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KODAI MATH. J.
11 (1988), 129—140

# ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF SEMIGROUPS OF LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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**Abstract**

Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$, $G$ a right reversible semitopological semigroup and $S=\{S(t): t\in G\}$ a continuous representation of $G$ as Lipschitzian self-mappings on $C$. We consider the asymptotic behavior of an almost-orbit $\{u(t): t\in G\}$ of $S=\{S(t): t\in G\}$. We show that if $E$ has a Fréchet differentiable norm and if $\limsup_t k_t \leqq 1$, then the closed convex set

$$\bigcap_{s\in G} \overline{co}\{u(t): t\geqq s\} \cap F(S)$$

consists of at most one point, where $k_t$ is the Lipschitzian constant of $S(t)$. This result is applied to study the problem of weak convergence of the net $\{u(t): t\in G\}$.

## 1. Introduction.

Let $C$ be a nonempty closed convex subset of a real Banach space $E$ and let $T$ be a mapping of $C$ into itself. $T$ is said to be a Lipschitzian mapping if for each $n\geqq 1$ there exists a positive real number $k_n$ such that

$$|T^n x - T^n y| \leqq k_n |x-y|$$

for all $x, y\in C$. A Lipschitzian mapping is said to be nonexpansive if $k_n=1$ for all $n\geqq 1$ and asymptotically nonexpansive if $\lim_n k_n=1$, respectively. Let $S=\{S(t): t\geqq 0\}$ be a family of nonexpansive mappings of $C$ into itself such that $S(0)=I$, $S(t+s)=S(t)S(s)$ for all $t, s\in[0, \infty)$ and $S(t)x$ is continuous in $t\in[0, \infty)$ for each $x\in C$. Then $S$ is said to be a nonexpansive semigroup on $C$. In [1], Bruck introduced the notion of an almost-orbit of a nonexpansive mapping. Miyadera and Kobayashi [11] extended the notion to the case of a nonexpansive semigroup; see also Takahashi and Park [14] for general commutative semigroups. Recently, the authors established the weak convergence of an almost-orbit of a noncommutative Lipschitzian semigroup in a Hilbert space [15]. In this paper, we shall extend the result in [15] to the case of Banach spaces.

Received November 19, 1987

Let $G$ be a right reversible semitopological semigroup and let $S=\{S(t):t\in G\}$ be a Lipschitzian representation of $G$ on $C$. We show that if $C$ is a nonempty closed convex subset of a uniformly convex Banach space $E$ and if $\limsup_t k_t\leqq 1$, where $k_t$ is the Lipschitzian constant of $S(t)$ $(t\in G)$, then the set $F(S)$ of all common fixed points of $S=\{S(t):t\in G\}$ is closed and convex. Moreover, if $E$ has a Fréchet differentiable norm and if $\{u(t):t\in G\}$ is an almost-orbit of $S=\{S(t):t\in G\}$, then the set

$$\bigcap_{s\in G}\overline{co}\{u(t):t\geqq s\}\cap F(S)$$

consists of at most one point, where $\overline{co}\{u(t):t\geqq s\}$ is the closed convex hull of $\{u(t):t\geqq s\}$. Using this result, we establish the weak convergence of an almost-orbit $\{u(t):t\in G\}$ of a right reversible Lipschitzian semigroup in a Banach space. We also show that if $P$ is the metric projection of $E$ onto $F(S)$, then the strong limit of $Pu(t)$ exists. These extend results in [10], [12], [14], [15]. Our proofs employ the methods of Hirano-Takahashi [7], Ishihara-Takahashi [9], Miyadera-Kobayashi [11], Takahashi [13] and Takahashi-Park [14].

