T. TADA
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ESSENTIAL SETS OF PICARD PRINCIPLE FOR ROTATION FREE DENSITIES

Dedicated to Professor Masanori Kishi on his 60th birthday

BY TOSHIMASA TADA

We denote by Ω the punctured unit disk $0<|z|<1$ and consider a Schrödinger equation

$$(1)\qquad (-\Delta+P(z))u(z)=0 \quad \left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ z=x+yi\right)$$

on Ω. The potential P is assumed to be nonnegative and locally Hölder continuous on $0<|z|\leqq 1$ and referred to as a *density* on Ω. We say that the *Picard principle* is valid for P (at the origin $z=0$) if the set $F_P(\Omega)$ of nonnegative solutions of (1) on Ω with vanishing boundary values on the unit circle Γ: $|z|=1$ is generated by one element u of $F_P(\Omega)$: $F_P(\Omega)=\{cu : c\geqq 0\}$. In other words the Picard principle is valid for P at the origin if and only if the Martin ideal boundary of Ω over the origin with respect to (1) consists of one point. Let P be a density on Ω for which the Picard principle is valid and Q a density on Ω with $Q\leqq P$ on Ω. The Picard principle for Q is generally invalid ([8], [9]). However the Picard principle for Q is valid if densities P and Q are *rotation free*, i.e. $P(z)=P(|z|)$ and $Q(z)=Q(|z|)$ on Ω ([7]). Moreover the Picard principle for Q is valid if $Q\leqq P$ on a subset of Ω for some densities P ([2]). In this note we will study this subset of Ω for the special densities $P(z)=|z|^{-2}$ and $P(z)=(\log|z|)^2/|z|^2$.

Hereafter *every* density P on Ω in consideration is assumed to be rotation free and is mainly viewed as a function $P(r)$ of r in the interval $(0, 1]$. In order to define the above subsets of Ω we take two sequences $\{a_n\}_1^\infty$, $\{b_n\}_1^\infty$ which are always supposed to satisfy

$$0<b_{n+1}<a_n<b_n<1 \quad (n=1, 2, \cdots), \qquad \lim_{n\to\infty} a_n=0$$

and we set

$$A=A(\{a_n\}, \{b_n\})=\bigcup_{n=1}^{\infty}[a_n, b_n].$$

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Then the set A ($\{z: |z| \in A\}$, more precisely) is called an *essential set* (of the Picard principle) for a density P on Ω if the Picard principle is valid for any density Q on Ω with

$$Q(r)=O(P(r)) \qquad (r\in A,\ r\to 0).$$

We remark that the Picard principle is valid for P if there exists an essential set for P. We also remark that the above condition can be replaced by

$$Q(r)\leqq P(r) \qquad (r\in A)$$

since the Picard principle for Q and cQ ($c>0$) are equivalent ([4]). A typical example of a density and an essential set for it are the density $P(r)=r^{-2}$ and the set A with $\limsup b_n/a_n>1$ ([2]). An essential set for $P(r)=r^{-2}$ which is smaller than the above essential set is given by M. Kawamura ([3]): The set A is an essential set for $P(r)=r^{-2}$ if

$$\sum_{n=1}^{\infty}\left(\log\frac{b_n}{a_n}\right)^2=\infty. \tag{2}$$

This result is a special case of the following generalization ([3]): For an arbitrary density P on Ω, the set A is an essential set for P if

$$\sum_{n=1}^{\infty}\frac{\left(\log\frac{b_n}{a_n}\right)^2}{1+\left(\log\frac{b_n}{a_n}\right)\int_{a_n}^{b_n}P(r)r\,dr+\log\frac{b_n}{a_n}}=\infty. \tag{3}$$

In this note we will show that (2) is not only sufficient but also *necessary* for A to be an essential set for $P(r)=r^{-2}$:

THEOREM 1. *The following statements are equivalent by pairs.*

(a) *The set A is an essential set for $P(r)=r^{-2}$,*

(b) $$\sum_{n=1}^{\infty}\left(\log\frac{b_n}{a_n}\right)^2=\infty,$$

(c) $$\sum_{n=1}^{\infty}\log\frac{1}{2}\left(\frac{b_n}{a_n}+\frac{a_n}{b_n}\right)=\infty.$$

