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HYPERSURFACES WITH HARMONIC CURVATURE IN A SPACE OF CONSTANT CURVATURE

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1. Introduction and theorems

A Riemannian curvature tensor R is said to be harmonic if it satisfies

$$\nabla_i R_{jk} - \nabla_j R_{ik} = 0$$

where R_{ij} means the component of the Ricci tensor, i.e. $R_{jk} = R^i{}_{jik}$. If the Ricci tensor is parallel, the curvature is harmonic. However the converse is generally not true, [2]. Concerning this matter, we obtain some results in the case of hypersurfaces in a space of non-negative constant curvature. The purpose of this note is to prove the next theorems:

We denote the k-dimensional Euclidean space and the k-dimensional sphere of curvature c by E^k and $S^k(c)$ respectively.

THEOREM 1. *Let M^n be a connected hypersurface with harmonic curvature, isometrically immersed in E^{n+1} by an isometric immersion ϕ with constant mean curvature. We denote the second fundamental form by h.*

(i) *If M^n is complete and trace h^4 is constant on M^n, then $\phi(M^n)$ is of the form $S^p \times E^{n-p}$, $0 \leqq p \leqq n$.*

(ii) *If M^n is compact, then $\phi(M^n)$ is S^n.*

THEOREM 2. *Let M^n be a connected hypersurface with harmonic curvature, isometrically immersed in $S^{n+1}(c)$ by an isometric immersion ϕ with constant mean curvature. If M^n is complete and trace h^4 is constant on M^n, or if M^n is compact, then $\phi(M^n)$ is of the form $S^p(r) \times S^{n-p}(s)$, $0 \leqq p \leqq n$, where $r = \alpha^2 + c$, $s = \beta^2 + c$, and α and β satisfy $\alpha\beta + c = 0$ and $p\alpha + (n-p)\beta = \text{trace}\, h$.*

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2. The proof of theorems

We consider a hypersurface M^n with harmonic curvature, isometrically

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immersed in an $(n+1)$-dimensional Riemannian space $\tilde{M}^{n+1}(c)$ of constant curvature c by an isometric immersion $\phi: M^n \to \tilde{M}^{n+1}(c)$, and denote the induced metric tensor, the induced connection, the curvature tensor of M^n and the second fundamental form by g, ∇, R and h respectively. We assume that the mean curvature $\operatorname{tr} h = h_k{}^k$ is constant. Under these conditions, the following formulae hold:

$$R_{ijkl} = c(g_{ik}g_{jl} - g_{il}g_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk} \tag{1}$$

(the Gauss equation),

$$\nabla_i h_{jk} - \nabla_j h_{ik} = 0 \quad \text{(the Codazzi equation)}, \tag{2}$$

$$\nabla_i h_k{}^k = 0 \quad \text{(mean curvature constant)}, \tag{3}$$

$$\nabla_i R_{jk} - \nabla_j R_{ik} = 0 \quad \text{(harmonic curvature)}, \tag{4}$$

where the indices $i, j, k, \cdots$ run from 1 to n. The formula (4) implies that the scalar curvature is constant, i. e.

$$\nabla_i R_k{}^k = 0\,. \tag{5}$$

On the other hand, we get from (1)

$$R_{jk} = (n-1)cg_{jk} + h_l{}^l h_{jk} - h_j{}^l h_{lk}\,. \tag{6}$$

For simplification, we shall write $h^2{}_{ij}$, $h^3{}_{ij}$, $\cdots$ instead of $h_i{}^k h_{kj}$, $h^2{}_i{}^k h_{kj}$, $\cdots$. And using (3),

$$\nabla_i R_{jk} = h_l{}^l \nabla_i h_{jk} - \nabla_i h^2{}_{jk}\,. \tag{7}$$

Hence we know from (2) and (7) that

$$\nabla_i h^2{}_{jk} - \nabla_j h^2{}_{ik} = 0 \tag{8}$$

is equivalent to (4). It is easy to see

$$\nabla_i h^2{}_k{}^k = 0 \tag{9}$$

from (7), (3) and (5).

First we shall give two formulae about $\|\nabla h^2\|^2 = \nabla_i(h_j{}^l h_{lk}) \cdot \nabla^i(h^{jm} h_m{}^k)$, where $\nabla^i = g^{ik}\nabla_k$.

LEMMA 1.

$$\|\nabla h^2\|^2 = \frac{1}{2}\nabla_i\nabla^i(\operatorname{tr} h^4) - nc \operatorname{tr} h^4 - \operatorname{tr} h \operatorname{tr} h^5 + c(\operatorname{tr} h^2)^2 + (\operatorname{tr} h^3)^2 \tag{10}$$

Proof.

(11)
$$\|\nabla h^2\|^2=(\nabla_i h^2{}_{jk})(\nabla^i h^{2jk})$$
$$=\nabla_i(h^2{}_{jk}\nabla^i h^{2jk})-h^2{}_{jk}\nabla_i\nabla^i h^{2jk}$$
$$=\frac{1}{2}\nabla_i\nabla^i(\mathrm{tr}\ h^4)-h^2{}_{jk}\nabla_i\nabla^i h^{2jk}$$

holds. Using (1), (3), (8), (9) and the Ricci identity, we get

(12)
$$\nabla_i\nabla^i h^{2jk}=\nabla_i\nabla^j h^{2ik}$$
$$=\nabla^j\nabla_i h^{2ik}+R_i{}^{jil}h^2{}_l{}^k+R_i{}^{jkl}h^{2i}{}_l$$
$$=\{(n-1)cg^{jl}+h_i{}^ih^{jl}-h_i{}^lh^{ji}\}h^2{}_l{}^k$$
$$+\{c(\delta_i{}^kg^{jl}-\delta_i{}^lg^{jk})+h_i{}^kh^{jl}-h_i{}^lh^{jk}\}h^{2i}{}_l\,.$$

From (11) and (12), the formula (10) follows. □

LEMMA 2.

