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ESTIMATES FOR THE HYPERBOLIC METRIC

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Abstract. Bounds for the density of the hyperbolic metric of a hyperbolic region X in the complex plane $\boldsymbol{C}$ or on the Riemann sphere $\boldsymbol{P}$ are given in terms of the euclidean or spherical distance to the boundary of X. Also, bounds for the infimum of the density of the hyperbolic metric are given in terms of the supremum of the radii of all disks in X. These bounds are related to various Landau constants and are implicit in previous work on finding lower bounds for Landau constants.

1. Introduction. Let X denote a hyperbolic region in the complex plane $\boldsymbol{C}$; that is, $\boldsymbol{C}\backslash X$ contains at least two points. The hyperbolic, or Poincaré, metric on X is denoted by $\lambda_X(z)|dz|$. It is a complete Riemannian metric on X with constant curvature -4. Recall that

$$\lambda_D(z)|dz|=\frac{|dz|}{1-|z|^2},$$

where $\boldsymbol{D}=\{z:|z|<1\}$ is the unit disk. Typically, there is no explicit formula for the density $\lambda_X(z)$ of the hyperbolic metric, so estimates are useful. However, there are few results that deal explicity with the size of the hyperbolic metric. Let us survey some of these. Ahlfors ([1], [2]) gave analytic bounds in case X is the thrice punctured sphere. Often, one is interested in bounds for $\lambda_X(z)$ in terms of the geometric quantity $\delta_X(z)$ which is the enclidean distance from z to the boundary of X. The upper bound $\lambda_X(z)\leqq 1/\delta_X(z)$ is a direct consequence of Schwarz' lemma [6, p. 45]. If X is simply connected, then $\lambda_X(z)\geqq 1/4\delta_X(z)$ [6, p. 45]. This lower bound is equivalent to the Koebe one-quarter theorem. If X is convex, then the factor 4 in the lower bound can be replaced by 2 [9]. Blevins [4] obtained a sharp lower bound for simply connected regions that are bounded by a quasiconformal circle. Beardon and Pommerenke ([3], [12]) investigated bounds in terms of $\delta_X(z)$ and another geometric quantity. In particular, they determined a necessary and sufficient condition on a region X for the existence of a positive constant c such that $\lambda_X(z)\geqq c/\delta_X(z)$. The condition is that there exists a positive constant M such that the modulus of any annulus in X that separates $\partial X\cup\{\infty\}$ is at most M. Hence, it is necessary that ∂X have no isolated points.

We are interested in obtaining a lower bound for $\lambda_X(z)$ in terms of $\delta_X(z)$ that is valid even if the boundary of X has isolated points. The clue to the

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form of the bound is provided by examining the hyperbolic metric of a punctured disk. If $X=\{z:0<|z-a|<R\}$, then

$$\lambda_X(z)=\frac{1}{2|z-a|\log(R/|z-a|)}.$$

Then $\delta_X(z)=|z-a|$ for $0<|z-a|\leqq R/2$ and so

$$\lambda_X(z)=\frac{1}{2\delta_X(z)\log(R/\delta_X(z))}$$

for z in X near a. This example also shows explicitly that $\lambda_X(z)\delta_X(z)$ has no positive lower bound as z approaches a. It also suggests that for a general hyperbolic region we consider the possibility of finding a lower bound of the form

$$\lambda_X(z)\geqq\frac{1}{2\delta_X(z)\log(b/\delta_X(z))}, \tag{1}$$

where b is a positive constant. Such a bound is implicit in Ahlfors' method for determining a lower bound for the Landau constant [1]. This idea is also used in [8]. Since bounds for the hyperbolic metric are relatively scarce, it seems worthwhile to make explicit these bounds.

Let $\Delta(X)$ be the supremum of $\delta_X(z)$ as z ranges over X. We only consider lower bounds of the form (1) with $b\geqq\Delta(X)$; this insures that the lower bound is positive in X. Since the right-hand side of (1) is a decreasing function of b on the interval $(\Delta(X), \infty)$, we let $b(X)$ denote the infimum of all constants $b>\Delta(X)$ such that (1) holds for all $z\in X$ in order to obtain the best possible lower bound of the form (1) when b is replaced by $b(X)$. Set $b(X)=\infty$ if there is no lower bound of the form (1). We shall show that $e^{1/2}\Delta(X)\leqq b(X)\leqq e\Delta(X)$. Consequently, there is a lower bound of the form (1) for the density of the hyperbolic metric if and only if there is a uniform bound on the size of the disks that are contained in X. As an application of this result we show that $\lambda_X(z)$ has a positive lower bound if and only if $\Delta(X)$ is finite. More precisely, if $\Lambda(X)=\inf\{\lambda_X(z):z\in X\}$, then $1/2\leqq\Delta(X)\Lambda(X)\leqq 1$. The upper bound is sharp, but the lower bound 1/2 is not. The best possible lower bound is related to Landau's constant. For convex regions we show that $\pi/4$ is the sharp lower bound.