The density $(\log r)^2/r^2$ is essentially different from the density r^{-2} ([2], [11]). For this reason the separate study of essential sets for $(\log r)^2/r^2$ is in order:

THEOREM 2. *The following statements are equivalent by pairs:*

(a) *The set A is an essential set for $P(r)=(\log r)^2/r^2$,*

(b) $$\sum_{n=1}^{\infty}\frac{1}{(\log a_n)^2}\log\frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\log a_n}+\left(\frac{a_n}{b_n}\right)^{\log a_n}\right\}=\infty,$$

(c) $$\sum_{n=1}^{\infty}\frac{1}{(\log b_n)^2}\log\frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\log b_n}+\left(\frac{a_n}{b_n}\right)^{\log b_n}\right\}=\infty.$$

We remark that the sequences $\{a_n\}$ and $\{b_n\}$ given by $a_n=2^{-2n}$ and $b_n=2^{-2n+1}$ $(n=1, 2, \cdots)$ do not satisfy (3) for $P(r)=(\log r)^2/r^2$ and nevertheless satisfy (b) in Theorem 2.

§1. Fundamental properties of P-units

1.1. For a density P on Ω, the unique bounded solution of

$$\left(-\Delta+P(z)+\frac{n^2}{|z|^2}\right)u(z)=0$$

on Ω with boundary values 1 on Γ is referred to as the *n-th P-unit* $(n=0,1,\cdots)$ ([6]). In particular we call the 0-th P-unit simply the P-unit. Since P is rotation free, the n-th P-unit e_n is also rotation free so that it may also be considered as a function $e_n(r)$ of r in $(0, 1]$. Then the n-th P-unit is the unique bounded solution of

$$l_{P,n}\phi(r)\equiv\phi''(r)+\frac{1}{r}\phi'(r)-\left(P(r)+\frac{n^2}{r^2}\right)\phi(r)=0$$

on $(0, 1)$ with $\phi(1)=1$, where we set $\phi'(r)=d\phi(r)/dr$ and $\phi''(r)=d^2\phi(r)/dr^2$. In particular the differential operator $l_{P,n}$ for $P\equiv0$ and $n=0$ is denoted simply by l. The n-th P-unit e_n is also positive and increasing, i.e. $e_n'(r)\geqq0$ on $(0, 1)$.

1.2. We recall fundamental properties of P-unit and first P-units which play an essential roll in the study of the Picard principle for densities P. Let P be a density on Ω and e_0, e_1 be the P-unit, the first P-unit, respectively. The Picard principle is valid for P if any only if

(4) $$\int_0^1\frac{e_1(r)}{r^2e_1'(r)}dr=\infty \quad ([10]).$$

For a test of the Picard principle for P, we only apply (4) to the first P-unit in the sequel. We also use fundamental properties of P-units mentioned below.

Suppose that two densities P, Q on Ω satisfy $P\leqq Q$ on a subinterval $(0, a]$ of $(0, 1]$ and denote by e_n, f_n the n-th P-unit, the n-th Q-unit, respectively $(n=0, 1, \cdots)$. Then we have

(5) $$\frac{e_n(r)}{e_n(s)}\geqq\frac{f_n(r)}{f_n(s)} \quad (0<r\leqq s\leqq a) \quad ([1])$$

by the maximum principle for (1) ([5], [6]). The inequality (5) means that the function $f_n(r)/e_n(r)$ is increasing so that we also have

(6) $$\frac{f_n(r)}{f_n'(r)}\leqq\frac{e_n(r)}{e_n'(r)} \quad (0<r<a,\ n\geqq1).$$

Ih particular $P(r)<P(r)+r^{-2}$ and $P(r)+r^{-2}\geqq r^{-2}$ imply that

$$\frac{e_0(r)}{e_0(s)} \geqq \frac{e_1(r)}{e_1(s)} \quad (0<r\leqq s\leqq 1) \quad ([6]), \tag{7}$$

$$\frac{e_1(r)}{e_1'(r)} \leqq r, \quad \text{i.e.} \quad \frac{e_1'(r)}{e_1(r)} \geqq \frac{1}{r}, \quad (0<r<1) \quad ([10]), \tag{8}$$

respectively since the first P-unit coincides with the $(P(r)+r^{-2})$-unit and the r^{-2}-unit is r.

