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NOTE ON HECKE OPERATORS AND COHOMOLOGY OF $PSL_2(Z)$

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Introduction.

In this note we study the action of Hecke operators on 1-dimensional cohomology group of the modular group $G=PSL_2(Z)$ with the coefficient module W, the even degree parts of the polynomial algebra $Z[x, y]$, or its reduction modulo a prime power l, $W/l=(Z/lZ)[x, y]$. The cohomology group $H^n(G; W/l)$ is a module over $H^0(G; W/l)=(W/l)^G$, the invariants of W/l. The ring $(W/l)^G$ is known by Dickson [1]. We notice the relation between the above module structure and the action of Hecke operators. Then we obtain some congruences for the eigenvalues of Hecke operators on modular forms.

THEOREM. *Let λ_l bn the eigenvalue of the Hecke operator T_l in $M_k{}^0(G)$; the set of all cusp forms of weight k. Then*

(1) $\lambda_7 \equiv 0 \bmod 7$ *if* $k \equiv 10, 14 \bmod 42$

(2) $\lambda_{11} \equiv 0 \bmod 11$ *if* $k \equiv 14 \bmod 110$.

where $\lambda_l \equiv 0 \bmod l$ means λ_l/l is an algebraic integer.

The above results are largely extended in E. Papier's up coming paper [5]. Papier kindly corrected many errors in the first version of this note. She also suggested Proposition 4.3. The authors wish to thank her heartly. They also thank to S. Mizumoto for conversations and suggestion, in particular the computation (1.6) is due to him.

§1. Hecke operators and the Eichler-Shimura isomorphism.

Let $G=PSL_2(Z)$ be the modular group and $V=Z[x, y]$, $|x|=|y|=1$, be the polynomial algebra over Z. If we denote the positive even degree parts of V by W. Then G acts on W by $gP(x, y)=P((x, y)g)$ for $g\in G$ and $P(x, y)\in W$. For any G-module E, the Eichler cohomology group $H_P^1(G; E)$ is defined to be the kernel of the restriction map $j^*: H^1(G; E)\to H^1(G_\infty; E)$, here G_∞ denotes

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the subgroup of G generated by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. From Lemma 2.2 in [9] or [3], in the case where $E=W^{+}\otimes R$, ; the positive degree parts of $W\otimes R$, the above map j^{*} is epic. Therefore

$$H^{1}(G; W\otimes R)\cong H_{P}^{1}(G; W\otimes R)\oplus H^{1}(G_{\infty}; W\otimes R). \tag{1.1}$$

Let us denote by $M_{k}(G)$ (resp. $M_{k}^{0}(G)$) the set of all automorphic (resp. cusp) forms of weight k with respect to G. Now we recall the actions of Hecke operators on cohomology groups and automorphic forms. Let α be an element of $M_{2}^{+}(Z)=\{A\in M_{2}(Z)\mid \det A>0\}$ and let $\alpha\mapsto\bar{\alpha}$ be the main involusion $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto\bar{\alpha}=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$. Then the double coset $G\alpha G$ decomposes into a disjoint union of finite number of left G cosets, $G\alpha G=\coprod_{i=1}^{d}G\alpha_{i}$. For $g\in G$, let $\alpha_{i}\bar{g}=\bar{g}_{i}\alpha_{i*}$ with some $1\leqq i^{*}\leqq d$ and some $g_{i}\in G$. Then for any G-module E, the Hecke operator $\tilde{T}_{\alpha}$ on $H^{1}(G; E)$ is defined by

$$\tilde{T}_{\alpha}u(g)=\textstyle\sum_{i=1}^{d}\bar{\alpha}_{i*}u(g_{i}) \qquad \text{for } u\in Z^{1}(G; E). \tag{1.2}$$

The Hecke operator T_{α} on $M_{k}(G)$ is defined by

$$T_{\alpha}f(z)=\det\alpha^{k-1}\textstyle\sum_{i=1}^{k}f(\alpha_{i}z)j(\alpha_{i}, z)^{-k} \qquad \text{for } f\in M(G). \tag{1.3}$$