## 2. Preliminaries.

Let $E$ be a real Banach space and let $E^*$ be its dual, that is, the space of all continuous linear functionals on $E$. The value of $f\in E^*$ at $x\in E$ will be denoted by $\langle x, f\rangle$. With each $x\in E$, we associate the set

$$J(x)=\{f\in E^*:\langle x, f\rangle=|x|^2=|f|^2\}.$$

Using the Hahn-Banach theorem, it is readily verified that $J(x)\neq\emptyset$ for any $x\in E$. The multi-valued map $J:E\to E^*$ is called the duality map of $E$. Let $U=\{x\in E:|x|=1\}$ be the unit sphere of $E$. Then a Banach space $E$ is said to be smooth provided the limit

$$\lim_{t\to 0}\frac{|x+th|-|x|}{t} \tag{1}$$

exists for each $x$, $h\in U$. In this case, the norm of $E$ is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each $x$ in $U$, limit (1) is attained uniformly for $h$ in $U$. The space $E$ is said to have a uniformly Gâteaux differentiable norm if for each $h\in U$, limit (1) is attained uniformly for $x\in U$. The norm of $E$ is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if limit (1) is attained uniformly for $(x, h)$ in $U\times U$. It is well known that if $E$ is smooth, then the duality map $J$ is single valued. It is also known that if $E$ has a Fréchet differentiable norm, $J$ is norm to norm continuous; see [2] and [4] for more details.

Let $G$ be a semitopological semigroup, i.e., $G$ is a semigroup with a Hausdorff topology such that for each $a\in G$ the mappings $g\to a\cdot g$ and $g\to g\cdot a$ from $G$ to $G$ are continuous. $G$ is said to be right reversible if any two closed left ideals of $G$ have nonvoid intersection. If $G$ is right reversible, $(G, \leqq)$ is a directed system when the binary relation "$\leqq$" on $G$ is defined by $a \leqq b$ if and only if $\{a\} \cup \overline{Ga} \supseteqq \{b\} \cup \overline{Gb}$.

### 3. Lemmas.

In this section, we prove several lemmas which are crucial in studying the asymptotic behavior of almost-orbits.

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $G$ be a semitopological semigroup.

DEFINITION 1. A family $S=\{S(t): t \in G\}$ of mappings from $C$ into itself is said to be a *(continuous) representation* of $G$ on $C$ if $S$ satisfies the following:

(1) $S(ts)x=S(t)S(s)x$ for all $t, s \in G$ and $x \in C$;

(2) For every $x \in C$, the mapping $s \rightarrow S(s)x$ from $G$ into $C$ is continuous.

DEFINITION 2. Let $S=\{S(t): t \in G\}$ be a representation of $G$ on $C$. $S$ is said to be *Lipschitzian* on $C$ if for each $t \in G$, there exists $k_t > 0$ such that $|S(t)x - S(t)y| \leqq k_t |x-y|$ for all $x, y \in C$.

See [5] and [8] for fixed point theorems of semigroups of Lipschitzian mappings. Denote by $F(S)$ the set of all common fixed points of mappings $S(t)$, $t \in G$ in $C$. Then we have the following:

THEOREM 1. *Let $C$ be a nonempty closed convex subset of a uniformly convex real Banach space $E$ and let $S=\{S(t): t \in G\}$ be a Lipschitzian representation of a right reversible semitopological semigroup $G$ on $C$. If $\limsup_t k_t \leqq 1$, then $F(S)$ is a closed and convex subset of $C$.*

*Proof.* The closedness of $F(S)$ is obvious. To show convexity it is sufficient to show that $z=(x+y/2) \in F(S)$ for all $x, y \in F(S)$. Let $x, y \in F(S)$, $x \neq y$. If $\lim_t S(t)z=z$, then for any $s \in G$,

$$S(s)z=\lim_t S(s)S(t)z=\lim_t S(st)z=\lim_t S(t)z=z,$$

i.e., $z \in F(S)$. Hence, it suffices to prove that $\lim_t S(t)z=z$. If not, there exists $\varepsilon>0$ such that for any $t \in G$, there is $t' \in G$ with $t' \geqq t$ and

$$4|S(t')z-z|=|2(S(t')z-x)-2(y-S(t')z)| \geqq \varepsilon .$$

Choose $d>0$ so small that

$$(R+d)\left(1-\delta\left(\frac{\varepsilon}{R+d}\right)\right)<R,$$

where $R=|x-y|>0$ and $\delta$ is the modulus of convexity of $E$. Since $\limsup_t k_t$

$\leqq 1$, it follows that there is $t_0 \in G$ such that $k_t|x-y| \leqq |x-y|+d$ for $t \geqq t_0$.

