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HELICAL IMMERSIONS AND NORMAL SECTIONS

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1. Introduction.

Let $f: M^n \to \bar{M}^{n+p}$ be an isometric immersion of a connected n-dimensional Riemannian manifold M into a Riemannian manifold $\bar{M}$ of dimension $n+p$. If $\gamma: I=[0, 1] \to M$ is a curve on M then $\sigma = f \circ \gamma$ is a curve on $\bar{M}$. Let σ be parametrized by its arc length, $\sigma^{(1)} = \dot{\sigma}$ be the unit tangent vector and $K_1 = \|\tilde{\nabla}_{\dot{\sigma}} \sigma^{(1)}\|$. $\tilde{\nabla}$ denotes the covariant differentiation of $\bar{M}$. If K_1 vanishes on $[0, 1]$ then σ is called of order 1. If K_1 is not identically zero, then we define $\sigma^{(2)}$ by $\tilde{\nabla}_{\dot{\sigma}} \sigma^{(1)} = K_1 \sigma^{(2)}$ on the set $I_1 = \{s \in [0, 1] : K_1(s) \neq 0\}$. Let $K_2 = \|\tilde{\nabla}_{\dot{\sigma}} \sigma^{(2)} + K_1 \sigma^{(1)}\|$. If $K_2 \equiv 0$ on I_1 then σ is called of order 2. If K_2 is not identically zero on I_1 then we define $\sigma^{(3)}$ by $\tilde{\nabla}_{\dot{\sigma}} \sigma^{(2)} = -K_1 \sigma^{(1)} + K_2 \sigma^{(3)}$. Inductively we put $K_d = \|\tilde{\nabla}_{\dot{\sigma}} \sigma^{(d)} + K_{d-1} \sigma^{(d-1)}\|$. If $K_d \equiv 0$ on I_{d-1} then σ is called of order d. It follows that if the curve σ is of order d we have the Frenet formula ([9]):

$$\tilde{\nabla}_{\dot{\sigma}}(\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(d)}) = (\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(d)}) K \tag{1.1}$$

where

$$K = \begin{bmatrix} 0 & -K_1 & 0 & \cdots & \cdots & 0 \\ K_1 & 0 & -K_2 & & & 0 \\ 0 & K_2 & 0 & \ddots & & \\ & 0 & \ddots & \ddots & & -K_{d-1} \\ & & & \ddots & K_{d-1} & 0 \end{bmatrix}$$

$K_1, K_2, \cdots, K_{d-1}$ are called the Frenet curvatures of σ. If, for each geodesic γ on M, the curve $f \circ \gamma$ on $\bar{M}$ has constant Frenet curvatures of order d, and they are independent of γ, then f is called a helical immersion of order d. In most cases the ambient space is considered as a Riemannian manifold of constant sectional curvature c, denoted by $\bar{M}^{n+p}(c)$. Sakamoto [9] and Nakagawa [8] have investigated helical immersions. The concept "helical immersion" originates from Besse [2]; it is important in the theory of harmonic manifolds.

Another important concept used in this paper called normal sections, originated from Chen [3]. In [3], [4], [7], submanifolds in E^m with (pointwise) planar normal sections were investigated. Chen and Verheyen [5] proved that

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a helical submanifold in E^m has geodesic normal sections. Verheyen [10] proved its inverse.

For a submanifold M^n immersed in a space form $\overline{M}^{n+p}(c)$, we can also define normal sections. For a point x in M and a unit vector $t\in T_xM$, the vector t and the normal space N_xM determine a $(p+1)$-dimensional subspace $E(x, t)$ of $T_x\overline{M}$, which determines a $(p+1)$-dimensional totally geodesic submanifold M_0. The intersection of M and M_0 gives rise a curve $\gamma(s)$ (in a neighborhood of x), called the normal section of M at x in the direction t.

For any two vector fields X, Y tangent to M, the second fundamental form h is given by $h(X, Y)=\tilde{\nabla}_XY-\nabla_XY$ where ∇ is the covariant differentiation in M. For any vector field ξ normal to M, put $\tilde{\nabla}_X\xi=-A_\xi X+\nabla^\perp_X\xi$, where $-A_\xi X$ and $\nabla^\perp_X\xi$ denote the tangential and normal components of $\tilde{\nabla}_X\xi$, respectively.