Here $j(\alpha_{i}, z)=c_{i}z+d_{i}$ for $\alpha_{i}=\begin{pmatrix}* & *\\ c_{i} & d_{i}\end{pmatrix}$. Then there exists an R-linear isomorphism called the Eichler-Shimura isomorphis,

$$\phi; M_{k+2}^{0}(G)\cong H_{P}^{1}(G; W^{k}\otimes R), \tag{1.4}$$

which commutes with Hecke operators (Shimura [10]). In (1.4) W^{k} denotes the k-degree parts of W. Let E_{k} be the Eisenstein series

$$E_{k}(z)=\frac{1}{2\zeta(k)}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^{k}}$$

PROPOSITION 1.5. *The map* ϕ *in* (1.4) *is extendable to an* R*-linear isomorphism*

$$\phi: M^{0}{}_{k+2}(G)\oplus R\cdot E_{k+2}\cong H^{1}(G; W^{k}\otimes R),$$

which commutes with Hecke operators.

Proof. From the proof of Proposition 8.5 in Shimura [10], (, more details see Papier [5]) we can extend ϕ to $M_{k+2}^{0}(G)\oplus R\cdot E_{k+2}$ by defining

$$\phi(f)(g)=\textstyle\sum_{j=0}^{k}\binom{k}{j}x^{j}y^{k-j}\displaystyle\int_{z_{0}}^{g(z_{0})}\mathrm{Re}\,(fz^{j}dz) \qquad \text{for } f\in M_{k+2}(G),$$

and the ϕ commutes with Hecke operators. From (1.1), it is easily seen that the extended ϕ is an isomorphism, if

(1.6) the coefficient of y^k in $\phi(E_{k+2})\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\int_i^{i+1}\mathrm{Re}\,(E_{k+2})dz\neq 0.$

We can prove that (1.6)>0 by direct computations for $k=2$ and 4, and by showing that $|B_k|/2k(k-1)>\sum_{n=1}^{\infty}\sigma_k(n)/e^{2\pi n}$ for $n\geqq 6$, where B is the k-th Bernoulli number and $\sigma_k(n)=\sum_{d|n}d^k$. Q.E.D.

PROPOSITION 1.7. *If there is a* $\lambda\in Z/lZ$ *such that* $\tilde{T}_\alpha x=\lambda x$ *for any* $x\in H^1(G;W^k/l)$, *then for any eigenvalue* λ_α *of* T_α *in* $M_{k+2}(G)$, *a conguence* $\lambda_\alpha\equiv\lambda$ (mod l) *holds.*

Proof. We have an exact sequence

$$H^1(G;W)\xrightarrow{\ l\ }H^1(G;W)\longrightarrow H^1(G;W/l)\,.$$

From the assumption, we have $\tilde{T}_\alpha x\equiv\lambda\tilde{x}$ (mod $l\,H^1(G;W)$) for any $\tilde{x}\in H^1(G;W)$. Note that the image of $H^1(G;W)$ is a lattice of $H^1(G;W\otimes R)$. Since $(T_\alpha-\lambda)/l$ stabilizes a lattice in $M(G)\oplus RE$, then all eigenvalues for $(T_\alpha-\lambda)/l$ must be algebraic integers. Q.E.D.

§2. Cohomology and invariants.

In this section we study the cohomology $H^1(G;W/l)$ and $(W/l)^G$. By the cup product, $H^1(G;W/l)$ is an $H^0(G;W/l)=(W/l)^G$ module. The action of $(W/l)^G$ is defined by $(wu)\,(g)=w\cdot u(g)$ for $w\in(W/l)^G$, $u\in Z^1(G;W/l)$ and $g\in G$.

PROPOSITION 2.1. *Let* $w\in(W/l)^G$ *and* $\alpha\in M_2^+(Z)$. *If there is a* $\lambda\in Z/lZ$ *such that* $\bar{\alpha}w=\lambda w$, *then* $\tilde{T}_\alpha wu=\lambda w\tilde{T}_\alpha\, u$ *for any* $u\in H^1(G;W/l)$.

Proof. By the definition (1.2),

$$\tilde{T}_\alpha wu(g)=\sum\bar{\alpha}_{i*}(wu)(g_i)$$
$$=\sum\bar{\alpha}_{i*}w\bar{\alpha}_{i*}u(g_i)\,.$$

Since $\bar{\alpha}_{i*}=g\bar{\alpha}_i g_i^{-1}\in G\bar{\alpha}G$, $\bar{\alpha}_{i*}w=\lambda w$ holds from the assumption. Therefore $\tilde{T}_\alpha wu(g)=\lambda w\sum\bar{\alpha}_{i*}u(g_i)=\lambda w\tilde{T}_\alpha u(g)$. Q.E.D.