§ 2. Characterization of essential sets

2.1. Consider a density P on Ω satisfying $P(r)=\alpha/r^2$ $(\alpha\geqq 0)$ on $[a, b]$ $(0<a<b<1)$. The first P-unit e_1 can be represented in the form $e_1(r)=\lambda r^{\beta}+\mu r^{-\beta}$ $(\beta=\sqrt{\alpha+1})$ on $[a, b]$. The inequalities $e_1(a)>0$ and $e_1'(a)>0$ by (8) yield that $\lambda>0$. By using $e_1(a)>0$ again and $e_1'(a)/e_1(a)\geqq a^{-1}$ we see that

$$-a^{2\beta} < \frac{\mu}{\lambda} \leqq \frac{\beta-1}{\beta+1} a^{2\beta}. \tag{9}$$

Then we apply (9) to the integral

$$\begin{aligned} \int_a^b \frac{e_1(r)}{r^2 e_1'(r)} dr &= \int_a^b \frac{1}{\beta}\left(\frac{2\lambda r^{2\beta-1}}{\lambda r^{2\beta}-\mu} - \frac{1}{r}\right) dr \\ &= \frac{1}{\beta^2} \log \frac{a^{\beta}(b^{2\beta}-\mu/\lambda)}{b^{\beta}(a^{2\beta}-\mu/\lambda)} \end{aligned} \tag{10}$$

to obtain the following evaluation:

$$\int_a^b \frac{e_1(r)}{r^2 e_1'(r)} dr > \frac{1}{\beta^2} \log \frac{1}{2}\left\{\left(\frac{b}{a}\right)^{\beta} + \left(\frac{a}{b}\right)^{\beta}\right\}, \tag{11}$$

$$\int_a^b \frac{e_1(r)}{r^2 e_1'(r)} \leqq \frac{1}{\beta^2} \log \frac{1}{2}\left\{\left(\frac{b}{a}\right)^{\beta} + \left(\frac{a}{b}\right)^{\beta} + \beta\left(\left(\frac{b}{a}\right)^{\beta} - \left(\frac{a}{b}\right)^{\beta}\right)\right\}. \tag{12}$$

Hence (4), (6) and (11) yield

LEMMA 3. *Let P be a density on Ω with $P(r)\leqq \alpha_n/r^2$ $(\alpha_n\geqq 0)$ on every interval $[a_n, b_n]$ $(n=1, 2, \cdots)$. Then the set A is an essential set for P if*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\beta_n} + \left(\frac{a_n}{b_n}\right)^{\beta_n}\right\} = \infty \quad (\beta_n=\sqrt{\alpha_n+1}). \tag{13}$$

2.2. Let us prove a converse of Lemma 3. To this end we prepare three lemmas in this no..

LEMMA 4. *For four numbers ε, c, a, b $(0<\varepsilon<1, 0<c<a<b<1)$ there exists a density $P_\varepsilon=P_{\varepsilon,c,a,b}$ on Ω with* $\operatorname{supp} P_\varepsilon \subset [c, a]$ *such that the P_ε-unit $e_{\varepsilon,0}$ satisfies that*

$$\frac{e_{\varepsilon,0}(a)}{e_{\varepsilon,0}(b)}<\varepsilon .$$

Proof. Consider the functions

$$h_\varepsilon(r)=1-(1-\varepsilon)\frac{\log(b/r)}{\log(b/a)} \qquad (c\leqq r\leqq b).$$

These functions satisfy $lh_\varepsilon=0$, $h_\varepsilon(a)=\varepsilon$, and $h_\varepsilon(b)=1$, where l is the differential operator defined in no. 1.1. There exists a small positive number δ_ε ($\delta_\varepsilon<(a-c)/4$) such that $h_\varepsilon(a-2\delta_\varepsilon)>0$. We also consider the functions $\phi_n(r)$ ($n=1, 2, \cdots$) on $[c, a]$ defined by

$$\phi_n(r)=\exp\left\{n\left(r-\frac{a+c}{2}\right)^2\right\}.$$

These functions satisfy

$$\frac{l\phi_n(r)}{\phi_n(r)}=4n^2\left(r-\frac{a+c}{2}\right)^2+2n+\frac{2n}{r}\left(r-\frac{a+c}{2}\right)$$

$$=\left\{2n\left(r-\frac{a+c}{2}\right)+\frac{1}{2r}\right\}^2+2n-\frac{1}{4r^2}\geqq 2n-\frac{1}{4c^2}.$$