Finally, we consider analogs of the preceding results for regions on the Riemann sphere $\boldsymbol{P}$. In this context we consider the "spherical" density $(1+|z|^2)\lambda_X(z)$ which is invariant under rotations of $\boldsymbol{P}$ and seek estimates of this quantity in terms of the size of the largest spherical disk in X with center z.

2. Lower bound for plane regions. In this section we consider lower bounds for the density of the hyperbolic metric in a plane region.

THEOREM 1. *Let X be a hyperbolic region in the complex plane $\boldsymbol{C}$. Then* $e^{1/2}\Delta(X)\leqq b(X)\leqq e\Delta(X)$.

Proof. We first establish the lower bound for $b(X)$. Of course, there is no harm in assuming that $b(X)<\infty$. Then (1) holds for $b=b(X)$. Since the upper bound $\lambda_X(z)\leqq 1/\delta_X(z)$ holds in any hyperbolic region in $\boldsymbol{C}$, we obtain from (1)

$$\frac{1}{2\log(b(X)/\delta_X(z))}\leqq 1,$$

or

$$e^{1/2}\delta_X(z)\leqq b(X).$$

This yields $e^{1/2}\Delta(X)\leqq b(X)$.

Next, we demonstrate the upper bound for $b(X)$ under the assumption that $\Delta=\Delta(X)<\infty$. We begin by assuming that we actually have the strict inequality $\delta_X(z)<\Delta$ for all $z\in X$. Define

$$\rho(z)|dz|=\frac{|dz|}{2\delta_X(z)\log(e\Delta/\delta_X(z))}.$$

We will show that $\rho(z)|dz|$ is an ultrahyperbolic metric on X. The inequality $\rho(z)\leqq\lambda_X(z)$ will then follow from Ahlfors' generalization of Schwarz' lemma ([1], [2, p. 13]). Since $\delta_X(z)$ is a continuous function, it is clear that $\rho(z)|dz|$ is a positive continuous metric on X. To show that $\rho(z)|dz|$ is an ultrahyperbolic metric on X, we must exhibit a supporting metric at each point z_0 of X. This is a metric $\lambda_0(z)|dz|$ defined in a neighborhood of z_0 with constant curvature -4 such that $\lambda_0(z)\leqq\rho(z)$ for z near z_0 with equality at z_0. Given $z_0\in X$, select $a\in\partial X$ with $|z_0-a|=\delta_X(z_0)$. Then $\delta_X(z)\leqq|z-a|<\Delta$ for z near z_0 with equality at z_0. The hyperbolic metric for $\{z:0<|z-a|<e\Delta\}$ is

$$\lambda_0(z)|dz|=\frac{|dz|}{2|z-a|\log(e\Delta/|z-a|)}.$$

Because the function $h(t)=1/2t\log(e\Delta/t)$ is decreasing on $(0,\Delta]$ and increasing on $[\Delta, e\Delta)$, the inequality $\delta_X(z)\leqq|z-a|<\Delta$ for z near z_0 yields $\lambda_0(z)\leqq\rho(z)$ for z close to z_0 with equality at z_0. Since the hyperbolic metric $\lambda_0(z)|dz|$ has constant curvature -4, it is a supporting metric for $\rho(z)|dz|$ at z_0. Then Ahlfors' lemma yields (1) with $b=e\Delta$. Hence, $b(X)\leqq e\Delta$ in case $\delta_X(z)<\Delta$ for all $z\in X$. In the general case, just replace Δ by $\Delta_n=\Delta+(1/n)$, n any positive integer, and obtain $b(X)\leqq e\Delta_n$. Finally, let n tend to infinity to conclude that $b(X)\leqq e\Delta$ in the general case.

As an application of Theorem 1 we obtain estimates for $\Lambda(X)$, the infimum of the hyperbolic metric in X, in terms of the quantity $\Delta(X)$.