Put $u=2(S(t_0')z-x)$, $v=2(y-S(t_0')z)$. Then $|u-v|=4|S(t_0')z-z| \geqq \varepsilon$. Further, since $t_0' \geqq t_0$, we have

$$|u|=2|S(t_0')z-x| \leqq k_{t_0'}|x-y| \leqq |x-y|+d=R+d\,,$$
$$|v|=2|y-S(t_0')z| \leqq k_{t_0'}|x-y| \leqq |x-y|+d=R+d\,.$$

So, we have

$$\left|\frac{u+v}{2}\right| \leqq (R+d)\left(1-\delta\left(\frac{\varepsilon}{R+d}\right)\right),$$

and hence

$$|x-y|=\left|\frac{u+v}{2}\right| \leqq (R+d)\left(1-\delta\left(\frac{\varepsilon}{R+d}\right)\right)<R=|x-y|\,.$$

This is a contradiction. Therefore, $\lim_t S(t)z=z$. The proof is completed.

DEFINITION 3. Let $G$ be right reversible and let $S=\{S(t): t \in G\}$ be a representation of $G$ on $C$. A function $u: G \to C$ is called an *almost-orbit* of $S=\{S(t): t \in G\}$ if

$$\lim_t (\sup_s |u(st)-S(s)u(t)|)=0\,.$$

LEMMA 1. *Let $G$ be right reversible and let $S=\{S(t): t \in G\}$ be Lipschitzian on $C$ with $\limsup_t k_t \leqq 1$. If $\{u(t): t \in G\}$ and $\{v(t): t \in G\}$ are almost-orbits of $S=\{S(t): t \in G\}$, then the limit of $|u(t)-v(t)|$ exists. In particular, for every $z \in F(S)$, the limit of $|u(t)-z|$ exists.*

*Proof.* Put

$$\phi(s)=\sup_t |u(ts)-S(t)u(s)|\,, \qquad \psi(s)=\sup_t |v(ts)-S(t)v(s)|$$

for $s \in G$. Then $\lim_s \phi(s)=\lim_s \psi(s)=0$. Since, for any $s, t \in G$,

$$|u(ts)-v(ts)| \leqq |u(ts)-S(t)u(s)|+|S(t)u(s)-S(t)v(s)|+|S(t)v(s)-v(ts)|$$
$$\leqq \phi(s)+\psi(s)+k_t|u(s)-v(s)|\,,$$

we have

$$\inf_t \sup_{t \leqq \tau} |u(\tau)-v(\tau)| \leqq \phi(s)+\psi(s)+(\inf_t \sup_{t \leqq \tau} k_\tau)|u(s)-v(s)|$$
$$\leqq \phi(s)+\psi(s)+|u(s)-v(s)|\,,$$

and then

$$\inf_t \sup_{t \leqq \tau} |u(\tau)-v(\tau)| \leqq \sup_t \inf_{t \leqq s} |u(s)-v(s)|\,.$$

Thus, $\lim_t |u(t)-v(t)|$ exists. Let $z \in F(S)$ and put $v(t) \equiv z$. Then $v(t)$ is an almost-orbit and hence the limit of $|u(t)-z|$ exists.

LEMMA 2. *Let $G$ be right reversible and let $S=\{S(t):t\in G\}$ be Lipschitzian on $C$ with* $\limsup_t k_t\leqq 1$. *Let $\{u(t):t\in G\}$ be an almost-orbit of $S=\{S(t):t\in G\}$. If $F(S)\neq\emptyset$, then there exists $t_0\in G$ such that $\{u(t):t\geqq t_0\}$ is bounded.*

*Proof.* Let $z\in F(S)$. Then, since $\lim_t|u(t)-z|$ exists by Lemma 1, there is $t_0\in G$ such that $\{|u(t)-z|:t\geqq t_0\}$ is bounded. Hence $\{u(t):t\geqq t_0\}$ is bounded.

