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DEFORMATIONS OF SOME ALGEBRAIC SURFACES WITH $q=0$ AND $p_g=1$

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§ 1. Introduction.

Let M^a be an affine algebraic surface in C^3 defined by $h(w)=1+\sum_{i=1}^{3} w^{A_i}=0$ where A_1, A_2, A_3 are linearly independent non-negative integral vectors. Let Δ be the simplex in R^3 spun by $\vec{0}$, A_1, A_2 and A_3. In [6, 7], Oka showed that M^a has a canonical smooth compactification in a toric variety W of dimension three. Let $A_4, \cdots, A_l$ be the other integral points on Δ and let $h_t(w)=h(w)+\sum_{i=4}^{l} t_i w^{A_i}$. There exists a Zariski open set U^e of C^{l-3} such that the family of affine algebraic surfaces $M_t^a=\{h_t(w)=0\}$ $(t\in U^e)$ has a simultaneous smooth compactification M_t in W $(M_0=M)$. This deformation is called the embedded deformation of M ([7]). Let ν_t be the sheaf of the germs of the holomorphic section of the normal bundle of M_t in W and let Θ_t and Θ_W be the sheaves of the germ of holomorphic vector fields of M_t and W respectively. We have the canonical exact sequence:

$$0 \longrightarrow \Theta_t \longrightarrow \Theta_W | M_t \longrightarrow \nu_t \longrightarrow 0 . \tag{1.1}$$

This induces the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0(M_t, \Theta_t) \longrightarrow H^0(M_t, \Theta_W | M_t) \longrightarrow H^0(M_t, \nu_t) \\ &\xrightarrow{\delta} H^1(M_t, \Theta_t) \longrightarrow H^1(M_t, \Theta_W | M_t) \longrightarrow H^1(M_t, \nu_t) \\ &\longrightarrow \cdots\cdots\cdots . \end{aligned} \tag{1.2}$$

In [7], Oka has studied the infinitesimal displacement map

$$\xi^e : T_t U^e \longrightarrow H^0(M_t, \nu_t) , \tag{1.3}$$

and the Kodaira-Spencer map $\delta\circ\xi^e$ where δ is the canonical homomorphism

$$\delta : H^0(M_t, \nu_t) \longrightarrow H^1(M_t, \Theta_t) . \tag{1.4}$$

The dimension of Ker δ is at least 3. He gives an example (See § 7, [7]) where dim Ker $\delta=3$ and δ is surjective.

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The purpose of this note is to give an example of an algebraic surface M embedded in a toric variety W such that $\dim \operatorname{Ker} \delta=12$ and δ is not surjective (Theorem 2.11). M is locally defined by $h(w)=1+w_1^8+w_1^2w_3^4+w_1^3w_2^2=0$.

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§2. Deformation of surfaces M.

In this section, we study the algebraic surface M introduced in §1 and its deformation $\{M_t\}$ through the infinitesimal displacement. We use the same notation as in [6, 7].

Let M^a be the affine algebraic surface in $\boldsymbol{C}^3$ which is defined by

$$h(w)=1+w_1^8+w_1^2w_3^4+w_1^3w_2^2=0\,. \tag{2.1}$$

Let Δ be as in §1. Δ has 27 other integral points $A_4, \cdots, A_{30}$ and let $h(w, t)=h(w)+\sum_{j=4}^{30} t_j w^{A_j}$. M_t^a is defined by $h(w, t)=0$. For the compactification of M_t^a, we consider the homogeneous polynomial $f_{\Xi}(z, t)$ which is defined by

$$f_{\Xi}(z, t)=h_{\Xi}(z_1/z_0, z_2/z_0, z_3/z_0, t)\cdot z_0^8\,. \tag{2.2}$$

and let

$$f(z, t)=f_{\Xi}(z, t)+z_2^L+z_3^L\,, \tag{2.3}$$

for sufficiently large L. Let M_t be the compactification of M_t^a through the troidal embedding theory as in [7]. M_t has the following numerical invariants.

$$K^2=0,\quad e(M_t)=24,\quad \pi_1(M_t)\cong Z/2Z \quad\text{and}\quad p_g=1\,. \tag{2.4}$$

