omega$ such that $S_k, H_k, E_{1, n}, H_k'$ and $S_{k+1}$ lie in this order and such that the hyperbolically convex domain $\omega_n$ surrounded by all sides of $\omega$ together with $H_k, E_{1, n}, H_k'$ and $E_k, \ldots, E_m$ is of a finite hyperbolic area. Let $\varepsilon_k\in PSL(2, \mathbb{R})$ (resp. $\varepsilon_k\in PSL(2, \mathbb{C})$) be an elliptic transformation of order 2 keeping $E_k$ (resp. $E_{1, n}$) and the middle point of $E_k$ (resp. $E_{1, n}$) invariant, $k=2, \ldots, m$. Let $\gamma_n$ be a hyperbolic transformation with $\gamma_n(H_k)=H_k'$ and $\gamma_n(\omega_n)\cap \omega_n=\emptyset$. Then $\Gamma$ and $\gamma_n$ and $\varepsilon_{1, n}, \varepsilon_2, \ldots, \varepsilon_m$ generate a finitely generated Fuchsian group $\Gamma_n$ of the first kind with the fundamental region $\omega_n$. We assume that $\{\gamma_n\}_{n=1}^\infty$ converges to a parabolic transformation. Then $\{E_{1, n}\}_{n=1}^\infty$ necessarily degenerates to a point.

3.2. Let $P_1, \ldots, P_{k, n}$ be a maximal list of primitive parabolic elements of $\Gamma$ whose fixed points lie on the boundary of $\omega_n$ such that $P_{r, n} \neq P_{s, n}^{-1}, 1\leq r<s\leq t$. Let $D_{s, n}={\xi}_{s, n}(\{z; \operatorname{Im} z<1\})$ be the horodisc of the primitive parabolic element $P_{s, n}$, where $\xi_{s, n}$ is the element of $PSL(2, \mathbb{R})$ such that $\xi_{s, n}\circ P_{s, n}\circ \xi_{s, n}^{-1}$ is of the form $z \rightarrow z+1$. The existence of such a horodisc is
immediate from Shimizu's lemma [5; p. 58]. For our later use, we prove a preliminary lemma.

**Lemma 2.** Let \( u_n \) be a point of \( \omega_n - \bigcup_{i=1}^{n} D_{i,n} \). Then \( \{d_L(u_n, \gamma_n(u_n))\}_{n=1}^{\infty} \) is bounded, where \( d_L(u_n, \gamma_n(u_n)) \) is the hyperbolic distance between \( u_n \) and \( \gamma_n(u_n) \) measured by \( \rho_0(z) \mid dz \).

**Proof.** The axis \( A_n \) of \( \gamma_n \) divides \( \omega_n \) into \( \omega_n^{-1} \) and \( \omega_n^* \) whose boundary includes \( E_{1,n} \). Let \( v_n \) be a point of the closure of \( \omega_n - \bigcup_{i=1}^{n} D_{i,n} \) such that \( d_L(v_n, A_n) \geq d_L(z, A_n) \) for all \( z \in \omega_n - \bigcup_{i=1}^{n} D_{i,n} \). Note the existence of a compact subset of \( L \) containing all \( v_n \in \omega_n^{-1} \). Then \( d_L(v_n, \gamma_n(v_n)) \) is less than a constant for all \( v_n \in \omega_n^{-1} \). Let \( \tau_n \) be the element of \( \text{PSL}(2, \mathbb{R}) \) such that \( \tau_n(z_n^*) = -1 \) and \( \tau_n(z_n^*) > 0 \), where \( z_n^* \) is the fixed point of \( \varepsilon_{1,n} \) in \( \omega_n \). Then \( \{\tau_n^{*} \gamma_n^{*} \tau_n^{-1}\}_{n=1}^{\infty} \) converges to a parabolic transformation and a compact subset of \( L \) contains all \( \tau_n(v_n) \) for all \( v_n \in \omega_n^* \). By the same reasoning as above we see that \( d_L(v_n, \gamma_n(v_n)) = d_L(\tau_n(v_n), \tau_n^{*} \gamma_n^{*} \tau_n^{-1}(\tau_n(v_n))) \) is less than a constant for all \( v_n \in \omega_n^* \). Note that \( d_L(u_n, A_n) \leq d_L(v_n, A_n) \). Then \( d_L(u_n, \gamma_n(u_n)) \leq d_L(v_n, \gamma_n(v_n)) \). Now our assertion is obvious. \( \square \)

