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PICARD SET OF A KIND OF DIFFERENTIAL POLYNOIMIALS

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Abstract

In the present paper we answer a problem about Picard sets of differential polynomial $F=f^nQ(f)$, which was raised by Anderson, Baker and clunie (cf. [1]).

1. Introduction and result

Let f be a transcendental meromorphic function. We denote by $S(r, f)$, as usual, any function satisfying

$$S(r, f)=O(\log r) \qquad \text{as} \quad r\to\infty$$

when f has finite order, and

$$S(r, f)=O(\log rT(r, f)) \qquad \text{as} \quad r\to\infty,\ r\notin E,\ \text{meas}\, E<\infty$$

when f has infinite order. We call a meromorphic function $a(z)$ "small" function if $a(z)$ satisfies $T(r, a)=S(r, f)$. We call $M(f)=f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}$ a monomial in f, $\nu_M=n_0+n_1+\cdots+n_k$ its degree and $\Gamma_M=n_0+2n_1+\cdots+(1+k)n_k$ its weight. Further, let $M_1(f), \cdots, M_l(f)$ denote monomials in f and $a_1, \cdots, a_l$ denote small functions, then $P(f)=a_1M_1(f)+\cdots+a_lM_l(f)$ is called a differential polynomial in f of degree $\nu_P=\max_{j=1}^{l}\nu_{M_j}$ and weight $\Gamma_P=\max_{j=1}^{l}\Gamma_{M_j}$. In paticular a_i $(1\leqq i\leqq l)$ are entire functions when f is an entire function.

J, M. Anderson, I. N. Baker and J. G. Clunie proved the following:

THEOREM A. [1] *Suppose that f is a transcendental entire function and $F=f^n$ $n\geqq 3$, $n\in \boldsymbol{N}$. Let $\mathfrak{F}=\{\lambda_n\}_{n=1}^{\infty}$ be an infinite point set in $\boldsymbol{C}$ with $\left|\dfrac{\lambda_{n+1}}{\lambda_n}\right|>q>1$ $(n=1.2, \cdots)$. Then $F'(z)$ assumes all values $w\in \boldsymbol{C}$, except possibly zero, infinitely often in $\boldsymbol{C}\setminus\mathfrak{F}$.*

The above authors asked the following two questions:

(a) Can the sets be made larger for entire functions at least? In particular,

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can they consist of small disks?

(b) Are there similar results for differential polynomials of the form $F(z)=f^nQ(f)$, where $Q(f)$ is a differential polynomial in f?

Question (a) was solved by Langley [2] for entire functions, but we have not seen any results about (b) so far.

We proved the following theorem:

THEOREM. *Let f be a transcendental entire function, $\mathfrak{T}=\{\lambda_n\}_{n=1}^{\infty}$ an infinite point set in $\boldsymbol{C}$ with $\left|\frac{\lambda_{n+1}}{\lambda_n}\right|>q>1$ $(n=1, 2, \cdots)$, set $F=f^nQ(f)$, $n\in\boldsymbol{N}$, where $Q(f)$ is a differential polynomial in f and $Q(f)\not\equiv 0$, Then $F'(z)$ assumes all values $w\in\boldsymbol{C}$, except possibly $w=0$, infinitely often in $\boldsymbol{C}\setminus\mathfrak{T}$, provided $n\geqq 3$.*

2. Lemmas.

LEMMA 1. *Suppose that f is a transcendental entire function and $F=f^nQ(f)$, $n\in\boldsymbol{N}$, $n\geqq 3$, where $Q(f)$ is a differential polynomial in f and $Q(f)\not\equiv 0$. Then*

$$T(r, f)<\bar{N}\left(r, \frac{1}{F'-1}\right)+S(r, f) \tag{1}$$

LEMMA 2. *Suppore that f is a nonconstant meromorphic function and $P(f)$ is a differential polynomial in f and $P(f)\not\equiv 0$, if z_0 is a pole of f of degree p $(p\geqq 1)$, and z_0 is not the pole of any small function a_j. Then z_0 is a pole of $P(f)$ of degree $p\cdot\nu_P+(\Gamma_P-\nu_P)$ at most.*