LEMMA 3. *Let $C$ be a nonempty closed convex subset of a uniformly convex real Banach space $E$. Let $G$ be right reversible and let $S=\{S(t):t\in G\}$ be Lipschitzian on $C$ with* $\limsup_t k_t\leqq 1$. *Let $\{u(t):t\in G\}$ be an almost-orbit of $S=\{S(t):t\in G\}$. Suppose $F(S)\neq\emptyset$. Let $y\in F(S)$ and $0<\alpha\leqq\beta<1$. Then for any $\varepsilon>0$, there is $t_0\in G$ such that*

$$|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)|<\varepsilon$$

*for all $t$, $s\geqq t_0$ and $\lambda\in[\alpha,\beta]$.*

*Proof.* By Lemma 1, $\lim_t|u(t)-y|$ exists. Let $r=\lim_t|u(t)-y|$. If $r=0$, then from $\limsup_t k_t\leqq 1$, there exists $t_0\in G$ such that

$$|u(t)-y|<\varepsilon \quad\text{and}\quad k_t\leqq 2$$

for all $t\geqq t_0$. Hence, for $s$, $t\geqq t_0$ and $0\leqq\lambda\leqq 1$,

$$\begin{aligned}
&|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)|\\
&\quad\leqq\lambda|S(t)(\lambda u(s)+(1-\lambda)y)-S(t)u(s)|+(1-\lambda)|S(t)(\lambda u(s)+(1-\lambda)y)-y|\\
&\quad\leqq\lambda k_t|\lambda u(s)+(1-\lambda)y-u(s)|+(1-\lambda)k_t|\lambda u(s)+(1-\lambda)y-y|\\
&\quad=2\lambda(1-\lambda)k_t|u(s)-y|<\varepsilon.
\end{aligned}$$

Now, let $r>0$. Then we can choose $d>0$ so small that

$$(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right)=r_0<r,$$

where $\delta$ is the modulus of convexity of $E$ and

$$c=\min\{2\lambda(1-\lambda):\ a\leqq\lambda\leqq\beta\}.$$

Let $a>0$ with $r_0+2a<r$. Then there is $t_0\in G$ such that

$$|u(s)-y|>r-a, \qquad \text{for}\quad s\geqq t_0,$$

$$|S(t)u(s)-u(ts)|<a, \qquad \text{for}\quad s\geqq t_0 \text{ and } t\in G,$$

$$k_t \leqq 2, \quad \text{for} \quad t \geqq t_0,$$

$$k_t |u(s)-y| \leqq r+d, \quad \text{for} \quad s, t \geqq t_0.$$

Suppose that

$$|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)| \geqq \varepsilon,$$

for some $s, t \geqq t_0$ and $\lambda \in [\alpha, \beta]$. Put $z=\lambda u(s)+(1-\lambda)y$, $u=(1-\lambda)(S(t)z-y)$ and $v=\lambda(S(t)u(s)-S(t)z)$. Then, we have

$$|u| \leqq (1-\lambda)k_t|z-y| = \lambda(1-\lambda)k_t|u(s)-y| \leqq \lambda(1-\lambda)(r+d),$$

$$|v| \leqq \lambda k_t |z-u(s)| = \lambda(1-\lambda)k_t|u(s)-y| \leqq \lambda(1-\lambda)(r+d).$$