The covariant differentiation D on the Whitney sum $T(M)\oplus N(M)$ is defined as follows (see [8]): For any $N(M)$-valued tensor field T of type $(1, k)$, C^∞-vector fields $X, X_1, X_2, \cdots, X_k$ tangent to M, put

$$
(1.2)\qquad \begin{aligned} DT(X, X_1, X_2, \cdots, X_k)&=(D_XT)(X_1, \cdots, X_k)\\ &=\nabla^\perp_X(T(X_1, \cdots, X_k))-\sum_{r=1}^{k} T(X_1, \cdots, \nabla_XX_r, \cdots, X_k). \end{aligned}
$$

We have the Ricci identity:

$$
(1.3)\qquad \begin{aligned} &(D^2T)(X, Y, X_1, \cdots, X_k)-(D^2T)(Y, X, X_1, \cdots, X_k)\\ &=R^\perp(X, Y)T(X_1, \cdots, X_k)-\sum_{r=1}^{k} T(X_1, \cdots, R(X, Y)X_r, \cdots, X_k) \end{aligned}
$$

where $R^\perp(X, Y)=\nabla^\perp_X\nabla^\perp_Y-\nabla^\perp_Y\nabla^\perp_X-\nabla^\perp_{[X,Y]}$ is the normal curvature tensor, R is the curvature tensor of M.

The following identity is well known ([2]):

$$
(1.4)\qquad \langle R^\perp(X, Y)\xi, \eta\rangle=\langle[A_\xi, A_\eta]X, Y\rangle.
$$

The following algebraic Lemma is a main tool in this paper.

LEMMA 1.1. *Let T_1, T_2 be tensors of (q, p)-type on a vector space V. Suppose for all $v\in V$*

$$
(1.5)\qquad T_1(v^p)=T_1(v, v, \cdots, v)=T_2(v^p),
$$

then for $v_1, \cdots, v_p\in V$,

$$
(1.6)\qquad \sum_{\sigma\in S_p} T_1(v_{\sigma(1)}, \cdots, v_{\sigma(p)})=\sum_{\sigma\in S_p} T_2(v_{\sigma(1)}, \cdots, v_{\sigma(p)}),
$$

where S_p is the symmetric group on p letters.

Proof. Let $\lambda_1, \cdots, \lambda_p$ be real parameters. Take $v=\sum_{i=1}^p\lambda_iv_i$ in (1.5). We have

$$\sum \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_p}T_1(v_{i_1}, v_{i_2}, \cdots, v_{i_p})=\sum \lambda_{i_1}\cdots\lambda_{i_p}T_2(v_{i_1}, v_{i_2}, \cdots, v_{i_p}).$$

Comparing the coefficients of $\lambda_1\cdot\lambda_2\cdots\lambda_p$ on both sides we have (1.6).

In §2 we discuss the relation between helical immersion and normal section. In §3 we consider helical immersed surfaces.

2. Helical immersions and normal sections.

The following theorem is a generalization of a theorem of Sakamoto ([9]).

THEOREM 2.1. *Let f be an isometric immersion $M^n \to \overline{M}^{n+p}(c)$. For all geodesics γ on M suppose $\sigma=f\circ\gamma$ have constant curvatures $K_1, K_2, \cdots K_j$ ($j\leqq d-1$, d the order of σ), then we have the Frenet frames:*

$(F_1)\qquad \sigma^{(1)}=X,$

$(F_i)\qquad \sigma^{(i)}=(K_1\cdots K_{i-1})^{-1}\sum_{l=0}^{[i/2]-1}a_{i,i-2l}(D^{i-2l-2}h)(X^{i-2l}), \qquad 2\leqq i\leqq j+1$

where $X=\dot{\gamma}$, $a_{i,i}=1$, $a_{i,i-2l}=\sum_{(i_1,\cdots,i_l)\in A_i}K_{i_1}^2K_{i_2}^2\cdots K_{i_l}^2$ for $l>0$, where A_i is the collection of subsets of $\{2, 3, \cdots, i-2\}$, any two numbers in such subsets have difference at least 2.