Then we take a large integer $n_\varepsilon=n_{\varepsilon,c,a,b}$ with $2n_\varepsilon-1/4c^2>0$ and

$$\frac{\phi_{n_\varepsilon}(a-2\delta_\varepsilon)}{\phi_{n_\varepsilon}(a-\delta_\varepsilon)}=\exp(-n_\varepsilon\delta_\varepsilon(a-c-3\delta_\varepsilon))<h_\varepsilon(a-2\delta_\varepsilon). \tag{14}$$

Construct a density $P_\varepsilon=P_{\varepsilon,c,a,b}$ on Ω with $P_\varepsilon(r)=l\phi_{n_\varepsilon}(r)/\phi_{n_\varepsilon}(r)$ on $[c+\delta_\varepsilon, a-\delta_\varepsilon]$ and supp $P_\varepsilon\subset[c, a]$. We denote by $e_{\varepsilon,0}$ the P_ε-unit. Since $e_{\varepsilon,0}$ is increasing as mentioned in no. 1.1, $e_{\varepsilon,0}(c+\delta_\varepsilon)/e_{\varepsilon,0}(b)\leqq 1$ and $e_{\varepsilon,0}(a-\delta_\varepsilon)/e_{\varepsilon,0}(b)\leqq 1$. Then the maximum principle yields that

$$\frac{e_{\varepsilon,0}(r)}{e_{\varepsilon,0}(b)}\leqq\frac{\phi_{n_\varepsilon}(r)}{\phi_{n_\varepsilon}(a-\delta_\varepsilon)}$$

on $[c+\delta_\varepsilon, a-\delta_\varepsilon]$ and hence $e_{\varepsilon,0}(a-2\delta_\varepsilon)/e_{\varepsilon,0}(b)<h_\varepsilon(a-2\delta_\varepsilon)$ by (14). Therefore the maximum principle again yields that

$$\frac{e_{\varepsilon,0}(r)}{e_{\varepsilon,0}(b)}<h_\varepsilon(r)$$

on $(a-2\delta_\varepsilon, b)$. Applying this inequality to $r=a$, we have Lemma 4. □

LEMMA 5. *For five numbers* $\varepsilon, c, a, b, \alpha$ ($0<\varepsilon<1$, $0<c<a<b<1$, $\alpha\geqq 0$) *there exists a density* $P_\varepsilon=P_{\varepsilon,c,a,b,\alpha}$ *on* Ω *with* supp $P_\varepsilon\subset[c, a]$ *such that the first* $(P+P_\varepsilon)$*-unit* $f_{\varepsilon,1}$ *satisfies*

$$\int_a^b\frac{f_{\varepsilon,1}(r)}{r^2f'_{\varepsilon,1}(r)}dr<\frac{1}{\beta^2}\log\frac{1}{2}\left\{\left(\frac{b}{a}\right)^\beta+\left(\frac{a}{b}\right)^\beta\right\}+\varepsilon \qquad (\beta=\sqrt{\alpha+1}),$$

for any density P on Ω with $P(r)=\alpha/r^2$ on $[a, b]$.

Proof. Let P be an arbitrary density on Ω with $P(r)=\alpha/r^2$ on $[a, b]$, δ a positive number with $\delta<1$, and $P_{\delta,c,a,b}$ the density in Lemma 4. We denote by $f_{\delta,1}$ the first $(P+P_{\delta,c,a,b})$-unit. On the interval $[a, b]$ the function $f_{\delta,1}(r)$ has the following form:

$$f_{\delta,1}(r)=\lambda_\delta r^{\beta}+\mu_\delta r^{-\beta} \qquad (\beta=\sqrt{\alpha+1}).$$

Let $e_{\delta,0}$ and $e_{\delta,1}$ be the $P_{\delta,c,a,b}$-unit and the first $P_{\delta,c,a,b}$-unit, respectively. Then by (5) the inequality $P_{\delta,c,a,b}\leqq P+P_{\delta,c,a,b}$ implies

$$\frac{e_{\delta,1}(a)}{e_{\delta,1}(b)}\geqq\frac{f_{\delta,1}(a)}{f_{\delta,1}(b)}.$$