We also have that

$$|u-v| = |S(t)z-(\lambda S(t)u(s)+(1-\lambda)y)| \geqq \varepsilon$$

and

$$\lambda u+(1-\lambda)v = \lambda(1-\lambda)(S(t)u(s)-y).$$

By lemma in [6], we have

$$\begin{aligned} \lambda(1-\lambda)|S(t)u(s)-y| &= |\lambda u+(1-\lambda)v| \\ &\leqq \lambda(1-\lambda)(r+d)\Big(1-2\lambda(1-\lambda)\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big) \\ &\leqq \lambda(1-\lambda)(r+d)\Big(1-c\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big) = \lambda(1-\lambda)r_0, \end{aligned}$$

and hence $|S(t)u(s)-y| \leqq r_0$. This implies that

$$\begin{aligned} |u(ts)-y| &\leqq |u(ts)-S(t)u(s)| + |S(t)u(s)-y| \\ &< a+r_0 < r-a. \end{aligned}$$

This contradicts the fact $|u(s)-y| > r-a$ for $s \geqq t_0$. The proof is completed.

For $x, y \in E$, we denote by $[x, y]$ the set $\{\lambda x+(1-\lambda)y : 0 \leqq \lambda \leqq 1\}$.

LEMMA 4 (Lau-Takahashi [10]). *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm and let $\{x_\alpha\}$ be a bounded net in $C$. Let $z \in \bigcap_\beta \overline{co}\{x_\alpha : \alpha \geqq \beta\}$, $y \in C$ and $\{y_\alpha\}$ a net of elements in $C$ with $y_\alpha \in [y, x_\alpha]$ and*

$$|y_\alpha - z| = \min\{|u-z| : u \in [y, x_\alpha]\}.$$

*If $y_\alpha \to y$, then $y=z$.*

By using Lemma 3 and Lemma 4, we prove the following:

LEMMA 5. *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $G$ be right reversible* *and let* $S=\{S(t):t\in G\}$ *be Lipschitzian on* $C$ *with* $\limsup_t k_t\leqq 1$. *Suppose* $F(S)\neq\emptyset$ *and let* $\{u(t):t\in G\}$ *be an almost-orbit of* $S=\{S(t):t\in G\}$. *If* $z\in\bigcap_s\overline{co}\{u(t):t\geqq s\}$ $\cap F(S)$ and $y\in F(S)$, *then for any positive number* $\varepsilon$, *there is* $s_0\in G$ *such that*

$$\langle u(t)-y,\ J(y-z)\rangle\leqq\varepsilon|y-z|$$

*for all* $t\geqq s_0$.

*Proof*. Since $F(S)\neq\emptyset$, we may assume that $\{u(t):t\in G\}$ is bounded. If $y=z$, then Lemma 5 is obvious. So, let $y\neq z$. For each $t\in G$, let $y_t$ be a unique element in $[y,\ u(t)]$ with

$$|y_t-z|=\min\{|u-z|:\ u\in[y,\ u(t)]\}.$$

Since $y\neq z$, by Lemma 4, $y_t$ does not converge to $y$. Thus, there is $c>0$ such that for any $t\in G$, there exists $t'\geqq t$ with $|y_{t'}-y|\geqq c$. Let

$$y_{t'}=a_{t'}u(t')+(1-a_{t'})y,\qquad 0\leqq a_{t'}\leqq 1.$$

Then there is $c_0>0$ such that $a_{t'}\geqq c_0$ all $t'$. In fact, since

$$c\leqq|y_{t'}-y|=a_{t'}|u(t')-y|\leqq a_{t'}\cdot\sup_t|u(t)-y|,$$

we may put $c_0=c/(\sup_t|u(t)-y|)$. Let $k=\lim_t|u(t)-y|$. Then $k>0$. Choose $r>0$ with $\varepsilon>r$ and $2r<k$, and take $a>0$ such that

$$(R+a)\Big(1-\delta\Big(\frac{c_0r}{R+a}\Big)\Big)<R,$$

where $\delta$ is the modulus of convexity of the norm and $R=|z-y|>0$. Fix $a'<a$. By Lemma 3, there exists $t_1\in G$ such that

$$|S(s)(c_0u(t)+(1-c_0)y)-(c_0S(s)u(t)+(1-c_0)y)|<a' \tag{2}$$

for all $s,\ t\geqq t_1$. Since $k=\lim_t|u(t)-y|>2r$ and $\{u(t):t\in G\}$ is an almost-orbit of $S=\{S(t):t\in G\}$. We can choose $t_2\in G$ so that

$$|u(t)-y|\geqq 2r,\qquad t\geqq t_2,$$
$$|u(st)-S(s)u(t)|<r,\qquad t\geqq t_2,\ s\in G.$$