3.3. Now we begin to make a proof of Theorem 3. Let \( x_n \) be the isomorphism of \( \Gamma_n \) onto a regular B-group \( \chi_n(\Gamma_n) \) on the boundary of \( T(\Gamma_n) \) such that \( \chi_n(\gamma) \) is parabolic if and only if \( \gamma \) is either parabolic or conjugate to \( \gamma_n \) in \( \Gamma_n \). Let \( w_n \) be a conformal homeomorphism of \( L \) onto the invariant component of \( \chi_n(\Gamma_n) \) such that \( \chi_n(\gamma) \cdot w_n(z) = w_n \cdot \gamma(z) \) for all \( z \in L \) and all \( \gamma \in \Gamma \).

The existence of such a \( x_n \) and a \( w_n \) is shown in Bers [4] and Abikoff [1]. Maskit [9] proved that there exist a Kobke group \( G_n \) and a conformal homeomorphism \( j_n \) of the invariant component of \( \chi_n(\Gamma_n) \) onto that \( A_n \) of \( G_n \) such that \( j_n \chi_n(\Gamma_n) j_n^{-1} = G_n \) and such that \( j_n \chi_n(\Gamma_n) j_n^{-1} \) is parabolic if and only if so is \( \chi_n(\gamma) \). Set \( f_n = j_n \cdot w_n \). Then \( \zeta = f_n(z) \) is a conformal homeomorphism of \( L \) onto \( \Delta_n \) and \( f_n^{*} \gamma_n^{*} f_n^{-1} \) is parabolic, so that \([f_n]\) does not belong to \( T(\Gamma_n) \). Since \( ||[f_n]|| = ||[\gamma_n f_n]\| \) for all \( \gamma \in \text{PSL}(2, \mathbb{R}) \), without loss of generality we may assume that \( g_n = f_n^{*} \gamma_n^{*} f_n^{-1} \) is of the form \( \zeta - a \cdot b_n \cdot b_n > 0 \), and that two non-invariant components \( D_n^{+} \) and \( D_n^{-} \) of \( G_n \) invariant under \( g_n \) are \( \{\xi; \text{Im} \zeta > \pi/2\} \) and \( \{\xi; \text{Im} \zeta < -\pi/2\} \), respectively. Let \( z_n \) be a point of both the axis of \( \gamma_n \) and the fundamental region \( w_n \) constructed in No. 3.1. Then by the same reasoning as above, we may also assume that \( \text{Re} f_n(z_n) = 0 \). From basic properties of the hyperbolic metric stated in No. 2.2 we have

\[
D_L(z_n, \gamma_n(z_n)) = d_{\Delta_n}(f_n(z_n), f_n(\gamma_n(z_n))) \\
\geq d_{[\xi; \text{Im} \xi < \pi/2]}(f_n(z_n), g_n(f_n(z_n))) \geq b_n/2\pi.
\]

Since \( \{\gamma_n\}_{n=1}^{\infty} \) converges to a parabolic transformation, the first term in the above inequalities converges to zero. Now we have the first assertion in the
LEMMA 3. (i) The sequence \( \{b_n\}_{n=1}^{\infty} \) of positive numbers converges to zero.

(ii) The invariant component \( \Delta_n \) of \( G_n \) includes the region \( \{\zeta; |\text{Im}\zeta| < (\pi/2) - b_n\} \).

Proof. We have only to prove (ii). By the assumptions on \( X_n \) we see that \( G_n \) is constructed from Fuchsian groups \( H_n^+ = \{g \in G_n; g(D_n^+)=D_n^+\} \) and \( H_n^- = \{g \in G_n; g(D_n^-)=D_n^-\} \) with the amalgamated parabolic cyclic subgroup generated by \( g_n \) via Maskit's combination theorem I. For terminologies see [6], [7] and [8].