Also, for $2\leqq k$, $l\leqq 2j+1$, $0\leqq r\leqq k-1$, $k+l\leqq 2j+3$, $X, Y\in U_xM$, the unit tangent sphere at x,

$$\langle (D^{k-2}h)(X^r, Y, X^{k-r-1}), (D^{l-2}h)(X^l)\rangle = \begin{cases} (-1)^{(k-l)/2}\nu_{(k+l)/2}\langle X, Y\rangle, & k+l=even, \\ 0 & k+l=odd. \end{cases} \tag{2.1}$$

Where $\nu_i=\|(D^{i-2}h)(X^i)\|^2$ only depend on $K_1, \cdots, K_{i-1}$, and for $k\leqq 2j+1$

$$A_{(D^{k-2}h)(X^k)}X=\begin{cases} (-1)^{k/2-1}\nu_{k/2+1}X, & if\quad k=even, \\ 0\qquad\qquad , & if\quad k=odd. \end{cases} \tag{2.2}$$

Proof. For $j=1$, K_1=constant implies that $\|h(X, X)\|=K_1$ is a constant. So

$$\langle h(X, X), h(X, Y)\rangle = K_1^2\langle X, Y\rangle$$

and $\sigma^{(1)}=X$, $\sigma^{(2)}=K_1^{-1}h(X, X)$. This proves (F_1), (F_2). Also

$$\langle (Dh)(X^2, Y), h(X^2)\rangle = 1/2\, Y\langle h(\tilde{X}^2), h(\tilde{X}^2)\rangle=0,$$

$$\langle (Dh)(X^3), h(X, Y)\rangle = -\langle h(X^2), (Dh)(X^2, Y)\rangle = 0$$

where $\tilde{X}$, $\tilde{Y}$ denote the vector fields adapted to X, Y, i.e. $\nabla_X\tilde{Y}$, $\nabla_Y\tilde{X}$, $\nabla_Y\tilde{Y}$ are 0.

Suppose the theorem is true for $j-1$. Assume that $K_1, \cdots, K_j$ are constant.

By inductive hypothesis we have (F_1), (F_2), $\cdots$, (F_j), also (2.1) for $k+l\leqq 2j+1$, $2\leqq k$, $l\leqq 2j-1$, (2.2) for $k\leqq 2j-1$. Then

$$K_j\sigma^{(j+1)}=\tilde{\nabla}_X\sigma^{(j)}+K_{j-1}\sigma^{(j-1)}$$

$$=(K_1\cdots K_{j-1})^{-1}\sum_{l=0}^{[j/2]-1}a_{j,j-2l}[-A_{(D^{j-2l-2}h)(X^{j-2l})}X+(D^{j-2l-1}h)(X^{j-2l+1})]$$

$$+K_{j-1}(K_1\cdots K_{j-2})^{-1}\sum_{l=0}^{[j-1/2]-1}a_{j-1,j-1-2l}(D^{j-3-2l}h)(X^{j-1-2l}).$$

Since $\sigma^{(j+1)}$ is orthogonal to X and $A_{(D^{j-2l-2}h)(X^{j-2l})}X\wedge X=0$, we have

$$K_j\sigma^{(j+1)}=(K_1\cdots K_{j-1})^{-1}\sum_{l=0}^{[j+1/2]-1}a_{j+1,j+1-2l}(D^{j-2l-1}h)(X^{j-2l+1})$$

where $a_{j+1,j+1}=a_{j,j}=1$, and for $l>0$ and $j-2l>1$

$$\text{(2.3)}\qquad a_{j+1,j+1-2l}=K_{j-1}^2a_{j-1,j+1-2l}+a_{j,j-2l}$$

$$=K_{j-1}^2\sum_{(i_1,\cdots,i_{l-1})\in A_{j-1}}K_{i_1}^2\cdots K_{i_{l-1}}^2+\sum_{(i_1,\cdots,i_l)\in A_j}K_{i_1}^2\cdots K_{i_l}^2$$