Furthermore, since $\limsup_t k_t\leqq 1$ and $R+a'<R+a$, we can choose $t_3\in G$ such that $k_sR+a'\leqq R+a$ for all $s\geqq t_3$.

Now, let $t_0\in G$ with $t_0\geqq t_i$, $i_0=1,2,3$. Fix $t'\geqq t_0$. Then, since $a_{t'}\geqq c_0$, we have

$$c_0u(t')+(1-c_0)y\in[y,\ a_{t'}u(t')+(1-a_{t'})y]=[y,\ y_{t'}].$$

Hence

$$|c_0u(t')+(1-c_0)y-z|\leqq \max\{|z-y|,\ |z-y_{t'}|\}=|z-y|=R.$$

By (2), we obtain

$$|c_0S(s)u(t')+(1-c_0)y-z|\leqq|S(s)(c_0u(t')+(1-c_0)y)-z|+a'$$
$$\leqq k_s|c_0u(t')+(1-c_0)y-z|+a'\leqq k_sR+a'\leqq R+a$$

for $s\geqq t_0$. On the other hand, since $|y-z|=R<R+a$ and

$$|(c_0S(s)u(t')+(1-c_0)y-z)-(y-z)|=|c_0S(s)u(t')+(1-c_0)y-y|$$
$$=c_0|S(s)u(t')-y|\geqq c_0(|u(st')-y|-|u(st')-S(s)u(t')|)\geqq c_0r$$

for any $s\in G$, it follows that

$$\left|\frac{1}{2}(c_0(S(s)u(t')+(1-c_0)y-z)+\frac{1}{2}(y-z)\right|=\left|\frac{c_0}{2}S(s)u(t')+\left(1-\frac{c_0}{2}\right)y-z\right|$$
$$\leqq(R+a)\left(1-\delta\left(\frac{c_0r}{R+a}\right)\right)<R$$

for all $s\geqq t_0$. This implies that if $u_s=(c_0/2)S(s)u(t')+(1-(c_0/2))y$, then $|u_s+\alpha(y-u_s)-z|\geqq|y-z|$ for all $\alpha\geqq 1$. By Theorem 2.5 in [3], we have

$$\langle u_s+\alpha(y-u_s)-y,\ J(y-z)\rangle\geqq 0$$

and hence $\langle u_s-y,\ J(y-z)\rangle\leqq 0$ for all $s\geqq t_0$. Then

$$\langle S(s)u(t')-y,\ J(y-z)\rangle\leqq 0$$

for $s\geqq t_0$. Therefore, for $s\geqq t_0$,

$$\langle u(st')-y,\ J(y-z)\rangle\leqq|u(st')-S(s)u(t')|\,|y-z|$$
$$+\langle S(s)u(t')-y,\ J(y-z)\rangle<r|y-z|<\varepsilon|y-z|.$$

Hence, for $t\geqq t_0t'$, there holds

$$\langle u(t)-y,\ J(y-z)\rangle\leqq\varepsilon|y-z|.$$

This completes the proof.

## 4. Asymptotic Behavior.

In this section, we study the asymptotic behavior of an almost-orbit $\{u(t): t\in G\}$ of $S=\{S(t): t\in G\}$.