Moreover from (7) and Lemma 4 it follows that

$$\delta>\frac{e_{\delta,0}(a)}{e_{\delta,0}(b)}>\frac{e_{\delta,1}(a)}{e_{\delta,1}(b)}$$

and hence we have

$$\frac{f_{\delta,1}(a)}{f_{\delta,1}(b)}<\delta.$$

This means

$$\frac{\mu_\delta}{\lambda_\delta}<\frac{\delta b^{\beta}-a^{\beta}}{a^{-\beta}-\delta b^{-\beta}}.$$

On the other hand $\mu_\delta/\lambda_\delta>-a^{2\beta}$ by (9). Hence we have

$$\lim_{\delta\to 0}\frac{\mu_\delta}{\lambda_\delta}=-a^{2\beta}$$

and the convergence is uniform for P. Therefore in view of (10) we obtain

$$\lim_{\delta\to 0}\int_a^b\frac{f_{\delta,1}(r)}{r^2 f'_{\delta,1}(r)}dr=\frac{1}{\beta^2}\log\frac{1}{2}\left\{\left(\frac{b}{a}\right)^{\beta}+\left(\frac{a}{b}\right)^{\beta}\right\}.$$

Here the convergence is also uniform for P so that there exists a positive constant $\delta=\delta_\varepsilon=\delta_{\varepsilon,c,a,b,\alpha}$ being independent of P such that

$$\int_a^b\frac{f_{\delta,1}(r)}{r^2 f'_{\delta,1}(r)}dr<\frac{1}{\beta^2}\log\frac{1}{2}\left\{\left(\frac{b}{a}\right)^{\beta}+\left(\frac{a}{b}\right)^{\beta}\right\}+\varepsilon.$$

Thus the desired density P_ε in Lemma 5 is the density $P_{\delta_\varepsilon,c,a,b}$. □

LEMMA 6. *For three numbers ε, c, a $(0<\varepsilon<1, 0<c<a<1)$ there exists a density $P_\varepsilon=P_{\varepsilon,c,a}$ on Ω with* supp $P_\varepsilon\subset[c, a]$ *such that the first P_ε-unit $e_{\varepsilon,1}$ satisfies*

$$\int_c^a\frac{e_{\varepsilon,1}(r)}{r^2 e'_{\varepsilon,1}(r)}dr<\varepsilon.$$

Proof. We take two numbers p, q with $0<c<p<q<a<1$, $p<ce^{\varepsilon/3}$, and $q>ae^{-\varepsilon/3}$. Construct a sequence $\{P_n\}_1^\infty$ of densities P_n on Ω with $\operatorname{supp} P_n \subset [c, a]$ and $P_n(r)=(n^2-1)/r^2$ on $[p, q]$. Every first P_n-unit $e_{n,1}$ ($n=1, 2, \cdots$) has the form $e_{n,1}(r)=\lambda_n r^n+\mu_n r^{-n}$ on $[p, q]$ and by (12) satisfies

$$\int_p^q \frac{e_{n,1}(r)}{r^2 e'_{n,1}(r)} dr \leqq \frac{1}{n^2} \log \frac{1}{2}\left\{\left(\frac{q}{p}\right)^n+\left(\frac{p}{q}\right)^n+n\left(\frac{q}{p}\right)^n\right\}$$

$$<\frac{1}{n} \log \frac{q}{p}+\frac{1}{n^2} \log \left(1+\frac{n}{2}\right)<\frac{1}{n} \log \frac{a}{c}+\frac{1}{n^2} \log \left(1+\frac{n}{2}\right).$$

Then there exists an integer $n=n_\varepsilon=n_{\varepsilon, c, a}$ such that

$$\int_p^q \frac{e_{n,1}(r)}{r^2 e'_{n,1}(r)} dr<\frac{\varepsilon}{3}.$$

On the other hand by (8) and the choice of p, q we have

$$\int_c^p \frac{e_{n,1}(r)}{r^2 e'_{n,1}(r)} dr \leqq \int_c^p \frac{1}{r} dr<\frac{\varepsilon}{3}, \quad \int_q^a \frac{e_{n,1}(r)}{r^2 e'_{n,1}(r)} dr<\frac{\varepsilon}{3}.$$

Therefore the density $P_{\varepsilon, c, a}=P_{n_\varepsilon}$ satisfies Lemma 6. □

2.3. We prove that the converse of Lemma 3 is true.

LEMMA 7. *Let P be a density on Ω with $P'(r) \geqq \alpha_n/r^2$ ($\alpha_n \geqq 0$) on every interval $[a_n, b_n]$ ($n=1, 2, \cdots$). If the set A is an essential set for P then* (13) *is valid.*

Proof. Consider a density P' on Ω with $P'(r)=\alpha_n/r^2$ on every $[a_n, b_n]$. If A is an essential set for P, then A is also an essential set for P' by the definition of essential sets. Therefore we may assume $P(r)=\alpha_n/r^2$ on every $[a_n, b_n]$ without loss of generality.