THEOREM 2. *Let X be a hyperbolic region in $\boldsymbol{C}$. Then*

$$\frac{1}{2\Delta(X)}\leqq\Lambda(X)\leqq\frac{1}{\Delta(X)}.$$

Proof. We begin by proving the upper bound. Since $\lambda_X(z)\leqq 1/\delta_X(z)$ for any hyperbolic plane region with equality at z if and only if X is a disk with center

z [6, p. 45], the upper bound is immediate. Also $\Lambda(X)=1/\Delta(X)$ for any disk.

Second, we establish the lower bound. If $\Delta=\Delta(X)=\infty$, then there is nothing to prove, so we may assume $\Delta<\infty$. Then from Theorem 1

$$\lambda_X(z)\geqq\frac{1}{2\delta_X(z)\log(e\Delta/\delta_X(z))}.$$

Since $h(t)=1/2t\log(e\Delta/t)$ has its minimum value $1/2\Delta$ on the interval $(0, e\Delta)$ at the point $t=\Delta$, we have $\lambda_X(z)>1/2\Delta$. Then $\Lambda(X)\geqq1/2\Delta$.

The best possible constant C such that $\Lambda(X)\geqq C/\Delta(X)$ for any hyperbolic region X in $\boldsymbol{C}$ is related to Landau's constant $\mathcal{L}$. We briefly recall the definition of $\mathcal{L}$. In our notation $\mathcal{L}=\inf\Delta(f(\boldsymbol{D}))$, where the infimum is taken over all holomorphic functions f defined in $\boldsymbol{D}$ and normalized by $f'(0)=1$. Assume that f is such a function, $X=f(\boldsymbol{D})$ and $\Delta(X)<\infty$. Then for $w\in X$

$$\frac{C}{\Delta(X)}\leqq\Lambda(X)\leqq\lambda_X(w).$$

The principle of hyperbolic metric [5, p. 336] yields

$$\lambda_X(f(z))|f'(z)|\leqq\lambda_D(z)=\frac{1}{1-|z|^2},$$

so that

$$\frac{C}{\Delta(X)}\leqq\lambda_X(f(0))\leqq1.$$

Consequently, $C\leqq\Delta(X)$ and so $C\leq\mathcal{L}$. From Theorem 2 we obtain the known lower bound $\mathcal{L}\geqq1/2$ that is due to Ahlfors [1]. This is not surprising since the lower bound in Theorem 1 is implicit in [1]. The best known upper bound for the Landau constant is [13]

$$\mathcal{L}\leqq\frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)}<.5433$$

and this bound is conjectured to be the actual value of the Landau constant. It seems plausible that $C=\mathcal{L}$. We now demonstrate that the analogous result is valid for convex regions.

THEOREM 3. *Let X be a convex hyperbolic region in $\boldsymbol{C}$. Then*

$$\frac{\pi}{4\Delta(X)}\leqq\Lambda(X)\leqq\frac{1}{\Delta(X)}$$

and both bounds are sharp.

Proof. In the proof of Theorem 2 we already noted that the upper bound is sharp for any disk. Now, we establish the lower bound and its sharpness. Let X be any convex hyperbolic region in $\boldsymbol{C}$ with $\Delta=\Delta(X)<\infty$. Then Minda [9] obtained the lower bound

$$\lambda_X(z) \geqq \frac{\pi}{4\Delta \sin\left(\frac{\pi \delta_X(z)}{2\Delta}\right)} \geqq \frac{\pi}{4\Delta} .$$

(Actually, in [9] the denominator in the lower bound has the factor 2 rather than 4. This is due to the fact that the hyperbolic metric was normalized to have curvature -1 rather than -4 in [9].) Thus, $\Lambda(X) \geqq \pi/4\Delta$. Finally, we demonstrate the sharpness. If $S=\{z: 0<\mathrm{Re}(z)<2M\}$, then

$$\lambda_S(z) = \frac{\pi}{4M \sin\left(\frac{\pi \,\mathrm{Re}(z)}{2M}\right)} \geqq \frac{\pi}{4M} = \frac{\pi}{4\Delta(S)}$$

with equality for $\mathrm{Re}(z)=M$. Thus, $\Lambda(S)=\pi/4\Delta(S)$. Actually, equality holds for any strip.

Recall that the Bloch-Landau constant for convex regions is $\pi/4$ ([9], [14]).