THEOREM 2. *Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm and let $C$ be a nonempty closed convex subset of $E$. Let $G$ be a right reversible semitopological semigroup and let $S=\{S(t): t\in G\}$ be a Lipschitzian* *representation of $G$ on $C$ with* $\limsup_t k_t \leqq 1$. *Suppose that* $\{u(t): t\in G\}$ *is an almost-orbit of* $S=\{S(t): t\in G\}$ *and* $F(S)\neq\emptyset$. *Then the set*

$$\bigcap_s \overline{co}\{u(t): t\geqq s\}\cap F(S)$$

*consists of at most one point.*

*Proof.* Let $y, z\in\bigcap_s \overline{co}\{u(t): t\geqq s\}\cap F(S)$. Then, by Theorem 1, $(y+z/2)\in F(S)$, it follows from Lemma 5 that for every $\varepsilon>0$, there is $t_0\in G$ such that

$$\left\langle u(tt_0)-\frac{y+z}{2}, J\left(\frac{y+z}{2}-z\right)\right\rangle\leqq\varepsilon\left|\frac{y+z}{2}-z\right|=\frac{\varepsilon}{2}|y-z|$$

for every $t\in G$. Since $y\in\overline{co}\{u(tt_0): t\in G\}$, we have

$$\left\langle y-\frac{y+z}{2}, J\left(\frac{y+z}{2}-z\right)\right\rangle\leqq\frac{\varepsilon}{2}|y-z|$$

and hence $\langle y-z, J(y-z)\rangle=|y-z|^2\leqq 2\varepsilon|y-z|$. Since $\varepsilon$ is arbitrary, we have $y=z$.

For a function $u: G\to C$, let $\omega(u)$ denote the set of all weak limit points of the net $\{u(t): t\in G\}$. If $\{u(t): t\in G\}$ is an almost-orbit of a Lipschitzian semigroup $S=\{S(t): t\in G\}$ and $F(S)\neq\emptyset$, then $\{u(t): t\geqq t_0\}$ is bounded for some $t_0\in G$ and hence $\omega(u)\neq\emptyset$. Using Theorem 2, we obtain the following results.

THEOREM 3. *Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm and let $C$ be a nonempty closed convex subset of $E$. Let $G$ be a right reversible semitopological semigroup and let $S=\{S(t): t\in G\}$ be a Lipschitzian representation of $G$ on $C$ with* $\limsup_t k_t\leqq 1$. *Suppose $F(S)\neq\emptyset$ and let $\{u(t): t\in G\}$ be an almost-orbit of $S=\{S(t): t\in G\}$. If $\omega(u)\subset F(S)$, then the net $\{u(t): t\in G\}$ converges weakly to some $z\in F(S)$.*

*Proof.* Let $z\in\omega(u)$. Then $z\in\bigcap_s \overline{co}\{u(t): t\geqq s\}$. By hypothesis, $\omega(u)\subset F(S)$ and hence $z\in\bigcap_s \overline{co}\{u(t): t\geqq s\}\cap F(S)$. It follows then from Theorem 2 that $\omega(u)=\{z\}$ and therefore $\{u(t): t\in G\}$ converges weakly to $z\in F(S)$.

The following theorem is a generalization of Takahashi and Park [14].

THEOREM 4. *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $G$ be a right reversible semitopological semigroup and let $S=\{S(t): t\in G\}$ be a Lipschitzian representation of $G$ on $C$ with* $\limsup_t k_t\leqq 1$. *Suppose $F(S)\neq\emptyset$ and let $\{u(t): t\in G\}$ be an almost-orbit of $S=\{S(t): t\in G\}$. Let $P$ denote the metric projection of $E$ onto $F(S)$. Then the strong limit of the net $\{Pu(t): t\in G\}$ exists and* $\lim_t Pu(t)=z_0$, *where $z_0$ is a unique element in $F(S)$ such that*