$$=\sum_{(i_1,\cdots,i_l)\in A_{j+1}}K_{i_1}^2\cdots K_{i_l}^2.$$

If j is odd and $j-2l=1$ we have $a_{j+1,2}=K_{j-1}^2a_{j-1,2}=K_2^2K_4^2\cdots K_{j-1}^2$. This proves (F_{j+1}). By $\langle\sigma^{(j+1)},\sigma^{(j+1)}\rangle=1$ and (F_{j+1}) we have

$$\left\langle\sum_{l=0}^{[j+1/2]-1}a_{j+1,j+1-2l}(D^{j-1-2l}h)(X^{j+1-2l}),\sum_{l=0}^{[j+1/2]-1}a_{j+1,j+1-2l}(D^{j-1-2l}h)(X^{j+1-2l})\right\rangle$$

$$=K_1^2K_2^2\cdots K_j^2.$$

But $\langle(D^{k-2}h)(X^k),(D^{i-2}h)(X^i)\rangle$ are constants, depending on $K_1,\cdots,K_{j-1}$ for $k+i\leqq 2j+1$, hence $\nu_{j+1}=\langle(D^{j-1}h)(X^{j+1}),(D^{j-1}h)(X^{j+1})\rangle$ is a constant, depending on $K_1,\cdots,K_j$.

For every l, $2\leqq l\leqq j+1$

$$0=X\langle(D^{2j-1-l}h)(\tilde{X}^{2j-l+1}),(D^{l-2}h)(\tilde{X}^l)\rangle$$

$$=\langle(D^{2j-l}h)(X^{2j-l+2}),(D^{l-2}h)(X^l)\rangle+\langle(D^{2j-1-l}h)(X^{2j-l+1}),(D^{l-1}h)(X^{l+1})\rangle.$$

So we have

$$\text{(2.4)}\qquad \langle(D^{2j-l}h)(X^{2j-l+2}),(D^{l-2}h)(X^l)\rangle=(-1)^{j-l+1}\nu_{j+1},\ 2\leqq l\leqq j+1.$$

Again,

$$\langle(D^jh)(X^{j+2}),(D^{j-1}h)(X^{j+1})\rangle=\frac{1}{2}X\langle(D^{j-1}h)(\tilde{X}^{j+1}),(D^{j-1}h)(\tilde{X}^{j+1})\rangle=0.$$

But

$$0=X\langle(D^{2j-l}h)(\tilde{X}^{2j-l+2}),(D^{l-2}h)(\tilde{X}^l)\rangle$$

$$=\langle(D^{2j-l+1}h)(X^{2j-l+3}),(D^{l-2}h)(X^l)\rangle+\langle(D^{2j-l}h)(X^{2j-l+2}),(D^{l-1}h)(X^{l+1})\rangle.$$

Therefore

(2.5) $$\langle (D^{2j-l}h)(X^{2j-l+2}), (D^{l-1}h)(X^{l+1})\rangle=0, \qquad 1\leqq l\leqq j.$$

To prove (2.1) is true for $k+l=2j+2$, $2j+3$, $2\leqq k$, $l\leqq 2j+1$, by (2.4) and (2.5) we need only to consider the case $\langle X, Y\rangle=0$.

Differentiating

$$\langle (D^{2j-1-l}h)(\tilde{X}^r, \tilde{Y}, \tilde{X}^{2j-l-r}), (D^{l-2}h)(\tilde{X}^l)\rangle=0, \qquad 2\leqq l\leqq 2j-1,\ 0\leqq r\leqq 2j-l$$

along the directions of X and Y respectively we have

$$\begin{aligned}&\langle (D^{2j-2}h)(X^r, Y, X^{2j-r-1}), h(X^2)\rangle\\ &=-\langle (D^{2j-3}h)(X^{r-1}, Y, X^{2j-1-r}), (Dh)(X^3)\rangle=\cdots\\ &=(-1)^r\langle (D^{2j-2-r}h)(Y, X^{2j-1-r}), (D^rh)(X^{r+2})\rangle\\ &=(-1)^{r+1}\langle (D^{2j-3-r}h)(X^{2j-1-r}), (D^{r+1}h)(Y, X^{r+2})\rangle=\cdots\\ &=\langle h(X^2), (D^{2j-2}h)(X^{2j-3-r}, Y, X^{r+2})\rangle.\end{aligned}$$