3. Lower bounds for spherical regions. For a plane region X the density of the hyperbolic metric can be viewed as the quotient of the hyperbolic metric $\lambda_X(z)|dz|$ and the euclidean metric $|dz|$. Note that $\lambda_X(z)$ is invariant under translations and rotations of $\boldsymbol{C}$. For a region on the Riemann sphere $\boldsymbol{P}$ we need to determine the proper analog of the density. Throughout the remainder of this section we assume that X denotes a hyperbolic region on $\boldsymbol{P}$. The spherical metric $|dz|/(1+|z|^2)$ is a Riemannian metric on $\boldsymbol{P}$ with constant curvature 4. This metric is invariant under the group of rotations of the sphere. Precisely, if either $T(z)=e^{i\theta}(z-a)/(1+\bar{a}z)$, $a \in \boldsymbol{C}$, or else $T(z)=e^{i\theta}/z$, where $\theta \in \boldsymbol{R}$, then

$$\frac{|T'(z)|}{1+|T(z)|^2} = \frac{1}{1+|z|^2} .$$

We define the spherical hyperbolic density of the hyperbolic metric to be the quotient of the hyperbolic metric and the spherical metric; in symbols,

$$\mu_X(z) = \frac{\lambda_X(z)|dz|}{\dfrac{|dz|}{1+|z|^2}} .$$

If T is a rotation of the sphere, then the conformal invariance of the hyperbolic metric [9] yields

$$\lambda_{T(X)}(T(z))|T'(z)| = \lambda_X(z) .$$

It follows that

$$\mu_{T(X)}(T(z)) = \mu_X(z) ,$$

so the spherical hyperbolic density is invariant under rotations of the sphere. Observe that if $0 \in X$, then $\mu_X(0)=\lambda_X(0)$. Set $M(X)=\inf\{\mu_X(z): z \in X\}$.

Next, we need a notion of distance on the sphere. Define

$$d(z, w)=\begin{cases}\left|\dfrac{z-w}{1+\bar{w}z}\right| & \text{if } z, w\in \boldsymbol{C}\,, \\ \dfrac{1}{|z|} & \text{if } z\in \boldsymbol{C},\ w=\infty\,.\end{cases}$$

The quantity $d(z, w)$ is invariant under all rotations T of $\boldsymbol{P}$; that is, $d(T(z), T(w))=d(z, w)$, but it is not a true distance function. However, it is related to the chordal distance χ and the spherical distance ψ by

$$\chi(z, w)=\frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}=\sin(\tan^{-1}(d(z, w)))\,,$$

$$\psi(z, w)=\tan^{-1}(d(z, w))\,.$$

Recall that $\psi(z, w)$ denotes half the angle at the center of the sphere subtended by the shorter arc of the great circle connecting z and w. Because of this connection with $\chi(z, w)$ and $\psi(z, w)$ we shall employ $d(z, w)$ as a measure of distance on $\boldsymbol{P}$. The advantage of this approach is a simplicity in the formulas and a clearer analogy with the results in the planar case. Of course, all results could be expressed in terms of χ or ψ instead of d by making use of their relationship. For $a\in \boldsymbol{P}$ and $r>0$ let $D(a, r)=\{z\in \boldsymbol{P}: d(a, z)<r\}$. This is a spherical disk with center a and radius r. The boundary of $D(a, r)$ is a euclidean circle when we view $D(a, r)$ on the Riemann sphere. For $r=1$ we obtain a hemisphere with center a. For $z\in X$ let $\varepsilon_X(z)$ denote the largest value of r such that $D(z, r)\subset X$. The geometric quantity $\varepsilon_X(z)$ is a measure of the spherical distance from z to ∂X. We are interested in estimating $\mu_X(z)$ in terms of $\varepsilon_X(z)$ and $M(X)$ by means of $E(X)=\sup\{\varepsilon_X(z): z\in X\}$.

THEOREM 4. *Let X be a hyperbolic region on the Riemann sphere $\boldsymbol{P}$. Then $\mu_X(z)\leqq 1/\varepsilon_X(z)$ for $z\in X$. If equality holds at a point z, then X is a spherical disk with center z.*

Proof. Because of the rotational invariance of the quantities $\mu_X(z)$ and $\varepsilon_X(z)$, there is no harm in assuming that $z=0$. In this case $\mu_X(0)=\lambda_X(0)$ and $\varepsilon_X(0)=\delta_X(0)$, so the conclusion $\mu_X(0)\leqq 1/\varepsilon_X(0)$ is equivalent to the known bound $\lambda_X(0)\leqq 1/\delta_X(0)$ [6, p. 45]. If equality holds in this latter inequality, then X is a disk centered at the origin. In the general case X would be a rotation of a disk centered at the origin; that is, X would be a spherical disk.