Proof. Let $\tau(z_0, P(f))$ be the degree of pole of $P(f)$ at z_0, then there are nonnegative integers $n_0, \cdots, n_k$ which satisfy $\tau(z_0, P(f))\leqq pn_0+(p+1)n_1+\cdots+(p+k)n_k=(p-1)(n_0+n_1+\cdots+n_k)+(n_0+2n_1+\cdots+(k+1)n_k)\leqq(p-1)\nu_P+\Gamma_P=p\cdot\nu_P+(\Gamma_P-\nu_P)$.

LEMMA 3. [3] *Let f be a nonconstant meromorphic function. If $Q(f)$ is a differential polynomial in f with arbitrary meromorphic coefficients q_j, $1\leqq j\leqq n$, then*

$$m(r, Q(f))\leqq\nu_Q m(r, f)+\textstyle\sum_{j=1}^{n} m(r, q_j)+S(r, f)$$

LEMMA 4. [3] *Let f be a nonconstant meromorphic function. And let $Q^*(f)$ and $Q(f)$ denote differential polynomials in f with arbitrary meromorphic coefficients $q_1^*, \cdots, q_k^*$ and $q_1, \cdots q_l$ respectively. Further, let $P[f]$ be a nonconstant polynomial in f of degree n. Then from $P[f]\cdot Q^*(f)=Q(f)$ we can infer the following:*

1) *if $\nu_Q\leqq n$ then $m(r, Q^*(f))\leqq\sum_{j=1}^{k} m(r, q_j^*)+\sum_{j=1}^{l} m(r, q_j)+S(r, f)$*
2) *if $\Gamma_Q\leqq n$ then $N(r, Q^*(f))\leqq\sum_{j=1}^{k} N(r, q_j^*)+\sum_{j=1}^{l} N(r, q_j)+O(1)$*

LEMMA 5. *Let f be a nonconstant meromorphic function. And let $Q(f)$ and*

$P(f)$ be differential polynomials in f satisfying $P(f)\not\equiv 0$, $Q(f)\not\equiv 0$. Then $g_0=-f^nQ(f)$ and $g_1=f^nQ(f)+P(f)$ are independent over $\boldsymbol{C}$, provided $n\geqq \Gamma_P+1$.

Proof. Assume that $c_0g_0+c_1g_1=0$, c_0, $c_1\in\boldsymbol{C}$, that is $f^nQ(f)(c_1-c_0)=-c_1P(f)$. Obviously, we have $c_1\neq 0$ and $c_0\neq c_1$, we get $T(r, Q(f))=S(r, f)$, $N(r, fQ(f))=S(r, f)$ from lemma 4, so $N(r, f)\leqq N\left(r, \frac{1}{Q(f)}\right)+N(r, fQ(f))=S(r, f)$, hence we have

$$nm(r, f)=m(r, -c_1P(f)/Q((f)(c_1-c_0))\leqq m(r, P(f))+m\left(r, \frac{1}{Q(f)}\right)+S(r, f)$$
$$\leqq \nu_P m(r, f)+N(r, Q(f))-N\left(r, \frac{1}{Q(f)}\right)+m(r, Q(f))+S(r, f)$$
$$\leqq \nu_P m(r, f)+S(r, f)$$

So we get

$$T(r, f)\leqq S(r, f)$$

which is impossible.

LEMMA 6. [1] *Let $G(z)$ be an entire function, assume that all the zeros of $G(z)$ lie in the set $\mathfrak{T}=\{\lambda_n\}_{n=1}^{\infty}$ and $\left|\frac{\lambda_{n+1}}{\lambda_n}\right|>q>1$. Then $\bar{n}\left(r, \frac{1}{G}\right)=0(\log r)$, $\bar{N}\left(r, \frac{1}{G}\right)=0((\log r)^2)$ as $r\to\infty$.*

Proof of Lemma 1. Suppose that $P(f)$ is a differential polynomial in f and $\Gamma_P\leqq n-1$, let $g_0=-f^nQ(f)$, $g_1=f^nQ(f)+P(f)$, we know $\frac{g_1'}{g_1}-\frac{g_0'}{g_0}\not\equiv 0$ from lemma 5. So, from $g_0+g_1=P(f)$ and $g_0'+g_1'=P'(f)$, we have

$$-f^n=\frac{P(f)(g_1'/g_1-P'(f)/P(f))}{(g_1'/g_1-g_0'/g_0)Q(f)} \tag{2}$$

we get

$$m(r, f^n)\leqq m(r, P(f))+m(r, g_1'/g_1-P'(f)/P(f))+m(r, 1/(g_1'/g_1-g_0'/g_0)Q(f))$$
$$\leqq \nu_P m(r, f)+N(r, (g_1'/g_1-g_0'/g_0)Q(f))-N(r, 1/(g_1'/g_1-g_0'/g_0)Q(f))$$
$$+m(r, g_1'/g_1)+m(r, P'(f)/P(f))+m(r, (g_1'/g_1-g_0'/g_0)Q(f))+S(r, f)$$

from lemma 3. From $T(r, g_i)=O(T(r, f))$ $(i=0, 1)$ and $T(r, P(f))=O(T(r, f))$, we have $S(r, g_i)\leqq S(r, f)$ $(i=0, 1)$ and $S(r, P(f))\leqq S(r, f)$. Thus

$$(n-\nu_P)m(r, f)\leqq N(r, (g_1'/g_1-g_0'/g_0)Q(f))-N(r, 1/(g_1'/g_1-g_0'/g_0)Q(f))$$
$$+m(r, (g_1'/g_1-g_0'/g_0)Q(f))+S(r, f)$$

We rewrite (2) as follows

$$-f^n(g_1'/g_1-g_0'/g_0)Q(f)=(g_1'/g_1-P'(f)/P(f))P(f)$$

It is easy to know that $m(r, (g_1'/g_1-g_0'/g_0)Q(f))=S(r, f)$ from lemma 4. Thus

$$(n-\nu_P)m(r, f)$$
$$\leqq N(r, (g_1'/g_1-g_1'/g_0)Q(f))-N(r, 1/(g_1'/g_1-g_0'/g_0)Q(f))+S(r, f) \qquad (3)$$

Assume that z_0 is a pole of f of order p, and z_0 is not a zero or pole of coefficients of $P(f)$. Suppose that

$$Q(f)(g_1'/g_1-g_0'/g_0)=c(z-z_0)^{\mu} \qquad (c=c(z)\not\equiv 0;\ \mu \text{ is an integer})$$

we know that $np\leqq p\nu_P+\Gamma_P-\nu_P+1+\mu$ from (2) and lemma 2, thus

$$\mu\geqq p(n-\nu_P)-(\Gamma_P-\nu_P+1) \qquad (4)$$

So, we have

$$N(r, 1/(g_1'/g_1-g_0'/g_0)Q(f))\geqq(n-\nu_P)N(r, f)-(\Gamma_P-\nu_P+1)\overline{N}(r, f)+S(r, f) \quad (4)'$$

from (4). Obviously, the poles of $Q(f)(g_1'/g_1-g_0'/g_0)$ occur only at poles of f, zeros of g_0 (except the zeros of $Q(f)$), zeros of g_1, zeros or poles of coefficients of $Q(f)$ and $P(f)$. If $n\geqq\Gamma_P+1$ and $p\geqq 1$, it is easy to see that $\mu\geqq 0$. Thus z_0 is not a pole of $Q(f)(g_1'/g_1-g_0'/g_0)$ provided z_0 is a pole of f. From the above anlyses, we have

$$N(r, Q(f)(g_1'/g_1-g_0'/g_0))\leqq\overline{N}(r, 1/g_1)+\overline{N}(r, 1/f)+S(r, f) \qquad (5)$$

Thus

$$(n-\nu_P)T(r, f)\leqq\overline{N}(r, 1/g_1)+\overline{N}(r, 1/f)+(\Gamma_P-\nu_P+1)\overline{N}(r, f)+S(r, f) \quad (6)$$

combining (3), (4)$'$ and (5).

Let $F=f^nQ(f)$, so $F'=f^{n-1}(nf'Q(f)+fQ'(f))=f^{n-1}Q_1(f)$, where $Q_1(f)$ is still a differential polynomial in f. Assume that $P(f)\equiv-1$, $g_0=-f^{n-1}Q_1(f)$ $(=-F')$, $g_1=f^{n-1}Q_1(f)-1=f^{n-1}Q_1(f)+P(f)$ $(=F'-1)$ such that g_0 and g_1 satisfy the conditions of lemma 5. Finally, we get $(n-2)T(r, f)\leqq\overline{N}\left(r, \dfrac{1}{F'-1}\right)+S(r, f)$ by applying equation (6) to g_0 and g_1 and noting $\Gamma_P=\nu_P=0$. Hence lemma 1 is proved.

3. Proof of theorem.

Without loss of geneaality we suppose that $w=1$. Obviously, $F'-1$ has infinitely many zeros from (1). If $F'-1$ has only finitely many zeros in $\boldsymbol{C}\setminus\mathcal{F}$, then $F'-1$ has infinitely many zeros in $\mathcal{F}$. We suppose that $F'=1$ at every point of $\mathcal{F}$ by deleting some of the points λ_n of $\mathcal{F}$ and adjusting notation if necessary. From lemma 6 and (1), we know that $T(r, f)=O((\log r)^2)$ as $r\to\infty$, $r\overline{\in}E$, meas $E<\infty$. So f has order zero (see [3, lemma 3]). Hence f has infinitely many zeros since f is transcendental. And we have $S(r, f)=O(\log r)$ as $r\to\infty$ since f has finite order, so we get

$$T(r,\ f)=O((\log r)^2) \qquad \text{as} \quad r\to\infty \tag{7}$$

For the convenience of presentation we set $f(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\mu_k}\right)$. It is easy to know that each μ_k is a zero of $F'(z)$ of order at least 2. And we have $T(r, F'-1)=O((\log r)^2)$ as $r\to\infty$ from (7). Given $\varepsilon>0$, for some $\varepsilon_k(0<\varepsilon_k<\varepsilon)$ and large k we know $|F'-1|>2$ (see [1, lemma 4]) and hence $|F'|>1$ on the boundary of or outside these discs $\Delta_k=\{z: |z-\lambda_k|<\varepsilon_k|\lambda_k|\}$, so μ_k lie in one of these discs, say Δ_k. If ε is chosen sufficiently small then the disc Δ_k contains no other $\lambda_m(m\neq k)$ from the condition $|\lambda_{n+1}/\lambda_n|>q>1$, and so no other z with $F'(z)=1$. Now, suppose that the equation $F'(z)=1$ has an m-fold root at λ_k and consider the level curves $|F'(z)|=1$ passing through λ_k, These lie in Δ_k and consist of m distinct loops with only the point λ_k in common. By the maximum and minimum modulus priciples, each loop contains at least one zero of $F'(z)$. So, F' has the same number of 1-points as zeros inside the Δ_k by Rouché theorem. Hence F' has only m simple zeros in the Δ_k. But that contradicts the presence of μ_k in the Δ_k which implies that F' has a zero of multiplicity at least 2 in the Δ_k. Hence the theorem is proved.

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