By Ricci identities for any $4\leqq k\leqq 2j+1$, $2\leqq l\leqq 2j-1$,

$$\begin{aligned}&(D^{k-2}h)(Y, X^{k-1})-(D^{k-2}h)(X, Y, X^{k-2})\\ &=-R^{\perp}(X, Y)(D^{k-4}h)(X^{k-2})+\sum_{s=0}^{k-3}(D^{k-4}h)(X^s, R(X, Y)X, X^{k-3-s}).\end{aligned}$$

Since $\langle R(X, Y)X, X\rangle=0$,

$$\langle (D^{k-4}h)(X^s, R(X, Y)X, X^{k-3-s}), (D^{l-2}h)(X^l)\rangle=0.$$

By (2.2)

$$\begin{aligned}&\langle R^{\perp}(X, Y)(D^{k-4}h)(X^{k-2}), (D^{l-2}h)(X^l)\rangle\\ &=\langle [A_{(D^{k-4}h)(X^{k-2})}, A_{(D^{l-2}h)(X^l)}]X, Y\rangle=0.\end{aligned}$$

Hence

(2.6) $$\langle (D^{k-2}h)(Y, X^{k-1}), (D^{l-2}h)(X^l)\rangle=\langle (D^{k-2}h)(X, Y, X^{k-2}), (D^{l-2}h)(X^l)\rangle$$

and then

$$\begin{aligned}&\langle (D^{2j-2-r}h)(Y, X^{2j-1-r}), (D^rh)(X^{r+2})\rangle\\ &=\langle (D^{2j-2-r}h)(X, Y, X^{2j-2-r}), (D^rh)(X^{r+2})\rangle\\ &=-\langle (D^{2j-3-r}h)(Y, X^{2j-2-r}), (D^{r+1}h)(X^{r+3})\rangle\\ &=-\langle (D^{2j-3-r}h)(X, Y, X^{2j-3-r}), (D^{r+1}h)(X^{r+3})\rangle=\cdots\\ &=(-1)^{2j-2-r}\langle h(X, Y), (D^{2j-2}h)(X^{2j})\rangle.\end{aligned}$$

Thus we have

$$\langle (D^{2j-2}h)(X^r, Y, X^{2j-1-r}), h(X^2)\rangle=\langle (D^{2j-2}h)(X^{2j}), h(X, Y)\rangle.$$

On the other hand if we write the identity $\langle (D^{2j-2}h)(X^{2j}),\ h(X^2)\rangle=(-1)^{j+1}\nu_{j+1}$ into the form

$$\langle (D^{2j-2}h)(X^{2j}),\ h(X^2)\rangle=(-1)^{j+1}\nu_{j+1}(\langle X,\ X\rangle)^{j+1},$$

by Lemma 1.1 we have

$$\sum_{\gamma=0}^{2j-1}\langle (D^{2j-2}h)(X^{\gamma},\ Y,\ X^{2j-\gamma-1}),\ h(X^2)\rangle+2\langle (D^{2j-2}h)(X^{2j}),\ h(X,\ Y)\rangle=0.$$

Hence we have

$$\langle (D^{2j-2}h)(X^{\gamma},\ Y,\ X^{2j-\gamma-1}),\ h(X^2)\rangle=\langle (D^{2j-2}h)(X^{2j}),\ h(X,\ Y)\rangle=0. \tag{2.7}$$

This shows that for $0\leqq s\leqq 2j-2,\ 0\leqq\gamma\leqq 2j-1-s$

$$\langle (D^{2j-2-s}h)(X^{\gamma},\ Y,\ X^{2j-1-s-\gamma}),\ (D^sh)(X^{s+2})\rangle=0. \tag{2.8}$$