Here K is a canonical divisor, $e(M_t)$ is the topological Euler characteristic and p_g is the geometric genus. For the calculation, we use §9 of [5]. M_t is a minimal surface. Let us recall the compactification M_t of M_t^a. Let $V_t=f^{-1}(0)$ (t is fixed). The dual Newton diagram $\Gamma^*(f)$ contains five particular vertices $P_1={}^t(5, 3, 0, 2)$, $P_2={}^t(3, 5, 0, 2)$, $P_3={}^t(1, 1, 1, 2)$, $P_4={}^t(1, 1, 2, 1)$ and $P={}^t(1, 1, 1, 1)$. Let Σ^* be a simplicial unimodular subdivision of $\Gamma^*(f)$ and let $\hat{\pi}: X\to\boldsymbol{C}^3$ be the associated birational proper morphism and let $\tilde{V}$ be the proper transform of V. For each strictly positive vertex Q of Σ^* with $\dim\Delta(Q)\geqq 1$, there is a corresponding exceptional divisor $\hat{E}(Q)$ and $E(Q)$ of $\hat{\pi}: X\to\boldsymbol{C}^3$ and $\hat{\pi}: \tilde{V}\to V$ respectively. $\hat{E}(Q)$ is a toric variety. Then it is shown in [5, 6] that the exceptional divisor $E(P)$ is a smooth compactification of M_t^a which is a hypersurface in the toric variety $\hat{E}(P)$. We denote $\hat{E}(P)$ by W hereafter. Let $\mathcal{S}$ be the 3-simplexes of Σ^* which contains P as a vertex. Then $\mathcal{S}$ gives a canonical affine coordinate system of W. In our case, $|\mathcal{S}|$ is 24. For a vertex Q which is adjacent to P and $\dim\Delta(P)\cap\Delta(Q)\geqq 1$, there is a corresponding divisor $C(Q)$ of M_t. In our case, we have the divisor $C(T_{12})$ besides $C(P_i)$ ($i=1, \cdots, 4$) where $T_{12}={}^t(2, 2, 0, 1)$.

Take the following 3-simplex $\sigma=(P, R, P_2, P_3)$ in $\mathcal{S}$ where $R={}^t(3, 4, 1, 3)$. σ is fixed hereafter. The defining equation of M_t in $\boldsymbol{C}^3_\sigma$ is

$$h_\sigma(y, t)=y_1+y_1^9y_2^{16}+y_1^3y_3^4+1+\sum_{j=4}^{30} t_j y^{B_j}=0\,. \tag{2.5}$$

where the monomials y^{B_j} $(j=4, \cdots, 30)$ are embedded monomials. As l is 30, the dimension of the embedded deformation is 27. Then by Theorem (5.1) of [7], we have the next Lemma.

LEMMA 2.6.

$$\dim H^0(M_t, \nu_t)=30\,.$$

By the Riemann-Roch theorem, we have the Euler-Poincare characteristics $\chi(\Theta_t)$ is -20.

LEMMA 2.7.

$$H^0(M_t, \Theta_W|M_t)\cong \boldsymbol{C}^{12} \quad \textit{and} \quad H^0(M_t, \Theta_t)=0\,.$$

Proof. We follow the method of calculation in §7 of [7]. Take the 3-simplex $\tau=(P, P_1, S, P_4)$ in $\mathcal{S}$ where $S={}^t(4, 3, 2, 3)$. We denote $y_{\sigma i}$, $y_{\tau i}$ by y_i, u_i respectively. Then we have $y_1=u_1^{16}u_2^7u_3^{-2}$, $y_2=u_1^{-9}u_2^{-4}u_3$ and $y_3=u_1^{-12}u_2^{-5}u_3$. Let $v\in H^0(M_t, \Theta_W|M_t)$. By the GAGA-principle, v can be expressed in $\boldsymbol{C}^3_\sigma\cap M_t$ as $\sum_{j=1}^{3} v_j\dfrac{\tilde\partial}{\partial y_i}$ where v_j is a Laurent polynomial in $y_1, \cdots, y_3$ and $\dfrac{\tilde\partial}{\partial y_j}$ is equal to $y_j\dfrac{\partial}{\partial y_j}$ by definition. We may assume that v_j has a regular form on $C(P_1)$ and $C(P_4)$ simultaneously (For the definitions of divisors $C(P_i)$ and regular forms, see Lemma (7.6) of [7]). Assume that the monomial y^ν has a non-zero coefficient in v_i. As we have