For a Möbius transformation \( h \) of the form \( z \mapsto (az+b)/(cz+d) \) with \( c \neq 0 \), that is, \( h^{-1}(\infty) = -d/c \neq \infty \), we define the isometric circle \( I(h) \) of \( h \) as \( \{z; |z-h^{-1}(\infty)| = 1/|c|\} \). Denote by \( \text{ext}(\mathcal{C}, I(h)) \) the unbounded component of \( \mathbb{C} - I(h) \). The region \( \omega_n^+ = \{\zeta; 0<\text{Re}\zeta < b_n\} \cap (\bigcap \text{ext} I(h)) \) (resp. \( \omega_n^- = \{\zeta; 0<\text{Re}\zeta < b_n\} \cap (\bigcap \text{ext} I(h)) \)) is a fundamental region for \( H_n^+ \) (resp. \( H_n^- \)), where the intersection \( \bigcap \) (resp. \( \bigcap \)) is taken over for all elements of \( J_n^+ = \{h \in H_n^+; h(\infty) \neq \infty\} \) (resp. \( J_n^- = \{h \in H_n^-; h(\infty) \neq \infty\} \)). Maskit's combination theorem I shows that \( \omega_n^+ \cup \omega_n^- \) is a fundamental region for \( G_n \). Note that centers \( h^{-1}(\infty) \) of the isometric circles of \( h_n \in J_n^+ \) (resp. \( J_n^- \)) lie on the line \( \{\zeta; |\text{Im}\zeta| = \pi/2\} \) (resp. \( \{\zeta; |\text{Im}\zeta| = -\pi/2\} \)). Since \( G_n \) contains the element \( g_n(z) = z+b_n \) the radius of the isometric circle of each element of \( J_n^+ \cup J_n^- \) is less than or equal to \( b_n \) by Shimizu's lemma. Therefore \( \Delta_n \) includes the region \( (\bigcap_{n=1}^{\infty} \omega_n^+ \cup \omega_n^-) \cap \{\zeta; |\text{Im}\zeta| < \pi/2\} \), which also does the region \( \{\zeta; |\text{Im}\zeta| < (\pi/2) - b_n\} \). \(\square\)

3.4. Denote by \( A_n \) the axis of \( \gamma_n \).

LEMMA 4. There exists a sequence \( \{t_n\}_{n=1}^{\infty} \) of positive numbers converging to zero such that \( f_n(A_n) \) is included in \( \{\zeta; |\text{Im}\zeta| < t_n\} \).

Proof. Assume that our assertion is false. Let \( a_n \) be the subarc of \( A_n \) bounded by \( z_n \) and \( \gamma_n(z_n) \). Let \( \zeta_n \) be a point of \( f_n(a_n) \) such that \( |\text{Im}\zeta_n| = \max_{\zeta \in f_n(a_n)} |\text{Im}\zeta| \). Then without loss of generality we may assume the existence of a subsequence, again denoted by \( \{\zeta_n\}_{n=1}^{\infty} \), of \( \{\zeta_n\}_{n=1}^{\infty} \) such that \( \{\text{Im}\zeta_n\}_{n=1}^{\infty} \) converges to a positive number \( v_0 \). By means of basic properties of the hyperbolic metric stated in No. 2.2, we have

\[
\int_{a_n} \rho_1(z) |dz| = \int_{f_n(a_n)} \rho_{f_n}(\zeta) |d\zeta| \\
\geq \int_{f_n(a_n)} \rho_{|\text{Im}\zeta|<v_0/2}(\zeta) |d\zeta| \geq (1/2\pi) \int_{f_n(a_n)} |d\zeta|.
\]