(13)
$$\|\nabla h^2\|^2=\frac{1}{3}\nabla_i\nabla^i(\mathrm{tr}\ h^4)+\frac{4}{3}[\mathrm{tr}\ h^4(\mathrm{tr}\ h^2-nc)+\mathrm{tr}\ h(c\ \mathrm{tr}\ h^3-\mathrm{tr}\ h^5)]\,.$$

Proof. We remark that

(14)
$$\nabla_i h^2{}_{jk}=2h_j{}^m\nabla_i h_{mk}$$

holds. In fact,

(15)
$$\nabla_i h^2{}_{jk}=(\nabla_i h_j{}^m)h_{mk}+h_j{}^m\nabla_i h_{mk}\,,$$

implies together with (2) and (8) that the second term of the right side of (15) is symmetric with respect to i, j and k, from which (14) follows. Hence

(16)
$$\|\nabla h^2\|^2=(\nabla_i h^2{}_{jk})(\nabla^i h^{2jk})=4h^{2l}{}_m(\nabla_i h_{lk})(\nabla^i h^{mk})\,.$$

On the other hand, we have

(17)
$$\|\nabla h^2\|^2=2h_j{}^l(\nabla_i h_{lk})(\nabla^i h^{2jk})$$
$$=2\nabla^i(h^{3kl}\nabla_i h_{lk})-2h^{2jk}(\nabla^i h_j{}^l)(\nabla_i h_{lk})-2h^{3kl}\nabla_i\nabla^i h_{lk}$$

by (14). The first and second terms of the right side of (17) are reduced to $\frac{1}{2}\nabla_i\nabla^i h^4{}_k{}^k$ and $-\frac{1}{2}\|\nabla h^2\|^2$ by (16) respectively. Using (1), (2), (3), (9) and the Ricci identity, we get

$$(18)\qquad \begin{aligned}\nabla^i\nabla_i h_{lk} &= \nabla^i\nabla_l h_{ik}\\ &= \nabla_l\nabla^i h_{ik} + R^i{}_{li}{}^m h_{mk} + R^i{}_{lk}{}^m h_{im}\\ &= \{(n-1)c\delta_l{}^m + h_i{}^i h_l{}^m - h^{im}h_{li}\}h_{mk}\\ &\quad + \{c(\delta^i{}_k\delta_l{}^m - g^{im}g_{lk}) + h^i{}_k h_l{}^m - h^{im}h_{lk}\}h_{im}\,.\end{aligned}$$

Therefore the third term of (17) can be reduced to $2(-nc\,\mathrm{tr}\,h^4 - \mathrm{tr}\,h\,\mathrm{tr}\,h^5 + c\,\mathrm{tr}\,h\,\mathrm{tr}\,h^3 + \mathrm{tr}\,h^2\,\mathrm{tr}\,h^4)$. Finally (17) becomes the formula (13). □

We eliminate the term of $\mathrm{tr}\,h\,\mathrm{tr}\,h^5$ from (10) and (13), and have

$$(19)\qquad \|\nabla h^2\|^2 = \nabla_i\nabla^i(\mathrm{tr}\,h^4) + 4[(\mathrm{tr}\,h^3)^2 - \mathrm{tr}\,h^2\,\mathrm{tr}\,h^4 + c(\mathrm{tr}\,h^2)^2 - c\,\mathrm{tr}\,h\,\mathrm{tr}\,h^3]\,.$$

Taking the suitable orthonormal frame, we diagonize h and denote its diagonal components by $\alpha_1, \cdots, \alpha_n$. Then the formula (19) can be rewritten to

$$(20)\qquad \|\nabla h^2\|^2 = \Delta(\mathrm{tr}\,h^4) - 2\sum_{i\neq j}\alpha_i\alpha_j(\alpha_i\alpha_j + c)(\alpha_i - \alpha_j)^2,$$

where Δ means $\nabla^i\nabla_i$.

In the case of $c=0$, if trace h^4 is constant on M^n, then all the nonzero eigenvalues of h have a constant unique value on M^n by (3) and (20). Therefore we can apply K. Nomizu and B. Smyth's argument if M^n is complete. Thus theorem 1(i) is proved.

If M^n is compact, then we obtain the same result by integrating (20) over M^n. By the compactness of M^n, theorem 1(ii) is concluded.

In order to consider the case of $c>0$, we recall the following formula appeared in [3]:

$$(21)\qquad \|\nabla h\|^2 = \frac{1}{2}\Delta(\mathrm{tr}\,h^2) - \frac{1}{2}\sum_{i\neq j}(\alpha_i - \alpha_j)^2(\alpha_i\alpha_j + c)\,.$$

This formula follows from (1), (2), (3) and the Ricci identity, and in our situation, the first term of the right side of (21) vanishes by (9). So we have

$$(22)\qquad \|\nabla h\|^2 = -\frac{1}{2}\sum_{i\neq j}(\alpha_i - \alpha_j)^2(\alpha_i\alpha_j + c)\,.$$

From (20) and (22), we have

$$\|\nabla h^2\|^2 + 4c\|\nabla h\|^2 = \Delta(\mathrm{tr}\,h^4) - 2\sum_{i\neq j}(\alpha_i\alpha_j + c)^2(\alpha_i - \alpha_j)^2,$$

and theorem 2 is proved as in the case of $c=0$.

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