Next we consider the invariant $(W/l)^G$ for a prime number l. We define two elements E_1 and E_2 in $V=Z[x, y]$ by

$$E_1=xy^l-x^ly\qquad(\text{where } l=2,\ E_1=(xy^2-x^2y)^2)\qquad\text{and}$$
$$E_2=x^{l(l-1)}+x^{(l-1)(l-1)}y^{(l-1)}+\cdots+y^{l(l-1)}\,.$$

Then the classical results of Dickson [1], are that $(W/l)^G=(Z/lZ)[E_1, E_2]$ and moreover W/l is a free $Z/lZ[E_1, E_2]$-module.

Let p be a prime number. Let us write $T_\alpha=T_p$ for $\alpha=\begin{pmatrix}1&0\\0&p\end{pmatrix}$. We need to know only the action T_p. It is immediate that

$$\bar{\alpha}E_1=(px)y^l-(px)^l y\equiv pE_1 \bmod l$$

$$\bar{\alpha}E_2=(px)^{l(l-1)}+\cdots+y^{l(l-1)}\equiv\begin{cases}E_2 \bmod l & \text{if } p\neq l,\\ y^{l(l-1)} \bmod l & \text{if } p=l.\end{cases}$$

Therefore we have

LEMMA 2.2. *If* $\tilde{T}_p x=\lambda x$ *for* $\lambda\in Z/l$ *and* $x\in H^1(G;W/l)$, *then* $\tilde{T}_p(E_1x)=p\lambda(E_1x)$ *and* $\tilde{T}_p(E_2x)=\lambda(E_2x)$ *for* $p\neq l$.

It is wellknown that $G=PSL_2(Z)$ is the free product $Z/2Z*Z/3Z$. Here $Z/2Z$ (resp. $Z/3Z$) is generated by $\tau=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (resp. $\sigma=\begin{pmatrix}1&0\\-1&0\end{pmatrix}$) [7]. Therefore, about the classifying space we have

$$BG=BZ/2Z\vee BZ/3Z$$

where $\vee$ denotes the one point union. For any G-module E, we have the Mayer-Vietories exact sequence

$$E^{Z/2Z}\oplus E^{Z/3Z}\longrightarrow E\longrightarrow H^1(G;E)\xrightarrow{i^*}$$
(2.3)
$$H^1(Z/2Z;E)\oplus H^1(Z/3Z;E)\longrightarrow 0.$$

Remark 2.4. If $l\geqq 5$, then $(W/l)/((W/l)^{Z/2Z}+(W/l)^{Z/3Z})\cong H^1(G;W/l)$ and this isomorpnism is given by $a\mapsto u_a$ where $u_a(\sigma)=(1-\sigma)a$ and $u_a(\tau)=0$ for $a\in W/l$.

PROPOSITION 2.5. *The* $Z/lZ[E_1, E_2]$*-module* $H^1(G;W/l)$ *is generated by generators of degree equal to or less than* l^2-1 (6 *for* $l=2$).

Proof. The free $Z/lZ[E_1, E_2]$-module W/l is generated by elements the degree equal to or less than

$$|E_1|+|E_2|-2=l^2-1 \quad (=6 \text{ for } l=2).$$

Hence so is the quotient module $(W/l)/((W/l)^{Z/2Z}+(W/l)^{Z/3Z})$. We also prove that $H^1(Z/2;W/2)$ (resp. $H^1(Z/3;W/3)$) is generated by elements degree $\leqq 6$ (resp. 8) from the explicit computation of the cohomology (see [9]). Q.E.D.