Since the first term converges to zero, so does the Euclidean length \( \int_{f_n(a_n)} |d\zeta| \) of \( f_n(a_n) \). Therefore for a sufficiently large \( n \) on, the arc \( f_n(a_n) \) is included
in \( \{ \zeta; \text{Im} \zeta > v_0/2 \} \), and so is \( f_n(A_n) = \bigcup_{n=1}^\infty g_n(f_n(a_n)) \). The geodesic \( f_n(A_n) \) in \( A_n \) divides \( A_n \) into the upper half \( A_n^+ \) and the lower half \( A_n^- \), both of which are invariant under \( \langle g_n \rangle \). The region \( A_n^+ \) is included in \( \Pi_n^+ = \{ \zeta; v_0/2 < \text{Im} \zeta < \pi/2 \} \) and by Lemma 2 \( A_n^- \) includes \( \Pi_n^- = \{ \zeta; -\pi/2 + b_n < \text{Im} \zeta < v_0/2 \} \). Let \( S_{1,n}, S_{2,n}, S_{3,n} \) and \( S_{4,n} \) be sets of all loops separating two boundary components of \( A_n^+ \langle g_n \rangle, \Pi_n^+ \langle g_n \rangle, \Pi_n^- \langle g_n \rangle \) and \( A_n^- \langle g_n \rangle \), respectively. Denote by \( \lambda_{k,n} \) the extremal length of \( S_{k,n} \). Then \( \lambda_{1,n}^{-1} \geq \lambda_{2,n}^{-1} \geq \lambda_{3,n}^{-1} \geq \lambda_{4,n}^{-1} \) if \( n \) is large enough so that \( v_0/2 > b_n \) [2; p. 15]. On the other hand, the Möbius transformation \( r_n \) of the form \( z \rightarrow -iz \) maps \( f_n^{-1}(A_n^+) = \{ z; -\pi/2 < \text{arg} z < 0 \} \) onto \( f_n^{-1}(A_n^-) = \{ z; -\pi < \text{arg} z < -\pi/2 \} \) and it holds that \( r_n \circ r_n = r_n \). Hence the conformal homeomorphism \( f_n \circ r_n \circ f_n^{-1} \) maps \( A_n^+ \) onto \( A_n^- \) and \( f_n \circ r_n \circ f_n^{-1} \) \( g_n = g_n \circ f_n \circ r_n \circ f_n^{-1} \). Therefore \( A_n^+ \langle g_n \rangle \) is conformal to \( A_n^- \langle g_n \rangle \) and \( \lambda_{1,n} = \lambda_{4,n} \). This contradiction yields us to conclude that our assertion is true. 

3.5. Let \( u_n \) be a point of the closure of \( w_n - \bigcup_{k=1}^\infty D_{k,n} \) with \( \rho_\bar{L}(u_n)^{-2} = \sup_{z \in \bar{L}(u_n)} |[f_n(u_z)]| \). The existence of such a point is immediate from Lemma 1. Without loss of generality we may assume that \( d_L(u_n, A_n) \leq d_L(u_n, \gamma(A_n)) \) for all \( \gamma \in \Gamma_n \) and that \( 0 \leq \Re f_n(u_n) < b_n \). As is stated in No. 3.2, the point \( z \in w_n \) lies on the axis of \( f_n \).

Now two cases can occur: (i) \( \{ d_L(u_n, z_n) \} \) is bounded. (ii) Otherwise.

We shall prove that (ii) never happens. Assume that (ii) does. Then since \( \{ d_L(f_n(u_n), f_n(z_n)) \} \) is unbounded, a subsequence, again denoted by \( \{ f_n(u_n) \} \), converges to a point \( \xi_0 \), which is either \( \pi i/2 \) or \(-\pi i/2 \). Let, say, \( \xi_0 = \pi i/2 \). Then each \( f_n(u_n) \) is contained in \( A_n^+ \). Set \( \eta_n(\xi) = (\xi - \Re f_n(u_n) - \pi i/2) / |\text{Im} f_n(u_n)|^{-\pi/2} \). Then \( \eta_n \) takes the point \( f_n(u_n) \) and the line \( \Re \text{Im} \xi = \pi/2 \) into \(-i\) and the real axis, respectively, and \( \eta_n(A_n) \subseteq L \). Note that \( \eta_n(A_n) \) includes the domain surrounded by \( \bigcup \eta_n(h(f_n(A_n))) \), where the union is taken over all \( h \in H_n^+ \). The parabolic transformation \( \eta_n \circ g_n \circ \eta_n^{-1} \) is of the form \( \zeta \rightarrow \zeta + e_n, e_n > 0 \). Note that

\[
d_L(u_n, \eta_n(u_n)) = d_{\eta_n \circ f_n}(\eta_n \circ f_n(u_n), \eta_n \circ f_n(\eta_n(u_n)))
\]

\[
\geq d_L(\eta_n \circ f_n(u_n), \eta_n \circ g_n(f_n(u_n))) = d_L(\eta_n \circ f_n(u_n), \eta_n \circ f_n(u_n) + e_n).
\]