Fix a sequence $\{\varepsilon_n\}_1^\infty$ of positive numbers ε_n such that $\sum_1^\infty \varepsilon_n<\infty$. We denote by Q_n the density $P_{\varepsilon, c, a, b, \alpha}$ on Ω in Lemma 5 for five numbers $\varepsilon=\varepsilon_n$, $c=b_{n+1}$, $a=a_n$, $b=b_n$, and $\alpha=\alpha_n$. We also denote by R_n the density $P_{\varepsilon, c, a}$ in Lemma 6 for three numbers $\varepsilon=\varepsilon_n$, $c=b_{n+1}$, and $a=a_n$. Consider the density $S=P+\sum_1^\infty(Q_n+R_n)$ on Ω. Then $S(r)=P(r)=\alpha_n/r^2$ on every $[a_n, b_n]$ ($n=1, 2, \cdots$). The first S-unit h_1, the first $(P+Q_n)$-unit $f_{n,1}$, and the first R_n-unit $g_{n,1}$ satisfy by (6)

$$\frac{h_1(r)}{h_1'(r)} \leqq \frac{f_{n,1}(r)}{f'_{n,1}(r)}, \quad \frac{h_1(r)}{h_1'(r)} \leqq \frac{g_{n,1}(r)}{g'_{n,1}(r)} \quad (n=1, 2, \cdots)$$

since $P+Q_n \leqq S$, $R_n \leqq S$ on $(0, 1]$. On the other hand Lemmas 5 and 6 imply that

$$\int_{a_n}^{b_n} \frac{f_{n,1}(r)}{r^2 f'_{n,1}(r)} dr<\frac{1}{\beta_n^2} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\beta_n}+\left(\frac{a_n}{b_n}\right)^{\beta_n}\right\}+\varepsilon_n \quad (\beta_n=\sqrt{\alpha_n+1})$$

and

$$\int_{b_{n+1}}^{a_n} \frac{g_{n,1}(r)}{r^2 g_{n,1}'(r)} dr < \varepsilon_n ,$$

respectively. Therefore we have the following evaluation:

$$\int_0^{b_1} \frac{h_1(r)}{r^2 h_1'(r)} dr < \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\beta_n} + \left(\frac{a_n}{b_n}\right)^{\beta_n}\right\} + 2\sum_{n=1}^{\infty} \varepsilon_n .$$

If A is an essential set for P then the Picard principle is valid for S so that the integral in the above inequality is ∞ by (4) and hence (13) is valid. □

Lemmas 3 and 7 yield the following

COROLLARY 8. *Let P be a density on Ω with $P(r)=\alpha_n/r^2$ $(\alpha_n \geqq 0)$ on every interval $[a_n, b_n]$ $(n=1, 2, \cdots)$. The set A is an essential set for P if and only if* (13) *is valid.*

§ 3. Proofs of theorems

3.1. We start with *the proof of Theorem* 1. First we show that the conditions (b) and (c) are equivalent. In the case that the sequence $\{a_n\}$, $\{b_n\}$ satisfy $\limsup b_n/a_n > 1$, the conditions (b) and (c) are valid. Observe that

$$\begin{cases} (\log x)^2 > \log \frac{1}{2}(x + x^{-1}) & (x>1), \\ (\log x)^2 < 3 \log \frac{1}{2}(x + x^{-1}) & (1<x<x_0) \end{cases}$$

for a positive constant x_0 with $x_0>1$. Then (b) and (c) are also equivalent in the case that $\lim b_n/a_n=1$.