$$y^\nu=u_1^{16\nu_1-9\nu_2-12\nu_3}u_2^{7\nu_1-4\nu_2-5\nu_3}u_3^{-2\nu_1+\nu_2+\nu_3}\,, \tag{2.8}$$

we must have $8\nu_2+8\nu_3+8\geqq 16\nu_1\geqq 9\nu_2+12\nu_3-1$. Combine this with $\nu_2\geqq-\delta_{i2}$, $\nu_3\geqq-\delta_{i3}$ where δ_{ij} is the Kronecker's symbol. The possible cases are $\dfrac{\tilde\partial}{\partial y_i}$, $y_1y_3\dfrac{\tilde\partial}{\partial y_i}$, $y_1y_2\dfrac{\tilde\partial}{\partial y_i}$, $y_1^2y_2y_3^2\dfrac{\tilde\partial}{\partial y_i}$, $y_1^2y_2^2y_3\dfrac{\tilde\partial}{\partial y_i}$, $y_1^2y_2^3\dfrac{\tilde\partial}{\partial y_i}$, $y_1^3y_2^4y_3\dfrac{\tilde\partial}{\partial y_i}$, $y_1^3y_2^5\dfrac{\tilde\partial}{\partial y_i}$, $y_1^4y_2^7\dfrac{\tilde\partial}{\partial y_i}$, $y_1^5y_2^9\dfrac{\tilde\partial}{\partial y_i}$ $(i=1, 2, 3)$, $y_2^{-1}\dfrac{\tilde\partial}{\partial y_2}$, $y_1y_2^{-1}y_3^2\dfrac{\tilde\partial}{\partial y_2}$, $y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_2y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1y_2^2y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1y_2^3y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1^2y_2^4y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1^2y_2^5y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1^3y_2^6y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1^4y_2^8y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$, $y_1^5y_1^{10}y_3\dfrac{\tilde\partial}{\partial y_3}$ and $y_1^6y_1^{12}y_3^{-1}\dfrac{\tilde\partial}{\partial y_3}$. After checking their linear combinations in detail, we have

(2.9) $$v=\alpha_1\frac{\tilde{\partial}}{\partial y_1}+\alpha_2 y_1 y_3\tilde{D}+\alpha_3 y_1 y_2\tilde{D}+\alpha_4 y_1^2 y_2^2 y_3+\tilde{D}+\alpha_5 y_1^2 y_2^3\tilde{D}$$
$$+\alpha_6 y_1^3 y_2^4 y_3\tilde{D}+\alpha_7 y_1^3 y_2^5\tilde{D}+\alpha_8 y_1^4 y_2^7\tilde{D}+\alpha_9\frac{\tilde{\partial}}{\partial y_2}+\alpha_{10}\frac{\tilde{\partial}}{\partial y_3}$$
$$+\alpha_{11} y_2 y_3^{-1}\frac{\tilde{\partial}}{\partial y_3}+\alpha_{12} y_1 y_2^3 y_3^{-1}\frac{\tilde{\partial}}{\partial y_3}, \qquad \text{in } \boldsymbol{C}_\sigma\cap M_t,$$

where $\alpha_i\in\boldsymbol{C}$ $(i=1, \cdots, 12)$ are arbitrary constants, and

$$\tilde{D}=2\frac{\tilde{\partial}}{\partial y_1}-\frac{\tilde{\partial}}{\partial y_2}-\frac{\tilde{\partial}}{\partial y_3}.$$

On the other hand, an easy calculation shows that v as in (2.9) is holomorphic on M_t. Thus, $H^0(M_t, \Theta_W|M_t)\cong\boldsymbol{C}^{12}$.