Since \( \{ d_L(u_n, \eta_n(u_n)) \} \) is less than a constant \( e_0 \) by Lemma 2, so is \( \{ e_n \} \). This together with Shimizu's lemma shows that each element of \( \eta_n f_n \eta_n^{-1} \) has the isometric circle whose radius is less than or equal to \( e_n \). Since \( K_n = \inf_{\xi \in \Gamma_n, f_n(A_n)} |\text{Im} \xi| \rightarrow \infty \) by Lemma 4 and since for each \( h \in H_n^+ \) the arc \( \eta_n(h(f_n(A_n))) \subseteq \eta_n(A_n) \) is included in \( \{ \xi \in L; \text{Im} \xi > -e_0^{-1}/K_n \} \), the kernel of \( \{ \eta_n(A_n) \} \) is \( L \). Let \( \xi_n \) be the element of \( \text{PSL}(2, \mathbb{R}) \) such that \( \xi_n(-1) = u_n \) and \( (\eta_n \circ f_n \circ \xi_n)(-i) > 0 \). Then by Carathéodory kernel theorem \( \eta_n \circ f_n \circ \xi_n \) converges locally uniformly to a conformal homeomorphism \( F \) which maps \( L \) onto the kernel \( L \) of \( \{ \eta_n(A_n) \} \). Obviously \( F \) is a Möbius transformation and \( [F](z) = 0 \). Using a theorem of Weierstrass, we have
This contradicts the fact \( \|f_n\| \geq 2 \) due to Ahlfors-Weill [3], and the case (ii) never happens.

3.6. Now we shall complete the proof of Theorem 3 under the condition (i). Since \( d_{\mathcal{H}}(f_n(u_n), f_n(A_n)) = d_L(u_n, A_n) \leq d_L(u_n, z_n) \) is less than a constant for each \( n \), Lemmas 3 and 4 show the existence of a subsequence, again denoted by \( \{f_n(u_n)\}_{n=1}^\infty \), of \( \{f_n(u_n)\}_{n=1}^\infty \) which converges to a point \( \zeta_0 \) with \( \Re \zeta_0 = 0 \) and \( |\Im \zeta_0| < \pi/2 \). Let \( \mu_n \) be the element of \( PSL(2, \mathbb{R}) \) such that \( \mu_n(-i) = z_n \) and \( (f_n \circ \mu_n)(-i) > 0 \). Carathéodory kernel theorem together with Lemma 3 shows that \( \{f_n \circ \mu_n(z) - f_n \circ \mu_n(-i)\}_{n=1}^\infty \) converges locally uniformly to \( F(z) = 3\pi i/2 + \log z \) which maps \( L \) onto the kernel \( \{\zeta; |\Im \zeta| < \pi/2\} \) of \( \{f_n \circ \mu_n(L)\}_{n=1}^\infty \), where we take the branch of \( \log z \) satisfying \( F(-i) = 0 \). Let \( E \) be a compact subset of \( L \) containing all \( \mu_n^{-1}(u_n) \). Then we see that

\[
\|f_n\| = \rho_L(u_n)^{-1} \|f_n\|(u_n) \\
= \rho_L^{-1}(u_n) \|f_n\|(u_n) \\
= \sup_{z \in \mathcal{E}} \rho_L(z)^{-1} \|f_n\|(z) \\
\rightarrow \sup_{z \in \mathcal{E}} \rho_L(z)^{-1} (3\pi i/2 + \log z) = 2.
\]

Recall that \( f_n \Gamma_n f_n^{-1} \) is a Koebe group. Then \( T(\Gamma_n) \) does not contain the point \( [f_n] \) and neither does \( T(\Gamma) \). Therefore \( 2 \leq i(\Gamma) \leq \|f_n\| \rightarrow 2 \). Now we obtain \( i(\Gamma) = 2 \) and complete the proof of Theorem 3.

Addendum. After this note was completed, Professor T. Nakanishi informed the author that T. Nakanishi and J.A. Velling know a proof of the following Theorem A which is a generalization of Theorems 1, 2 and 3.

**Theorem A.** Let \( \Gamma \) be a Fuchsian group keeping \( L \) invariant. Then \( i(\Gamma) \) is equal to 2 if \( \Gamma \) satisfies one of the following:

1. For any positive number \( d \), there exists a hyperbolic disc of radius \( d \) which is precisely invariant under the trivial subgroup of \( \Gamma \).
2. For any positive number \( d \), there exists the collar of width \( d \) about the axis of a hyperbolic element of \( \Gamma \).

He also informed the author that their proof of Theorem A is different from the proof of Theorem 3 and depends on properties of a family of functions constructed in Kalme [11].
REFERENCES


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