This theorem helps to show that the geometric quantity $\varepsilon_X(z)$ is a reasonable candidate for estimating $\mu_X(z)$. The following example will motivate the form of our lower bound for $\mu_X(z)$.

EXAMPLE 1. Let us calculate $\mu_X(z)$ for a punctured spherical disk. Let $X=\{z: 0<d(a, z)<R\}$. Let T be a rotation of $\boldsymbol{P}$ that sends a to the origin. Then $T(X)=\{z: 0<|z|<R\}$ and

$$\mu_X(z)=\mu_{T(X)}(T(z))=(1+|T(z)|^2)\lambda_{T(X)}(T(z))$$
$$=\frac{1+|T(z)|^2}{2|T(z)|\log(R/|T(z)|)}$$
$$=\frac{1+d^2(a,z)}{2d(a,z)\log(R/d(a,z))},$$

since $d(a, z)=d(T(a), T(z))=d(0, T(z))=|Tz|$. For $d(a,z)$ small we have $d(a,z)=\varepsilon_X(z)$ and so

$$\mu_X(z)=\frac{1+\varepsilon_X^2(z)}{2\varepsilon_X(z)\log(R/\varepsilon_X(z))}.$$

In view of the preceding example we seek a lower bound of the form

$$\mu_X(z)\geqq\frac{1+\varepsilon_X^2(z)}{2\varepsilon_X(z)\log(c/\varepsilon_X(z))}, \tag{2}$$

where $c\geqq E(X)$ is a positive constant. Let $c(X)$ be the smallest such constant. We wish to estimate $c(X)$ in terms of $E(X)$. Since $\boldsymbol{C}\setminus\{0\}$ is not hyperbolic and $E(\boldsymbol{C}\setminus\{0\})=1$ because $\boldsymbol{C}\setminus\{0\}$ contains a hemisphere but no larger spherical disk, it is plausible that a restriction $E(X)<1$ be imposed in order to obtain a bound of the form (2).

THEOREM 5. *Let X be a hyperbolic region on $\boldsymbol{P}$. Then $E(X)\exp((1+E^2(X))/2)\leqq c(X)$. If $E(X)<1$, then $c(X)\leqq E(X)\exp((1+E^2(X))/2(1-E^2(X)))$.*

Proof. We start by establishing the lower bound for $c(X)$ under the assumption that $c(X)<\infty$. Then (2) holds with $c=c(X)$. Because $\mu_X(z)\leqq 1/\varepsilon_X(z)$, we obtain from (2)

$$\frac{1+\varepsilon_X^2(z)}{2\log(c(X)/\varepsilon_X(z))}\leqq 1,$$

or

$$\varepsilon_X(z)\exp((1+\varepsilon_X^2(z))/2)\leqq c(X).$$

The lower bound follows immediately.

Now we derive the upper bound under the assumption that $E=E(X)<1$. Initially, we assume that $\varepsilon_X(z)<E$ for all $z\in X$. We shall show that

$$\rho(z)|dz|=\frac{1+\varepsilon_X^2(z)}{2\varepsilon_X(z)\log(A/\varepsilon_X(z))}\frac{|dz|}{1+|z|^2}$$

is an ultrahyperbolic metric on X, where $A=E\exp((1+E^2)/2(1-E^2))$. Fix $z_0\in X$; we will construct a supporting metric at z_0. Select $a\in\partial X$ with $\varepsilon_X(z_0)=d(a, z_0)$. Then $\varepsilon_X(z)\leqq d(a, z)<E$ for z near z_0 with equality at z_0. From Example 1 it follows that the hyperbolic metric on the punctured disk $\{z: 0<d(a, z)<A\}$ is

$$\lambda_0(z)|dz|=\frac{1+d^2(a,z)}{2d(a,z)\log(A/d(a,z))}\frac{|dz|}{1+|z|^2}.$$

The fact that the function $k(t)=(1+t^2)/2t\log(A/t)$ is decreasing on $(0, E]$ and increasing on $[E, A)$ together with the inequality $\varepsilon_X(z)\leqq d(a, z)<E$ for z near z_0 imply that $\lambda_0(z)|dz|\leqq\rho(z)|dz|$ for z near z_0 with equality at z_0. Thus $\lambda_0(z)|dz|$ is a supporting metric for $\rho(z)|dz|$ at z_0. Ahlfors' lemma yields $\lambda_X(z)|dz|\geqq\rho(z)|dz|$. If we divide both sides of this inequality by the spherical metric $|dz|/(1+|z|^2)$, then we obtain (2) with $c=A$. Hence, $c(X)\leqq A$ in case $\varepsilon_X(z)<E$ for all $z\in X$. In the general case, replace E by $E_n=E+(1/n)$, where the positive integer n is taken so large that $E_n<1$. Then obtain $c(X)\leqq A_n$. Let n tend to infinity to get $c(X)\leqq A$ in the general case.