Now we consider $H^0(M_t, \Theta_t)$. Let v be as in (2.9). We can write v as $v=\sum_{i=1}^{12}\alpha_i X_i$. We show that the mapping $\theta: H^0(M_t, \Theta_W|M_t)\to H^0(M_t, \nu_t)$ is injective. Assume that $\theta(v)_\sigma=\sum_{i=1}^{12}\alpha_i X_i(h_\sigma)\equiv 0$ modulo $h_\sigma(y, t)$. We claim that all α_i $(i=1, \cdots, 12)$ vanish. We have,

$$\sum_{i=1}^{12}\alpha_i X_i(h_\sigma)=\sum_{i=1}^{12}\alpha_i X_i(h(y)+\sum_{j=4}^{30}t_j y^{A_j})$$
$$=\sum_{j=1}^{30}t_j\sum_{i=1}^{12}\alpha_i X_i(y^{A_j}),$$

where $t_1=t_2=t_3=1$. We can see that the support of $X_i(y^{A_j})$ is included in the support of h_σ. As the right hand side of the above equality has no constant term, this implies $\sum_{i=1}^{12}\alpha_i X_i(h_\sigma)\not\equiv 0$ modulo h_σ. This shows that the mapping θ is injective, completing the proof of Lemma (2.7).

LEMMA 2.10.
$$H^2(M_t, \Theta_W|M_t)=0.$$

Proof. By the Serre duality, we have isomorphism

$$H^2(M_t, \Theta_W|M_t)\cong H^0(M_t, \Omega_W^1(K))$$
$$\cong H^0(M_t, \Omega_W^1|M_t(6C(P_2)-C(P_3)+C(T_{12}))),$$

as we have $K=6C(P_2)-C(P_3)+C(T_{12})$ by an easy calculation. Ω_W^1 is the sheaf of the germs of 1-forms on W. Let $\omega=\sum_{i=1}^{3}Y_i\tilde{d}y_i$ be a rational 1-form and assume that the restriction of ω is in $H^0(M_t, \Omega_W^1|M_t(6C(P_2)-C(P_3)+C(T_{12}))$. Let y^ν be a monomial with non-zero coefficient in Y_i (i: fixed). Then by Lemma (7.4) of [7], we have $\nu_2\geqq -6+\delta_{i2}$, $\nu_3\geqq 1+\delta_{i3}$ and $8\nu_2+8\nu_3\geqq 16\nu_1\geqq 9\nu_2+12\nu_3$. This has the unique integral solution $\nu=(-2, -5, 1)$. Let $\tau'=(P, P_1, T_{12}, S')$ where $S'={}^t(5, 4, 1, 3)$. Using $K=2C(P_2)-2C(P_3)+C(P_4)$ on $\boldsymbol{C}_{\tau'}\cap M_t$, and assuming that the restriction of ω is in $H^0(M_t, \Omega^1_w(K))$, we have $\nu_2\geqq -2+\delta_{i2}$, $\nu_3\geqq 2+\delta_{i3}$, $16\nu_1-9\nu_2-12\nu_3\geqq 0$ and $4\nu_1-2\nu_2-3\nu_3\geqq 0$. The above integral solution does not satisfy these inequalities. Hence, we have $H^2(M_t, \Theta_W|M_t)=0$. This completes the proof of Lemma (2.10).

Now we are ready to show that

THEOREM 2.11. *The Kodaira-Spencer map*

$$\delta\circ\xi^e : T_t U^e \longrightarrow H^1(M_t, \Theta_t)\,,$$

is neither injective nor surjective.

Using Theorem (5.1) of [7], we get

COROLLARY 2.12. *The canonical homomorphism*

$$\delta : H^0(M_t, \nu_t) \longrightarrow H^1(M_t, \Theta_t)\,.$$

is neither injective nor surjective.

Proof of Theorem 2.11. We consider the exact sequence (1.2). Considering the section $\phi\in H^0(M_t, \nu_t)$ such that $\phi_\sigma=1$, we have that the normal bundle N_t is defined by the divisor $(\phi)=[16C(P_1)+4C(T_{12})]$. The notation ϕ_σ is the same as in §7, [7]. By the Riemann-Roch theorem, we have $\chi(\nu_t)=30$, $\chi(\Theta_t)=-20$ and $\chi(\Theta_W|M_t)=10$. Then we get $H^2(M_t, \nu_t)=0$, and using the Lemmas (2.6), (2.7) and (2.10), $H^1(M_t, \nu_t)=H^2(M_t, \Theta_t)=0$, $\dim H^1(M_t, \Theta_W|M_t)=2$ and $\dim H^1(M_t, \Theta_t)=20$. This completes the proof of Proposition (2.11).