PROPOSITION 2.6. *Assume* $p\equiv 1 \bmod l$ (*resp.* $p\neq l$) *and there is a* $\lambda\in Z/lZ$ *such that* $\tilde{T}_p x=\lambda x$ (*resp.* $\tilde{T}_p x=0$) *for any* $x\in H^1(G;W^k/l)$ *with* $0\leqq k\leqq l^2-1$ (*when* $l=2$, $0\leqq k\leqq 6$). *Then for any eigenvalues* λ_p *of* T_p *in* $M_{K+2}(G)$, *with any* $K\geqq 0$, *the congruence* $\lambda_p=\lambda \bmod l$ (*resp.* $\lambda_p=0 \bmod l$) *holds.*

Proof. Any element $f \in H^1(G; W/l)$ can be written as $f=\sum a_i f_i$ here $a_i \in Z/lZ[E_1, E_2]$ and $|f_i| \leqq l^2-1$. Then from the assumption and Lemma 2.2, we have

$$\tilde{T}_p f=\sum a_i \tilde{T}_p f_i=\sum a_i \lambda f_i=\lambda f, \quad \text{if } p \in 1 \bmod l. \qquad \text{Q.E.D.}$$

§ 3. Congruence of eigenvalues of T_p.

In this section we obtait some results about congruence properties of eigenvalues of Hecke operators on modular forms by studying the cohomology of G_∞. Recall $\gamma=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $G_\infty=\langle\gamma\rangle$ and $j: G_\infty \hookrightarrow G$ is the inclusion map. The cohomology of G_∞ is easily computed.

LEMMA 3.1.

$$H^1(G_\infty; V/l) \cong (V/l)/\mathrm{Im}(\gamma-1)$$

$$\cong Z/lZ[v] \otimes (Z/lZ\{1, y, \cdots, y^{l-2}\} \oplus Z/lZ[x]\{y^{l-1}\})$$

where $v=y^l-yx^{l-1}$ and $Z/lZ\{a, b, \cdots\}$ is the Z/l-module generated by $a, b, \cdots$.

It is easily seen that $GaG=\coprod Ga_i$, $a_i=\begin{pmatrix}1 & i\\ 0 & p\end{pmatrix}$ for $0 \leqq i \leqq p-1$, $a_p=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$.

LEMMA 3.2. *$j^*(\tilde{T}_p\phi)=(a_0+\cdots+a_p)j^*\phi$ in $H^1(G_\infty; W/l)$ for $\phi \in H^1(G; W/l)$.*

Proof. Direct computation shows that $\gamma\bar{a}_j=\bar{a}_{j-1}\cdot 1$ for $1 \leqq j \leqq p-1$, $\gamma\bar{a}_0=\bar{a}_{p-1}\gamma$ and $\gamma\bar{a}_p=\bar{a}_p\gamma^p$. Therefore we have

$$\begin{aligned}\tilde{T}_p\phi(\gamma)&=\bar{a}_{p-1}\phi(\gamma)+\textstyle\sum_{j=1}^{p-1}\bar{a}_{j-1}\phi(1)+\bar{a}_p\phi(\gamma^p)\\ &=\bar{a}_{p-1}\phi(\gamma)+\bar{a}_p(1+\gamma+\cdots+\gamma^{p-1})\phi(\gamma)\\ &=(\gamma^{1-p}-1)a_p\phi(\gamma)+(a_p+a_0+\cdots+a_{p-1})\phi(\gamma)\end{aligned}$$

since $\bar{a}_{p-1}=\gamma^{1-p}a_p$ and $\bar{a}_p\gamma^j=a_j$. Q.E.D.

COROLLARY 3.3. (i) *If $p=\pm 1 \bmod l$, then $j^*(\tilde{T}_p\phi)=(1+p)j^*(\phi)$.* (ii) *If $p=0 \bmod l$, then $j^*(\tilde{T}_p\phi)=j^*(\phi)$ for $j^*(\phi)=v^s y^i$ and $j^*(\tilde{T}_p\phi)=0$ for $j^*(\phi)=v^s x^i y^{l-1}$, $i \geqq 1$.*

From Theorem 2.6, if $j^*: H^1(G; W^k/l) \to H^1(G_\infty; W^k/l)$ is injective for $2 \leqq k \leqq l^2-1$, then

(3.4) $$\lambda_p \equiv p+1 \bmod l \quad \text{if } p \equiv \pm 1 \bmod l.$$

Hatada [4] proved

THEOREM 3.5. (3.4) *holds for* $l\leqq 5$.

Indeed, Haberland proved ([3] Lemma 1, 5.1.) that j^* is injective for all $k\geqq 2$ when $l\leqq 5$. However, Papier showed that (3.4) does not hold in $M_{50}^0(G)$ for $l=7$, that follows that there is an $x\in H^1(G;W^{48}/7)$ such that $T_px\not\equiv(1+p)x$ mod 7 [10].