THEOREM 6. *Let X be a hyperbolic region on $\boldsymbol{P}$. Then*

$$\frac{1}{E(X)}-E(X)\leqq M(X)\leqq\frac{1}{E(X)}.$$

Proof. The upper bound follows easily from Theorem 4. Equality holds for any spherical disk. The lower bound is nonpositive for $E(X)\geqq 1$, so we may assume that $E=E(X)<1$ in the course of establishing it. Then Theorem 5 gives

$$\mu_X(z)\geqq\frac{1+\varepsilon_X^2(z)}{2\varepsilon_X(z)\log(A/\varepsilon_X(z))},$$

where $A=E\exp((1+E^2)/2(1-E^2))$. Since $k(t)=(1+t^2)/2t\ \log(A/t)$ attains its minimum value $E^{-1}-E$ on the interval $(0, A)$ at the point $t=E$, we have $\mu_X(z)\geqq E^{-1}-E$. Hence, $M(X)\geqq E^{-1}-E$.

The lower bound for $M(X)$ in terms of $E(X)$ is not sharp. The best possible lower bound is related to Landau constants for meromorphic functions; for more information about these constants the reader is directed to [8].

EXAMPLE 2. For each positive integer n let X_n denote the complex plane punctured at both the origin and all the n^{th} roots of unity. For instance, X_1 is the Riemann sphere punctured at 0, 1 and ∞. We wish to determine a lower bound for $\lambda_n(z)$, the hyperbolic density on X_n. Theorem 1 is of no help in this situation because X_n contains arbitrarily large euclidean disks. We shall make use of Theorem 6. Set $E_n=E(X_n)$. Triviallv, $E_1=1$. Elementary geometric considerations show that for $n\geqq 2$ $E_n=(\sqrt{3+\cos(2\pi/n)}-\sqrt{1+\cos(2\pi/n)})/\sqrt{2}$. In particular, $E_2=1$, $E_3=(\sqrt{5}-1)/\sqrt{2}$ and $E_4=(\sqrt{3}-1)/\sqrt{2}$. We see that $E_n<1$ only for $n\geqq 3$ so we can apply Theorem 6 to obtain a meaningful lower bound in these cases. For $n\geqq 3$ we obtain

$$\lambda_n(z)\geqq\left(\frac{1}{E_n}-E_n\right)\Big/(1+|z|^2).$$

As special cases we have

$$\lambda_3(z)\geqq\frac{1}{1+|z|^2},$$

$$\lambda_4(z) \geqq \frac{\sqrt{2}}{1+|z|^2}.$$

For $n \geqq 3$ we have shown that the hyperbolic metric on X_n dominates a constant multiple of the spherical metric.

It is possible to obtain similar lower bounds for $\lambda_1(z)$ and $\lambda_2(z)$ by making use of the following device. The function $p(z)=z^4$ is a covering of X_4 onto X_1 so the invariance of the hyperbolic metric under a covering [7] yields

$$\lambda_4(z)=\lambda_1(p(z))|p'(z)|=\lambda_1(z^4)4|z|^3.$$

If we set $w=z^4$, then

$$\begin{aligned}\lambda_1(w)&=\lambda_4(z)/4|z|^3\\ &\geqq \frac{1}{2\sqrt{2}}\frac{1}{1+|w|^2}\frac{1+|z|^8}{|z|^3+|z|^5}\\ &\geqq \frac{1}{2\sqrt{2}}\frac{1}{1+|w|^2}.\end{aligned}$$

This lower bound is implicit in the work of Pommerenke [11]. By making use of the covering $q(z)=z^2$ of X_2 onto X_1 one can demonstrate in a similar manner that

$$\lambda_2(z) \geqq \frac{1}{\sqrt{2}}\frac{1}{1+|z|^2}.$$

Thus, in all cases $\lambda_n(z)$ dominates an explicit scalar multiple of $1/(1+|z|^2)$.

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