Next we apply the above assertion to the sequences $\{a_n^{\sqrt{2}}\}_1^\infty$, $\{b_n^{\sqrt{2}}\}_1^\infty$. Then the condition (b) is equivalent to

$$\sum_{n=1}^{\infty} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\sqrt{2}} + \left(\frac{a_n}{b_n}\right)^{\sqrt{2}}\right\} = \infty .$$

On the other hand this condition is equivalent to (a) by Lemma 3 and 7 and hence the proof of Theorem 1 is complete.

3.2. We turn to *the proof of Theorem* 2 which is carried over in 3.2 and 3.3. Consider auxiliary conditions

$$\text{(15)} \qquad \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\alpha_n} + \left(\frac{a_n}{b_n}\right)^{\alpha_n}\right\} = \infty \quad (\alpha_n = \sqrt{(\log a_n)^2+1}) ,$$

$$\text{(16)} \qquad \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \log \frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\beta_n} + \left(\frac{a_n}{b_n}\right)^{\beta_n}\right\} = \infty \quad (\beta_n = \sqrt{(\log b_n)^2+1}) .$$

In this no. we show (b)⇒(15)⇒(a)⇒(16)⇒(c). Since

$$\left(\frac{b_n}{a_n}\right)^{\log a_n}+\left(\frac{a_n}{b_n}\right)^{\log a_n}<\left(\frac{b_n}{a_n}\right)^{\alpha_n}+\left(\frac{a_n}{b_n}\right)^{\alpha_n},$$

$$\frac{1}{(\log a_n)^2}<\frac{(\log a_1)^{-2}+1}{\alpha_n^2} \qquad (n=1, 2, \cdots),$$

we have (b)⇒(15). Observe that the density $P(r)=(\log r)^2/r^2$ in Theorem 2 satisfies $(\log b_n)^2/r^2 \leqq P(r) \leqq (\log a_n)^2/r^2$ on every interval $[a_n, b_n]$ $(n=1, 2, \cdots)$. Then Lemma 3 assures (15)⇒(a); Lemma 7 assures (a)⇒(16).

We consider the function

$$\phi(x)=\phi(x\,;\,\rho)=\frac{1}{x^2}\log\frac{1}{2}(\rho^x+\rho^{-x})$$

of x in $(0, \infty)$ for every positive constant ρ with $\rho>1$. This function ϕ satisfies

$$\lim_{x\to 0} x^3\phi'(x)=\lim_{x\to 0}\{x^3\phi'(x)\}'=0\,,$$

$$\{(\rho^x+\rho^{-x})^2(x^3\phi'(x))'\}'=2(\log\rho)^2(2-\rho^{2x}-\rho^{-2x})$$

and hence ϕ is decreasing. Therefore $\phi(\log b_n^{-1}\,;\,b_n/a_n)>\phi(\beta_n\,;\,b_n/a_n)$ $(n=1,2,\cdots)$. This means (16)⇒(c).

3.3. If we prove that (c) yields (b), then the proof of Theorem 2 is complete. In the case sequences $\{a_n\}$ and $\{b_n\}$ satisfy $\lim(\log b_n^{-1})/(\log a_n^{-1})=1$, there exists a positive constant M with $(\log b_n^{-1})^{-1}\leqq M(\log a_n^{-1})^{-1}$ $(n=1, 2, \cdots)$ so that (c) yields (b) by inequalities

$$\left(\frac{b_n}{a_n}\right)^{\log b_n}+\left(\frac{a_n}{b_n}\right)^{\log b_n}<\left(\frac{b_n}{a_n}\right)^{\log a_n}+\left(\frac{a_n}{b_n}\right)^{\log a_n} \quad (n=1, 2\cdots).$$

Therefore we assume

$$\liminf_{n\to\infty}\frac{\log b_n^{-1}}{\log a_n^{-1}}<1\,.$$

However, in this case we see that the upper limit of

$$\frac{1}{(\log a_n)^2}\log\frac{1}{2}\left\{\left(\frac{b_n}{a_n}\right)^{\log a_n}+\left(\frac{a_n}{b_n}\right)^{\log a_n}\right\}$$

is positive since this term is greater than

$$\frac{1}{(\log a_n)^2}\log\left\{\frac{1}{2}\left(\frac{a_n}{b_n}\right)^{\log a_n}\right\}=\frac{\log 2^{-1}}{(\log a_n)^2}+1-\frac{\log b_n^{-1}}{\log a_n^{-1}}$$

and $(\log a_n)^2\to\infty$ as n tens to ∞. Thus (b) is valid. □

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Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan