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# SELF-HOMOTOPY EQUIVALENCES OF $H_*(-; Z/p)$-LOCAL SPACES

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**Abstract**

Under certain finiteness conditions, $p$-completion commutes with the formation of a certain group of self-homotopy equivalences.

## 1. Introduction.

Let $X$ be a pointed, 0-connected topological space, $\mathrm{Aut}(X)$ the group of based homotopy classes of based self-homotopy equivalences of $X$, and $\mathrm{Aut}_\#(X)$ the kernel of the obvious homomorphism

$$\mathrm{Aut}(X) \longrightarrow \prod_{i=1}^{d} \mathrm{Aut}\, \pi_i(X)$$

where we further assume either that $X$ is a $CW$-complex of dimension $d$ or that $\pi_*(X)=0$ for $*>d$, $1\leqq d<\infty$. The purpose of this paper is to investigate the behaviour of $\mathrm{Aut}_\#$ under $H_*(-; \boldsymbol{Z}/p)$-localization of the space.

To explain the main result, let $X$ be a finite, connected, and nilpotent $CW$-complex and $X_{Z/p}$ its $H_*(-; \boldsymbol{Z}/p)$-localization in the sense of Bousfield [1]. Then $\mathrm{Aut}_\#(X)$ is nilpotent [3] and

$$\mathrm{Aut}_\#(X_{Z/p})=\mathrm{Ext}(\boldsymbol{Z}/p^\infty, \mathrm{Aut}_\#(X))$$

where $\mathrm{Ext}(\boldsymbol{Z}/p^\infty, -)$ is the Ext$-p$-completion functor defined for all nilpotent groups by Bousfield and Kan [2].

This paper can be viewed as a parallel, not only in subject but also in method, to [7]. I am very grateful to Prof. K. Maruyama for sending me a preprint of his paper, to Prof. C. U. Jensen who kindly supplied the proof of Proposition 2.4, and to the topologists at Memorial University for the invitation to their conference on Spaces of Self-homotopy Equivalences, Montreal, August 1988.

## 2. Completions of nilpotent groups.

In this section I collect for later reference some fundamental facts about

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$p$-complete (or Ext$-p$-complete, in the terminology of [2]) nilpotent groups. The prime sources of information are [1], [2], and [8].

Let $p$ be a prime number. In the pointed homotopy category of $CW$-complexes, Bousfield [1] constructed a $H_*(-;\boldsymbol{Z}/p)$-localization functor which agrees with the $\boldsymbol{Z}/p$-completion functor of Bousfield and Kan [2] for nilpotent complexes. This $H_*(-;\boldsymbol{Z}/p)$-localization of a complex $X$ will here be written as $\eta_X: X \to X_{Z/p}$.

Let $N$ be a nilpotent group. Define

$$E_p N = \pi_1(K(N, 1)_{Z/p})$$

$$H_p N = \pi_2(K(N, 1)_{Z/p}).$$

$E_p$ and $H_p$ are endofunctors of the category of nilpotent groups; $H_p N$ is of course even an abelian group. The induced map

$$\eta_*: N = \pi_1 K(N, 1) \longrightarrow \pi_1(K(N, 1)_{Z/p}) = E_p N$$

is called the completion map.

DEFINITION 2.1. *$N$ is $p$-complete if the following two conditions are satisfied:*
(i) $H_p N = 0$
(ii) *The completion map $\eta: N \to E_p N$ is an isomorphism.*

The first condition is satisfied by any finitely generated nilpotent group; $E_p N$ and $H_p N$ are $p$-complete groups; the completion map $N \to E_p N$ is universal for homomorphisms from $N$ into $p$-complete nilpotent groups.

PROPOSITION 2.2. ([2], VI. 2.5). *Any short exact sequence*

$$1 \longrightarrow K \longrightarrow N \longrightarrow Q \longrightarrow 1$$

*of nilpotent groups induces an exact sequence*

$$0 \longrightarrow H_p K \longrightarrow H_p N \longrightarrow H_p Q \longrightarrow E_p K \longrightarrow E_p N \longrightarrow E_p Q \longrightarrow 1.$$

COROLLARY 2.3. (a) *If $H_p K = 0 = H_p Q$, also $H_p N = 0$*
(b) *If $K$ and $Q$ are $p$-complete, so is $N$*
(c) *If $H_p Q = 0$ and $N$ is $p$-complete, $K$ is $p$-complete.*

For an abelian group $A$,

$$E_p A = \mathrm{Ext}_Z(\boldsymbol{Z}/p^\infty, A)$$

$$H_p A = \mathrm{Hom}_Z(\boldsymbol{Z}/p^\infty, A)$$

and the completion map $\eta: A \to E_p A$ equals the boundary map $\delta$ of the exact sequence

$$0 \longrightarrow H_pA \longrightarrow \mathrm{Hom}\Big(\boldsymbol{Z}\Big[\frac{1}{p}\Big], A\Big) \longrightarrow A \overset{\delta}{\longrightarrow} E_pA \longrightarrow \mathrm{Ext}\Big(\boldsymbol{Z}\Big[\frac{1}{p}\Big], A\Big) \longrightarrow 0$$

induced from the short exact sequence

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \boldsymbol{Z}\Big[\frac{1}{p}\Big] \longrightarrow \boldsymbol{Z}/p^\infty \longrightarrow 0$$

of abelian groups.

Any $p$-complete abelian group is in a canonical way a module over the ring $E_p\boldsymbol{Z}=\boldsymbol{Z}_p^\wedge=\varprojlim \boldsymbol{Z}/p^n\boldsymbol{Z}$ of $p$-adic integers ([2], VI. 4.3). Conversely,

PROPOSITION 2.4. *Any finitely generated module over $\boldsymbol{Z}_p^\wedge$ is $p$-complete.*

*Proof.* Since $\boldsymbol{Z}_p^\wedge$ is a *PID*, all finitely generated $\boldsymbol{Z}_p^\wedge$-modules are finite direct sums of cyclic $\boldsymbol{Z}_p^\wedge$-modules. But the only cyclic $\boldsymbol{Z}_p^\wedge$-modules are $\boldsymbol{Z}_p^\wedge$ itself and the cyclic groups $\boldsymbol{Z}/p^j\boldsymbol{Z}$, $1\leqq j<\infty$, and these groups are $p$-complete.

Not all $\boldsymbol{Z}_p^\wedge$-modules are $p$-complete ([2], VI. 4.4).

Let now $N$ be a nilpotent group acting nilpotently on an abelian group $H$. If $H_pN=0$ the short split exact sequence

$$0 \longrightarrow H \longrightarrow H\rtimes N \rightleftarrows N \longrightarrow 1$$

will induce a short split exact sequence

$$0 \longrightarrow E_pH \longrightarrow E_p(H\rtimes N) \rightleftarrows E_pN \longrightarrow 1$$

of $p$-complete groups. Thus $E_pN$ acts on $E_pH$. Also $N$ itself acts on $E_pH$; this follows from the universal property. All actions are compatible in the sense that the following diagram of action maps and completion maps

$$\begin{array}{ccc} N\times H & \longrightarrow & H \\ \downarrow & & \downarrow \\ N\times E_pH & \longrightarrow & E_pH \\ \downarrow & & \| \\ E_pN\times E_pH & \longrightarrow & E_pH \end{array}$$

commutes.

Consider an element $\theta\in H$ and its image $\eta\theta\in E_pH$ under completion. The completion map on $N$ restricts to a homomorphism

$$N^\theta \longrightarrow (E_pN)^{\eta\theta}$$

of the respective isotropy subgroups determined by $\theta$ and $\eta\theta$.

THEOREM 2.5. *Suppose that $H_pN=0$ and that $H$ is finitely generated. Then $H_p(N^\theta)=0$, $(E_pN)^{\eta\theta}$ is $p$-complete, and the slanted arrow*

$$\begin{tikzcd} N^{\theta} \arrow[r] \arrow[d] & (E_pN)^{\eta\theta} \\ E_p(N^{\theta}) \arrow[ur] & \end{tikzcd}$$

*generated by the universal property, is an isomorphism.*

*Proof.* In [4] Hilton proved a theorem like this one for $p$-localization instead of $p$-completion. I simply adapt his proof to the present context.

The proof is by induction on $c=nil_N(H)$. If $c=1$, the action is trivial, and there is nothing to prove. Suppose now that $nil_N(H)=c$ and that the theorem holds for all actions of nilpotency$<c$. $N$ acts trivially on $\Gamma$: $=\Gamma_N^{c-1}(H)$ so there is an induced action on $H/\Gamma$ and $nil_N(H/\Gamma)=c-1$. As $H_p(H/\Gamma)=0$, the short exact sequence

$$0 \longrightarrow \Gamma \longrightarrow H \xrightarrow{\kappa} H/\Gamma \longrightarrow 0$$

of $N$-modules induces a short exact sequence

$$0 \longrightarrow E_p\Gamma \longrightarrow E_pH \xrightarrow{E_p\kappa} E_p(H/\Gamma) \longrightarrow 0$$

of $E_pN$-modules. Their associated exact orbit sequences ([4], (1.5) p. 189)

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & N^{\theta} \arrow[r] \arrow[d] & N^{\kappa\theta} \arrow[r, "\delta"] \arrow[d] & \Gamma \arrow[d] \\ 1 \arrow[r] & (E_pN)^{\eta\theta} \arrow[r] & (E_pN)^{E_p\kappa(\eta\theta)} \arrow[r, "\delta"] & E_p\Gamma \end{tikzcd}$$

are connected by (restrictions of) completion maps. (The homomorphism $\delta$ is defined by $\delta n=\theta-n\theta$). Applying the functor $E_p$ to the upper exact orbit sequence yields the sequence

$$1 \longrightarrow E_p(N^{\theta}) \longrightarrow E_p(N^{\kappa\theta}) \xrightarrow{E_p\delta} E_p\Gamma,$$

easily seen to be exact. By the induction hypothesis, $(E_pN)^{E_p\kappa(\eta\theta)}$ is $p$-complete and the vertical arrow

$$\begin{tikzcd} 1 \arrow[r] & E_p(N^{\theta}) \arrow[r] & E_p(N^{\kappa\theta}) \arrow[r] \arrow[d] & E_p\Gamma \arrow[d, equal] \\ 1 \arrow[r] & (E_pN)^{\eta\theta} \arrow[r] & (E_pN)^{E_p\kappa(\eta\theta)} \arrow[r] & E_p\Gamma, \end{tikzcd}$$

generated by the universal property, an isomorphism; so is then the restriction $E_p(N^{\theta})\to(E_pN)^{\eta\theta}$.

Finally, in the short exact sequence

$$1 \longrightarrow N^{\theta} \longrightarrow N^{\kappa\theta} \longrightarrow \operatorname{im}\delta \longrightarrow 0,$$

$H_p(\operatorname{im}\delta)=0$, so $H_p(N^{\theta})\cong H_p(N^{\kappa\theta})$, and the latter group is trivial by induction hypothesis.

### 3. Spaces of self-homotopy equivalences.

In this chapter, I investigate the relation between $\mathrm{Aut}_{\#}(Y)$ and $\mathrm{Aut}_{\#}(X)$ where $r: Y\to X$ is one step on a Postnikov ladder.

The basic situation of this chapter is depicted in the following diagram

$$\begin{tikzcd} K(M, n) \arrow[r] & Y \arrow[r] \arrow[d, "r"'] & \bar{P}L \arrow[d] \\ & X \arrow[r, "\theta"] \arrow[dr, "\theta_0"'] & L \arrow[d, shift right, "\hat{k}"'] \\ & & B \arrow[u, shift right, "\check{k}"'] \end{tikzcd}$$

Here, $X$ is a connected and $n$-anticonnected (i.e. $\pi_i(X)=0$ for $i\geqq n$) space, the square is a pull back, $L=L(M, n+1)=E\pi\times_{\pi}K(M, n+1)$ where $M$ is a $\pi$-module, $\pi=\pi_1(X)$, $\bar{P}L\to L$ is the path fibration [6] over and under $B=B\pi=K(\pi, 1)$, $\check{k}$ is the zero-section of the fibration $\hat{k}: L\to B$, $\theta_0$ is a fibration inducing the identity on $\pi_1$, and $\theta$ is a map over $B$ whose vertical homotopy class also will be denoted by $\theta\in H^{n+1}(X; M)$.

The notation for mapping spaces is as in Switzer [10]. $\mathrm{Aut}_{\#}(Z)$, $Z\in\{X, Y\}$, can be thought of as a set of path-components of $F_1(Z, *; Z)$; write $\mathrm{aut}_{\#}(Z)\subset F_1(Z, *; Z)$ for the subspace made up by these path-components such that $\pi_0\,\mathrm{aut}_{\#}(Z)=\mathrm{Aut}_{\#}(Z)$.

The homotopy sequence for the standard fibration

$$\prod_{i=0}^{n-1} K(H^{n-i}(X; M), i) \longrightarrow \mathrm{aut}_{\#}(Y) \longrightarrow \mathrm{aut}_{\#}(X)$$

terminates with the following exact sequence of groups

$$\cdots \longrightarrow \pi_1\,\mathrm{aut}_{\#}(X) \xrightarrow{\partial} H^n(X; M) \longrightarrow \mathrm{Aut}_{\#}(Y) \longrightarrow \mathrm{Aut}_{\#}(X)^{\theta} \longrightarrow 1$$

where $\mathrm{Aut}_{\#}(X)^{\theta}$ is the isotropy subgroup of $\theta$ under the action of $\mathrm{Aut}_{\#}(X)$ on $H^{n+1}(X; M)$. (Write $\mathrm{aut}_{\#}(X)^{\theta}\subset\mathrm{aut}_{\#}(X)$ for the corresponding space such that $\pi_0\,\mathrm{aut}_{\#}(X)^{\theta}=\mathrm{Aut}_{\#}(X)^{\theta}$).

In order to obtain a useful description of $\partial$, I shall now present an alternative fibration resulting in the same long exact homotopy sequence. First, I introduce the spaces

$$\mathrm{aut}_{\#}^{B}(Y)=\{g\in\mathrm{aut}_{\#}(Y)\,|\,\theta_0 rg=\theta_0 r\}$$

$$\mathrm{aut}_{\#}^{B}(X)=\{f\in\mathrm{aut}_{\#}(X)\,|\,\theta_0 f=\theta_0\}$$

of self-homotopy equivalences over $B$.

LEMMA 3.1. *The inclusion* $\mathrm{aut}_{\#}^{B}(Z)\subset\mathrm{aut}_{\#}(Z)$ *is a weak homotopy equivalence,* $Z\in\{X, Y\}$.

*Proof.* The fibration $\theta_0$ induces a fibration of mapping spaces

$$\underline{\theta}_0:\mathrm{aut}_{\#}(X)\longrightarrow F_{\theta_0}^{0}(X, *; B),$$

where $F_{\theta_0}^{0}(X, *; B)$, the identity component of the space of based maps $X\to B$, is contractible. Hence the fibre $\underline{\theta}_0^{-1}(\theta_0)=\mathrm{aut}_{\#}^{B}(X)$ is equivalent to the total space $\mathrm{aut}_{\#}(X)$. Similarly for $Y$.

Consider also $\mathrm{aut}_{\#}^{B}(X)^{\theta}=\mathrm{aut}_{\#}^{B}(X)\cap\mathrm{aut}_{\#}(X)^{\theta}$.

Let $I=[0, 1]$ denote the unit interval, $\dot{I}=\{0, 1\}$, and $p_2: I\times X\to X$ the projection onto the second factor. There is a fibration [9]

$$\prod_{i=0}^{n-1} K(H^{n-i}(X; M), i)\longrightarrow F_{\theta p_2}(I\times(X, *); L, B)\longrightarrow F_{\theta p_2}(\dot{I}\times(X, *); L, B)$$

as well as a map of $\mathrm{aut}_{\#}^{B}(X)^{\theta}$ into the base space taking $f\in\mathrm{aut}_{\#}^{B}(X)^{\theta}$ into the pair $(\theta, \theta f)$.

PROPOSITION 3.2. *The pull back of the diagram*

$$\begin{array}{ccc} & & F_{\theta p_2}(I\times(X, *); L, B) \\ & & \downarrow \\ \mathrm{aut}_{\#}^{B}(X)^{\theta} & \longrightarrow & F_{\theta p_2}(\dot{I}\times(X, *); L, B) \end{array}$$

*is weakly homotopy equivalent to* $\mathrm{aut}_{\#}(Y)$.

*Proof.* Recall first that

$$Y=\{(x, u)\in X\times\bar{P}L\,|\,\theta x=u(1)\}$$

where $u: I\to L$ is a path in the fibre $\hat{k}^{-1}\theta_0(x)\subset L$ with $u(0)=\check{k}\theta_0(x)$.

The pull back, $E$, of the diagram in the proposition consists of pairs $(f, H)$ such that $f\in\mathrm{aut}_{\#}^{B}(X)$ and $H: I\times X\to L$ is a based vertical homotopy of $\theta$ to $\theta f$. Associate to any such pair the self-map $[f, H]: Y\to Y$ given by

$$[f, H](x, u)=(f(x), u*H(-, x))$$

where the symbol $*$ stands for multiplication of paths. Note that $*$ is defined here, the resultant path stays inside the fibre over $\theta_0(x)$ and ends at $H(1, x)=\theta f(x)$. Since $\pi_n[f, H]=1$, we have a fibre map

$$\begin{array}{ccc} E & \xrightarrow{[-,-]} & \mathrm{aut}_{\#}(Y) \\ \downarrow & & \downarrow \\ \mathrm{aut}_{\#}^{B}(X)^{\theta} & \longrightarrow & \mathrm{aut}_{\#}(X)^{\theta} \end{array}$$

Since the maps between the base spaces and the fibres are weak homotopy equivalences, so is the map between the total spaces.

COROLLARY 3.3. *The map* $\partial\colon \pi_1 \mathrm{aut}_{\#}(X) \to H\colon = H^n(X; M)$ *is the composition*

$$\pi_1 \mathrm{aut}_{\#}(X) \longrightarrow \pi_1 F_{\theta p_2}(\dot{I} \times (X, *); L, B) = H \times H \longrightarrow H$$

*where the second map takes* $(a, b) \in H \times H$ *to* $b - a$.

*Proof.* The second homomorphism is computed in ([9], p. 184).

## 4. Proof of the main theorem.

After some general remarks concerning the effect of $H_*(- ; \boldsymbol{Z}/p)$-localization on function spaces, the main theorem is formulated and proved by an inductive argument.

Let $X$ be a nilpotent connected space which is either a finite $CW$-complex or finitely anticonnected of finite type (i.e. $\pi_i(X)=0$ for $i \gg 0$ and finitely generated for all $i$).

For each based self-map $f$ of $X$ there exists a based self-map $f_p$ of $X_{Z/p}$, unique up to homotopy, with $f_p \eta = \eta f$. The completion map $\eta\colon X \to X_{Z/p}$ induces maps of function spaces

$$F_f^0(X, *; X) \xrightarrow{\underline{\eta}} F_{\eta f}^0(X, *; X_{Z/p}) \xleftarrow{\bar{\eta}} F_{f_p}^0(X_{Z/p}, *; X_{Z/p})$$

where the superscript 0 indicates the path-component containing the function occuring as subscript.

LEMMA 4.1. (a) $F_f^0(X, *; X)$ *is nilpotent of finite type.*

(b) $F_{\eta f}^0(X, *; X_{Z/p})$ *is* $H_*(- ; \boldsymbol{Z}/p)$*-local.*

(c) $\underline{\eta}$ *induces a homotopy equivalence* $F_f^0(X, *; X)_{Z/p} \cong F_{\eta f}^0(X, *; X_{Z/p})$.

(d) $\bar{\eta}$ *is a weak homotopy equivalence.*

*Proof.* (a), (b), and (c) are special cases of ([2], VI. 7.1); in case $X$ is finitely anticonnected of finite type, replace the source space $X$ by its $N$-skeleton $X^N$ for $N \gg 0$. (d) is a special case of ([1], Proposition 12.2).

Let now $M$ be a finitely generated abelian group on which $\pi$ acts, $\pi = \pi_1(X)$. Then $\pi$ also acts on $E_p M$ and since $H_p \pi = 0$, even $E_p \pi = \pi_1(X_{Z/p})$ acts on $E_p M$. Consider the induced homomorphisms

$$\bar{H}^i(X; M) \xrightarrow{\eta_*} \bar{H}^i(X; E_pM) \xrightarrow{\eta^*} \bar{H}^i(X_{Z/p}; E_pM)$$

of cohomology groups with local coefficients. As a $p$-complete abelian group, $E_pM$ is a $\boldsymbol{Z}_p^{\hat{}}$-module, so also $\bar{H}^i(Z; E_pM)$, $Z=X$, $X_{Z/p}$, is a $\boldsymbol{Z}_p^{\hat{}}$-module through coefficient group homomorphisms.

LEMMA 4.2. (a) *$\bar{H}^i(X; M)$ is a finitely generated abelian group.*
(b) *$\bar{H}^i(X; E_pM)$ is $p$-complete*
(c) *$\eta_*$ induces an isomorphism $E_p\bar{H}^i(X; M)\cong\bar{H}^i(X; E_pM)$ of $\boldsymbol{Z}_p^{\hat{}}$-modules.*
(d) *$\eta^*$ is an isomorphism of $\boldsymbol{Z}_p^{\hat{}}$-modules.*

*Proof.* If either finiteness condition on $X$ is satisfied the cellular cohomology groups $\Gamma^*(X; M)$ are finitely generated ([5], Theorem II. 4.2), so (a) is obvious. Statements (b) and (c) follow from the observations that $\Gamma^*(X; E_pM)=E_p\Gamma^*(X; M)$ and $E_p$ is an exact functor on the category of finitely generated abelian groups (Proposition 2.2). To prove (d), consider the fibration

$$L_{Z/p}=L(E_pM, i+1) \longrightarrow B_{Z/p}=K(E_p\pi, 1)$$

constructed from the $E_p\pi$-action on $E_pM$. Let $u: X_{Z/p}\to L_{Z/p}$ be any map. The completion map $\eta: X\to X_{Z/p}$ induces a map $\bar{\eta}$ of fibrations

$$\begin{array}{ccc}
F_{u\eta}(X_{Z/p}, *; L_{Z/p}, B_{Z/p}) & \xleftarrow{\bar{\eta}} & F_u(X, *; L_{Z/p}, B_{Z/p}) \\
\downarrow & & \downarrow \\
F_{u\eta}(X_{Z/p}, *; L_{Z/p}) & \xleftarrow{\bar{\eta}} & F_u(X, *; L_{Z/p}) \\
\downarrow & & \downarrow \\
F_{u\eta}(X_{Z/p}, *; B_{Z/p}) & \xleftarrow{\bar{\eta}} & F_u(X, *; B_{Z/p}).
\end{array}$$

According to ([1], Proposition 12.2), the two lower horizontal maps are weak homotopy equivalences. So is then the map between the fibres. It follows in particular, that the induced map of $\pi_1F_u(X, *; L_{Z/p}, B_{Z/p})=\bar{H}^i(X; E_pM)$ into $\pi_1F_{u\eta}(X, *; L_{Z/p}, B_{Z/p})=\bar{H}^i(X_{Z/p}; E_pM)$ is an isomorphism of $\boldsymbol{Z}_p^{\hat{}}$-modules.

The main result of this paper is the following.

THEOREM 4.3. *Let $X$ be a based, nilpotent, connected space which is either a finite CW-complex or finitely anticonnected of finite type. Then $H_p\,\mathrm{Aut}_\#(X)=0$, $\mathrm{Aut}_\#(X_{Z/p})$ is $p$-complete, and the slanted arrow*

$$\begin{tikzcd}
\mathrm{Aut}_\#(X) \arrow[r] \arrow[d, "\eta"'] & \mathrm{Aut}_\#(X_{Z/p}) \\
E_p\,\mathrm{Aut}_\#(X) \arrow[ur] &
\end{tikzcd}$$

*generated by the universal property, is an isomorphism.*

By a standard trick it suffices to prove Theorem 4.3 for finitely anticonnected nilpotent spaces of finite type in which case induction over the height of a Postnikov tower for $X$ is a natural approach.

To start the induction, note that the theorem certainly is true in case $X=K(\pi, 1)$ for then $\mathrm{Aut}_{\#}(X)=1=\mathrm{Aut}_{\#}(X_{Z/p})$. As to the inductive step, suppose now further that $\pi_i(X)=0$ for $i\geqq n$, that Theorem 4.3 holds for $X$, and that $Y$, as in Chapter 3, is the total space

$$K(M, n) \longrightarrow Y \longrightarrow X$$

of a nilpotent Eilenberg-MacLane fibration over $X$ with $k$-invariant $\theta\in H^{n+1}(X; M)$. Then $Y$ is nilpotent, $(n+1)$-anticonnected of finite type, and I must show that Theorem 4.3 is true for $Y$.

According to Lemma 4.2, there is a unique class

$$\theta_p\in H^{n+1}(X_{Z/p}\,; E_pM)$$

with $\eta^*\theta_p=\eta_*\theta$ and corresponding to a vertical homotopy class $\theta_p: X_{Z/p}\to L_{Z/p}$ of maps over $B_{Z/p}=K(E_p\pi, 1)$. Using the characterization of $\eta: Y\to Y_{Z/p}$ of ([8], Theorem 2, Theorem 3) it is easy to see that

LEMMA 4.4. *The pull back of the diagram*

$$\begin{array}{ccc} & & \bar{P}L_{Z/p} \\ & & \downarrow \\ X_{Z/p} & \xrightarrow{\theta_p} & L_{Z/p} \end{array}$$

*is* $Y_{Z/p}$.

The fibre map

$$\begin{array}{ccc} K(M, n) & \longrightarrow & K(E_pM, n) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_{Z/p} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_{Z/p} \end{array}$$

induces a map of exact sequences

$$\begin{array}{ccccccccc} \pi_1\mathrm{aut}_{\#}(X) & \xrightarrow{\partial} & H^n(X; M) & \longrightarrow & \mathrm{Aut}_{\#}(Y) & \longrightarrow & \mathrm{Aut}_{\#}(X)^{\theta} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_1\mathrm{aut}_{\#}(X_{Z/p}) & \xrightarrow{\partial} & H^n(X_{Z/p}\,; E_pM) & \longrightarrow & \mathrm{Aut}_{\#}(Y_{Z/p}) & \longrightarrow & \mathrm{Aut}_{\#}(X_{Z/p})^{\theta_p} & \longrightarrow & 1 \end{array}$$

The two vertical homomorphisms to the left are of the form $(\bar{\eta})_*^{-1}\circ\underline{\eta}_*$ and $(\eta^*)^{-1}\circ\eta_*$, respectively; cf. Lemma 4.1 and 4.2. The two non-trivial homomorphisms to the right are of the form $f\to f_p$ where $f_p\eta=\eta f$.

The idea is now to apply the functor $E_p$ to the upper exact sequence and compare the result to the lower exact sequence. But first I make two observations concerning the lower sequence.

Lemma 4.1 shows that $\pi_1 \mathrm{aut}_\#(X_{Z/p})$ is a $p$-complete abelian group, in particular a $\boldsymbol{Z}_p^\wedge$-module ([2], VI. 4.3). Also $H^n(X_{Z/p}; E_pM)$ is a $\boldsymbol{Z}_p^\wedge$-module and in fact.

LEMMA 4.5. *$\partial\colon \pi_1 \mathrm{aut}_\#(X_{Z/p})\to H^n(X_{Z/p}; E_pM)$ is a $\boldsymbol{Z}_p^\wedge$-module homomorphism.*

*Proof.* Use Corollary 3.3.

LEMMA 4.6. *$\mathrm{Aut}_\#(Y_{Z/p})$ and $\mathrm{Aut}_\#(X_{Z/p})^{\theta_p}$ are nilpotent $p$-complete groups.*

*Proof.* Surely $\mathrm{Aut}_\#(Y_{Z/p})$ is nilpotent ([3], Theorem A) as $Y_{Z/p}$ is finitely anticonnected.

As explained on ([3], p. 189), the nilpotent group $\mathrm{Aut}_\#(X)$ acts nilpotently on the finitely generated abelian group $H^{n+1}(X, M)$. Since $H_p \mathrm{Aut}_\#(X)=0$ by induction hypothesis, Theorem 2.5 can be applied to show that $(E_p \mathrm{Aut}_\#(X))^{\eta*\theta}$ is $p$-complete nilpotent and to yield the first of the isomorphisms

$$E_p(\mathrm{Aut}_\#(X)^\theta)\cong(E_p \mathrm{Aut}_\#(X))^{\eta*\theta}\cong \mathrm{Aut}_\#(X_{Z/p})^{\theta_p}$$

while the second isomorphism is by the induction hypothesis. In particular, all three groups are nilpotent and $p$-complete.

Since $H^n(X_{Z/p}; E_pM)$ is a finitely generated $\boldsymbol{Z}_p^\wedge$-module and $\partial$ a $\boldsymbol{Z}_p^\wedge$-module homomorphism by Lemma 4.5, the cokernel of $\partial$ is also a finitely generated $\boldsymbol{Z}_p^\wedge$-module and thus $p$-complete by Proposition 2.4.

To conclude the proof, consider the short exact sequence of nilpotent groups

$$0\longrightarrow \operatorname{cok}\partial\longrightarrow \mathrm{Aut}_\#(Y_{Z/p})\longrightarrow \mathrm{Aut}_\#(X_{Z/p})^{\theta_p}\longrightarrow 1$$

and apply Corollary 2.3. b.

Next follow two remarks on the upper exact sequence.

LEMMA 4.7. *$H_p \mathrm{Aut}_\#(Y)=0$.*

*Proof.* Apply Corollary 2.3. a to the short exact sequence

$$0\longrightarrow \operatorname{cok}\partial\longrightarrow \mathrm{Aut}_\#(Y)\longrightarrow \mathrm{Aut}_\#(X)^\theta\longrightarrow 1$$

noting that $H_p(\operatorname{cok}\partial)=0$, since the cokernel of $\partial$ is finitely generated, and that $H_p(\mathrm{Aut}_\#(X)^\theta)=0$ by Theorem 2.5.

LEMMA 4.8. *The sequence*

$$E_p(\pi_1 \mathrm{aut}_\#(X))\xrightarrow{E_p\partial} E_pH^n(X; M)\longrightarrow E_p \mathrm{Aut}_\#(Y)\longrightarrow E_p(\mathrm{Aut}_\#(X)^\theta)\longrightarrow 1,$$

*obtained by applying the functor $E_p$ to the upper exact sequence, is exact.*

*Proof.* Note first that this is not entirely obvious since $E_p$ is not an exact functor.

Application of $E_p$ to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{im}\partial & \longrightarrow & H^n(X;M) & \longrightarrow & \operatorname{cok}\partial & \longrightarrow & 0 \\
 & & \uparrow & \nearrow & & \searrow & \downarrow & & \\
 & & \pi_1 \operatorname{aut}_\#(X) & & & & \operatorname{Aut}_\#(Y) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \operatorname{Aut}_\#(X)^\theta & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

with exact row and columns, results in a similar commutative diagram with $E_p$ in front of all occuring groups. The exactness is preserved since $H_p(\operatorname{Aut}_\#(X)^\theta)=0=H_p(\operatorname{cok}\partial)$. An easy diagram chase now finishes the proof.

Since all groups in the lower exact sequence are $p$-complete and nilpotent by Lemma 4.1, 4.2, and 4.6, the universal property will generate a map of exact sequences

$$\begin{array}{ccccccccc}
E_p(\pi_1 \operatorname{aut}_\#(X)) & \longrightarrow & E_p H^n(X;M) & \longrightarrow & E_p \operatorname{Aut}_\#(Y) & \longrightarrow & E_p(\operatorname{Aut}_\#(X)^\theta) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1 \operatorname{aut}_\#(X_{Z/p}) & \longrightarrow & H^n(X_{Z/p}; E_p M) & \longrightarrow & \operatorname{Aut}_\#(X_{Z/p}) & \longrightarrow & \operatorname{Aut}_\#(X_{Z/p})^{\theta_p} & \longrightarrow & 1
\end{array}$$

The upper sequence is exact by Lemma 4.8, the outer homomorphisms are isomorphisms by Lemma 4.1, 4.2, and Theorem 2.5. The Five Lemma now implies that the middle homomorphism

$$E_p \operatorname{Aut}_\#(Y) \longrightarrow \operatorname{Aut}_\#(Y_{Z/p})$$

is also an isomorphism. This finishes the inductive step and thus the proof of Theorem 4.3.

*Remark* 4.9. For a family of primes $P$, let $N_P$ denote the $P$-localization [5] of the nilpotent group or space $N$. The above method can also be used to prove that

$$\operatorname{Aut}_\#(X_P)=\operatorname{Aut}_\#(X)_P$$

under the same finiteness assumptions on $X$ as in Theorem 4.3. This is a slight extension of Maruyama's result ([7], Theorem 0.1).

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