$$\lim_t |u(t)-z_0| = \min\{\lim_t |u(t)-z| : z\in F(S)\}.$$

*Proof.* Since $F(S)\neq\emptyset$, we know that $\{u(t): t\in G\}$ is bounded and $\lim_t |u(t)-z| = g(z)$ exists for each $z\in F(S)$. Let $R=\inf\{g(z): z\in F(S)\}$ and $M=\{u\in F(S): g(u)=R\}$. Then, since $g(z)$ is convex and continuous on $F(S)$ and $g(z)\to\infty$ as $|z|\to\infty$, $M$ is a nonempty closed convex bounded subset of $F(S)$. Fix $z_0\in M$ with $g(z_0)=R$. Since $P$ is the metric projection of $E$ onto $F(S)$, we have $|u(t)-Pu(t)|\leqq|u(t)-y|$ for all $t\in G$ and $y\in F(S)$, and hence

$$\inf_t \sup_{t\leqq s} |u(s)-Pu(s)| \leqq R.$$

Suppose that $\inf_t \sup_{t\leqq s} |u(s)-Pu(s)|<R$. Then we may choose $\varepsilon>0$ and $t_0\in G$ so that $|u(s)-Pu(s)|\leqq R-\varepsilon$ for all $s\geqq t_0$. Since

$$|u(ts)-Pu(s)|\leqq\phi(s)+k_t|u(s)-Pu(s)|$$

for all $s, t\in G$ and $\lim_s \phi(s)=0$, where $\phi(s)=\sup_t |u(ts)-S(t)u(s)|$, we can choose $s\geqq t_0$ such that

$$|u(ts)-Pu(s)|\leqq k_t|u(s)-Pu(s)|+\frac{\varepsilon}{2}\leqq k_t(R-\varepsilon)+\frac{\varepsilon}{2}$$

for all $t\in G$. Therefore, we obtain that

$$\lim_t |u(t)-Pu(s)| = \inf_t \sup_{t\leqq\tau} |u(\tau)-Pu(s)| \leqq (\limsup_t k_t)(R-\varepsilon)+\frac{\varepsilon}{2}$$

$$\leqq R-\varepsilon+\frac{\varepsilon}{2}=R-\frac{\varepsilon}{2}<R.$$

This is a contradiction. So we conclude that

$$\inf_t \sup_{t\leqq s} |u(s)-Pu(s)|=R.$$

Now, we claim that $\lim_t Pu(t)=z_0$. If not, then there exists $\varepsilon>0$ such that for any $t\in G$, $|Pu(t')-z_0|\geqq\varepsilon$ for some $t'\geqq t$. Choose $a>0$ so small that

$$(R+a)\Big(1-\delta\Big(\frac{\varepsilon}{R+a}\Big)\Big)=R_1<R,$$

where $\delta$ is the modulus of convexity of the norm of $E$. We have $|u(t')-Pu(t')|\leqq R+a$ and $|u(t')-z_0|\leqq R+a$ for large enough $t'$. Therefore we have

$$\Big|u(t')-\frac{Pu(t')+z_0}{2}\Big|\leqq(R+a)\Big(1-\delta\Big(\frac{\varepsilon}{R+a}\Big)\Big)=R_1.$$

Since $w_{t'}=(Pu(t')+z_0)/2\in F(S)$, as in the above,

$$|u(tt')-w_{t'}|\leqq k_t|u(t')-w_{t'}|+\phi(t')$$

for all $t \in G$. Since $\lim_s \phi(s)=0$, there is $t'$ such that

$$|u(tt')-w_{t'}| \leqq k_t |u(t')-w_{t'}| + \frac{R-R_1}{2} \leqq k_t R_1 + \frac{R-R_1}{2},$$

and hence

$$\lim_t |u(t)-w_{t'}| = \inf_t \sup_{t \leqq s} |u(s)-w_{t'}| \leqq (\limsup_t k_t) R_1 + \frac{R-R_1}{2}$$

$$\leqq R_1 + \frac{R-R_1}{2} = \frac{R+R_1}{2} < R.$$

This contradicts the fact $R=\inf\{g(z): z \in F(S)\}$. Therefore, we have $\lim_t Pu(t)=z_0$. Consequently, it follows that the element $z_0 \in F(S)$ with $g(z_0)=\min\{g(z): z \in F(S)\}$ is unique. The proof is completed.

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