Now

$$\begin{aligned}
&\langle (D^{2j-1}h)(X^{\gamma},\ Y,\ X^{2j-\gamma}),\ h(X^2)\rangle\\
&=X\langle (D^{2j-2}h)(\tilde{X}^{\gamma-1},\ \tilde{Y},\ \tilde{X}^{2j-\gamma}),\ h(\tilde{X}^2)\rangle\\
&\quad-\langle (D^{2j-2}h)(X^{\gamma-1},\ Y,\ X^{2j-\gamma}),\ (Dh)(X^3)\rangle\\
&=-\langle (D^{2j-2}h)(X^{\gamma-1},\ Y,\ X^{2j-\gamma}),\ (Dh)(X^3)\rangle=\cdots\\
&=(-1)^{\gamma}\langle (D^{2j-\gamma-1}h)(Y,\ X^{2j-\gamma}),\ (D^{\gamma}h)(X^{\gamma+2})\rangle\\
&=(-1)^{\gamma}\langle (D^{2j-\gamma-1}h)(X,\ Y,\ X^{2j-\gamma-1}),\ (D^{\gamma}h)(X^{\gamma+2})\rangle\\
&=(-1)^{\gamma+1}\langle (D^{2j-\gamma-2}h)(Y,\ X^{2j-\gamma-1}),\ (D^{\gamma+1}h)(X^{\gamma+3})\rangle=\cdots\\
&=-\langle h(X,\ Y),\ (D^{2j-1}h)(X^{2j+1})\rangle.
\end{aligned}$$

By (2.5) and Lemma 1.1 we have

$$\sum_{\gamma=0}^{2j}\langle (D^{2j-1}h)(X^{\gamma},\ Y,\ X^{2j-\gamma}),\ h(X^2)\rangle+2\langle (D^{2j-1}h)(X^{2j+1}),\ h(X,\ Y)\rangle=0.$$

Since $j>1$,

$$\langle (D^{2j-1}h)(X^{\gamma},\ Y,\ X^{2j-\gamma}),\ h(X^2)\rangle=\langle (D^{2j-1}h)(X^{2j+1}),\ h(X,\ Y)\rangle=0$$

and for $0\leqq s\leqq 2j-2,\ 0\leqq\gamma\leqq 2j-1-s$,

$$\langle (D^{2j-1-s}h)(X^{\gamma},\ Y,\ X^{2j-s-\gamma}),\ (D^sh)(X^{s+2})\rangle=0. \tag{2.9}$$

This proves (2.1) for $k+l=2j+2,\ 2j+3$. (2.2) is a consequence of (2.1).

Remark. In proving (F_{j+1}), (2.4) and (2.8) we only need the assumption that K_j is a function of the point x, not depending on the direction X.

COROLLARY 2.2. *If for every geodesic γ the Frenet curvatures $K_1,\ \cdots,\ K_j$ of*

$\sigma=f\circ\gamma$ are constants, then $\sigma^{(2)}, \cdots, \sigma^{(j+1)}\in N_xM$. Especially if $K_1, \cdots, K_{d-1}$ are constants then f is an immersion with geodesic normal sections.

Proof. The first conclusion follows from theorem 2.1. For the second conclusion assume $K_1, \cdots, K_{d-1}$ are constants then $\sigma^{(2)}, \cdots, \sigma^{(d)}\in N_xM$. By the theory of ordinary differential equations we know that the geodesic γ is contained in the totally geodesic submanifold M_0, whose tangent space at x is spanned by $\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(d)}$, which is contained in $E(x, X)$. This means σ is a normal section of M at x in the direction X.

The second assertion, i. e. a helical submanifold has geodesic normal sections was proved by Chen and Verheyen ([5]). The inverse of theorem 2.1 is also true.