Remark 2.13. Our toric variety W has many "symmetries", i.e. we have $\dim H^0(W, \Theta_W)=12$.

We give another example of an algebraic surface N in which the surjectivity of δ fails but $\dim\ker\delta=3$.

Example 2.14. Let N^a be the affine algebraic surface in $\boldsymbol{C}^3$ which is defined by

$$h(w)=1+w_2^5w_3^3+w_2^4w_3^4+w_1^4w_2^3w_3=0\,. \tag{2.15}$$

As the homogeneous polynomial $f_\Xi(z)$, we take

$$f_\Xi(z)=z_0^8+z_2^5z_3^3+z_2^4z_3^4+z_1^4z_2^3z_3\,. \tag{2.16}$$

N has the following invariants.

$$K^2=0,\quad e(N)=24 \quad\text{and}\quad \pi_1(N)\cong Z/2Z\,.$$

As $K \sim C(P_1)+C(P_3)$, N is minimal, and $p_g=1$. Then we have the following exact sequence:

$$0 \longrightarrow H^0(N_t, \Theta_W | N_t) \longrightarrow H^0(N_t, \nu_t) \xrightarrow{\delta} H^1(N_t, \Theta_t)$$
$$\longrightarrow H^1(N_t, \Theta_W | N_t) \longrightarrow 0,$$

and $H^0(N_t, \Theta_W | N_t) \cong \boldsymbol{C}^3$, $\dim H^0(N_t, \nu_t)=14$, $\dim H^1(N_t, \Theta_t)=20$ and $\dim H^1(N_t, \Theta_W | N_t)=9$. Hence we get that the Kodaira-Spencer map $\delta \circ \xi^e : T_t U^e \to H^1(N_t, \Theta_t)$ is not surjective, and δ is neither injective and surjective as the case of M.

Remark 2.17. Minimal surfaces M and N with $q=0$, $p_g=1$, Euler number $=24$, $K^2=0$ and non-trivial fundamental group are classified as the minimal properly elliptic surface, and by Theorem (7.1) of p. 201, [1], the deformation of such surfaces is also minimal.

Remark 2.18. $\{B_j\}$ in (2.5) are (1, 0, 1), (1, 1, 0), (2, 0, 2), (2, 1, 1), (2, 2, 0), (2, 2, 1), (2, 3, 0), (3, 1, 3), (3, 2, 2), (3, 3, 1), (3, 4, 0), (3, 4, 1), (3, 5, 0), (4, 3, 3), (4, 4, 2), (4, 5, 1), (4, 6, 0), (4, 7, 0), (5, 6, 2), (5, 7, 1), (5, 8, 0), (6, 8, 2), (6, 9, 1), (6, 10, 0), (7, 11, 1), (7, 12, 0) and (8, 14, 0).

REFERENCES

[1] W. BARTH, C. PETERS AND A. VAN DE VEN, Compact complex surface, Springer, Berlin-Heidelberg-New York-Tokyo, 1984.

[2] G. KEMPF, F. KNUDSEN, D. MUMFORD AND B. SAINT-DONAT, Toroidal Embeddings, Lecture Notes in Math., **339**, Springer (1973).

[3] K. KODAIRA, Complex Manifolds and Deformation of Complex Structures, Springer, Berlin-Heidelberg-New York, 1985.

[4] K. KODAIRA AND D.C. SPENCER, On deformations of complex structures I, II, Annals of Math., **67** (1958), 328-466.

[5] M. OKA, On the Resolution of Hypersurface Singularities, Advanced Studies in Pure Mathematics **8** (1986), 405-436.

[6] M. OKA, Examples of Algebraic Surfaces with $q=0$ and $p_g \leqq 1$ which are locally Hypersurfaces, preprint.

[7] M. OKA, On the deformation of a certain type of algebraic varieties, preprint.

[8] A.N. VARCHENKO, Zeta-Function of Monodromy and Newton's Diagram, Invent. Math., **37** (1976), 253-262.

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