Consider the maps

$$H^1(G;W)\xrightarrow{r}H^1(G;W/l)\xrightarrow{i^*}H^1(G_\infty;W/l)/(v^sx^iy^{l-1}|i\geqq 1)\cong Z/l[y]$$

Then $H^1(G;W^s/l)$ decompose as $\operatorname{Ker} i^*\oplus E'$ with $E'\cong Z/l$ or $\cong 0$. Let us take $H^1(G;W^s)=K^s\oplus E$ with $r(K^s)\subset\operatorname{Ker} i^*$ and $r(E)=E'$. Then $E\cong Z$ because for the Eisenstein series, the Hecke action operates $T_l(E_{s+2})=E_{s+2}$ and from Proposition 1.5, $E\otimes R\cong R$. Since $K\otimes R\cong H_p^1(G;W^s)\otimes R$ and E is torsion free, K^s is closed under $\tilde{T}_l$. From Corollary 3.3, if j^* is injective for all $k\geqq 2$, then $\tilde{T}_lx=0$ in K, and so we get

$$\lambda_l\equiv 0 \bmod l \qquad \text{in } M^0(G). \tag{3.5}$$

From the Haberland result, we get

THEOREM 3.6. (3.5) *holds for* $l\leqq 5$.

The above theorem is also proved by Hatada for $l\leqq 3$ and by Papier $l\leqq 5$ [5]. Papier also proved

$$\lambda_p^2=p^k(1+p)^2 \bmod 5 \quad \text{in } M_{k+2}^0(G) \qquad \text{if } p=\pm 2 \bmod 5. \tag{3.7}$$

The injectivity of j^* shows the above congruence from the fact that

$$j^*(T_p\Phi)=\pm p^{k/2}(1+p)j^*(\Phi) \qquad \text{in } H^1(G_\infty;W^k/l)$$

for elements $j^*(\Phi)=v^sx^iy^j$.

THEOREM 3.8. *Let* $l-1<k<l^2-l$ *and* $M_{k+2}^0(G)=0$. *Then the eigenvalue* $\lambda_l=0 \bmod l$ *for the Hecke action* T_l *in* $M_{s+2}^0(G)$ *where* $s=k \bmod l(l-1)$.

Proof. By the assumption $M_{k+2}^0(G)=0$, K^k is torsion. The l-torsion in $H^1(G;W)$ is isomorphic to $H^0(G;W^+/l)=Z/l[E_1,E_2]^+$ from the exact sequence

$$H^0(G;W)\xrightarrow{r}H^0(G;W/l)\xrightarrow{\delta}H^1(G;W)\xrightarrow{l}H^1(G;W)\xrightarrow{r}$$

and $H^0(G;W)=W^G$. Therefore $K^k=Z/l\{E_1^m\}$ where we only need to consider the l-torsion since the lowest dimensional l^2-torsion element is E_1^l.

Let $f\in\operatorname{Ker} i^*$. Since $l-1<k<l^2-l$ and $s=k \bmod l(l-1)$, we can take

$$f=E_2^tf_1+E_1\dot{g}, \qquad |f_1|=k,\ t>1.$$

From Proposition 2.2, $\tilde{T}_l E_1 g=(\bar{\alpha}E_1)\tilde{T}_l g=0$. Since $f_1=\lambda r\delta(E_1^m)$, $\lambda\in Z/l$, we have

$$E_2^t f_1=E_2^t r\delta(E_1^m)=E_1^m E_2^{t-1} r\delta(E_2).$$

Hence $\tilde{T}_l(E_2^t f_1)=0$. Therefore $\tilde{T}_l(\mathrm{Ker}\, i^*)=0$. Since K is closed under $\tilde{T}_l$, $\tilde{T}_l(K)=0 \bmod l(K)$. Since $K\otimes R\cong H_P^1(G;W)\otimes R$, we have the theorem. Q.E.D.

The fact $M_{s+2}^0(G)=0$ for $s+2\leqq 14$ and $\neq 12$ implies the theorem in the introduction.

§ 4. System of eigenvalues mod l.

Let $M_k(G)_Z$ be the subset of $M_k(G)$ consisting of all forms whose q-coefficient at infinity are integral. The space of modular forms mod l of weight k is an $\bar{F}_l$-vector space

$$\bar{M}_k(G)=\{\bar{f}=\sum\bar{a}_n q^n \mid f=\sum a_n q\in M_k(G)_Z\}\subset\bar{F}_l[[q]]$$

where $\bar{a}_n$ denotes the reduction of $a_n \bmod l$.

An element $(\bar{\lambda}_p)\in\prod_{p\neq l;\,\mathrm{prime}}\bar{F}_l$ (resp $(\tilde{\lambda}_p)\in\prod_{p\neq l}\bar{F}_l$) is called a system of eigenvalues mod l (resp in $H^1(G;W/l)\otimes\bar{F}_l$) except for $p=l$ if there exists a non zero form $f\in\bar{M}(G)\otimes\bar{F}_l$ (resp. $f\in H^1(G;W/l)\otimes\bar{F}_l$) such that $T_p f=\bar{\lambda}_p f$ (resp. $\tilde{T}_p f=\tilde{\lambda}_p f$) for all $p\neq l$. Let $\bar{\Phi}_k$ (resp. $\tilde{\Phi}_k$) be the set of systems of eigenvalues mod l (resp. in $H^1(G;W/l)\otimes\bar{F}_l$) except for $p=l$ of weight k.

Since $M_k(G)_Z$ is a lattice of $M_k(G)$ by the arguments similar to Proposition 1.7, we get

LEMMA 4.1. $\bar{\Phi}_{k+2}\subset\tilde{\Phi}_k$.

Let us write $\bar{\Phi}_k(s)=\{(p^s\bar{\lambda}_p) \mid (\bar{\lambda}_p)\in\bar{\Phi}_k\}$ and $\tilde{\Phi}_k(s)=\{(p^s\tilde{\lambda}_p) \mid (\tilde{\lambda}_p)\in\tilde{\Phi}_k\}$,

LEMMA 4.2. $\bigcup_{k=0}^{\infty}\tilde{\Phi}_k=\bigcup_{s=0}^{l-2}\bigcup_{k=0}^{l^2-1}\tilde{\Phi}_k(s)$.

Proof. Let $\tilde{T}_p f=\tilde{\lambda}f$. Then from Lemma 2.2,

$$\tilde{T}_p(E_1^i E_2^j f)=p^i\tilde{\lambda}E_1^i E_2^j f.$$

From Proposition 2.5, we have the lemma. Q.E.D.

PROPOSITION 4.3. $\bigcup_{k=2}^{\infty}\bar{\Phi}_k=\bigcup_{s=0}^{l-2}\bigcup_{k=0}^{l^2-1}\bar{\Phi}_k(s)$.

Proof. We need only prove that eigenvalues of l-torsion part of $H^1(G;W)$ are contained in $\bar{\Phi}_k$. Recall the proof of Theorem 3.8. l-torsion part is generated as a $(W/l)^G$-module by $\delta(E_1)$ and $\delta(E_2)$. The boundary map is defined

$$\delta(E_1)(\gamma)=(1/l)(1-\gamma)(E_1)\equiv x^2y^{l-1} \bmod (x^3),$$

where we consider $E_1 \in W$. Therefore $j^*(\delta(E_1)) \neq 0$ in $H^1(G_\infty; W/l)$ and $T_p(\delta(E_1)) = (p+p^2)(\delta(E_1))$ by Lemma 3.2. Since $E_1\delta(E_2)=\delta(E_1E_2)=E_2\delta(E_1)$ and its j^*-image is non zero, we see $T_p(\delta E_2)=(p+1)$. Therefore eigenvalues of l-torsion part are $p^s(p+1)$, $s \geqq 0$. Since the eigenvalue of Eisenstein series of weight $l+1$ is $1+p$, we have the proposition. Q.E.D.

Remark 4.4. The above proposition is very weaker version of wellknown theorem; $\bigcup_{k=2}^{\infty}\Phi_k = \bigcup_{s=2}^{l-1}\bigcup_{k=2}^{l+1}\Phi_k(s)$ where Φ is the set of system of eigenvalues mod l for all prime p, (without the restriction $p \neq l$). It was originally proved by Tate-Serre [11] with the lower bound $l+1$.

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