THEOREM 2.3. *Let $f: M^n\to\overline{M}^{n+p}(c)$ be an isometric immersion, $j\geqq 1$. If at each point $x\in M$, for every unit vector $X\in U_x(M)$, $A_{(D^{k-2}h)(X^k)}X\wedge X=0$ for $2\leqq k\leqq 2j+1$, then the Frenet curvatures $K_1, K_2, \cdots, K_j$ are constants, and so* (2.1), (2.2) *hold.*

Proof. If $A_{h(X^2)}X=\mu_1X$ holds for some μ_1 (μ_1 may depend on X), then for $Y\in U_xM$, $\langle h(X^2), h(X, Y)\rangle=\mu_1\langle X, Y\rangle$. This implies that $K_1^2=\|h(X^2)\|^2=\mu_1$ is constant on U_xM. By the assumption $A_{(Dh)(X^3)}X=\mu_2X$ for some μ_2. So for $Y\in U_x(M)$,

$$\langle (Dh)(X^3), h(X, Y)\rangle=\mu_2\langle X, Y\rangle$$

$$\begin{aligned}\langle (Dh)(X^3), h(X, Y)\rangle&=X\langle h(\tilde{X}^2), h(\tilde{X}, \tilde{Y})\rangle-\langle h(X^2), (Dh)(X^2, Y)\rangle\\&=(X\mu_1)\langle X, Y\rangle-1/2\,Y\langle h(\tilde{X}^2), h(\tilde{X}^2)\rangle=(X\mu_1)\langle X, Y\rangle-1/2\,Y\mu_1\,.\end{aligned}$$

Since Y is arbitrary and X can be chosen such that $\langle X, Y\rangle=0$ we see that $Y\mu_1=0$, μ_1 is a constant on M. By Lemma 1.1, we also have

$$\langle (Dh)(X^2, Y), h(X^2)\rangle=0\,.$$

It is easy to see $\mu_2=0$. This proves the theorem in case $j=1$.

Suppose the theorem is true for $j-1$. Assume that $A_{(D^{k-2}h)(X^k)}X=\mu_{k-1}X$ for $2\leqq k\leqq 2j+1$, $X\in U_xM$. By inductive hypothesis, $K_1, \cdots, K_{j-1}$ are constants and so $\mu_1, \mu_2, \cdots, \mu_{2j-2}$ are constants. By differentiating the identity (when $Y\in U_xM$):

$$\langle (D^{2j-3}h)(\tilde{X}^{2j-1}), h(\tilde{X}, \tilde{Y})\rangle=0$$

along the direction of X we have

$$\langle (D^{2j-3}h)(X^{2j-1}), (Dh)(X^2, Y)\rangle=-\mu_{2j-1}\langle X, Y\rangle\,. \tag{2.10}$$

Suppose we have proved that for $0\leqq\gamma\leqq i-1$, $2\leqq i\leqq k$, $k\leqq 2j-2$

$$\langle (D^{i-2}h)(X^\gamma, Y, X^{i-1-\gamma}), (D^{2j-i}h)(X^{2j-i+2})\rangle=(-1)^i\mu_{2j-1}\langle X, Y\rangle\,. \tag{2.11}$$

Then by differentiating

$$\langle (D^{k-2}h)(\tilde{X}^{\gamma}, \tilde{Y}, \tilde{X}^{k-1-\gamma}), (D^{2j-1-k}h)(\tilde{X}^{2j-k+1})\rangle = 0$$

along the direction of X we have

$$\begin{aligned}&\langle (D^{k-1}h)(X^{\gamma+1}, Y, X^{k-1-\gamma}), (D^{2j-1-k}h)(X^{2j-k+1})\rangle \\ &\quad + \langle (D^{k-2}h)(X^{\gamma}, Y, X^{k-\gamma-1}), (D^{2j-k}h)(X^{2j-k+2})\rangle = 0\end{aligned}$$

and thus

$$\langle (D^{k-1}h)(X^{\gamma+1}, Y, X^{k-\gamma-1}), (D^{2j-1-k}h)(X^{2j-k+1})\rangle = (-1)^{k+1}\mu_{2j-1}\langle X, Y\rangle$$

for $0 \leqq \gamma \leqq k-1$. Besides by Ricci identity

$$\begin{aligned}&(D^{k-1}h)(X, Y, X^{k-1}) - (D^{k-1}h)(Y, X^{k}) \\ &= R^{\perp}(X, Y)(D^{k-3}h)(X^{k-1}) - \sum_{s=0}^{k-2}(D^{k-3}h)(X^{s}, R(X, Y)X, X^{k-s-2})\end{aligned}$$

Using the same argument as before we see

$$\begin{aligned}&\langle (D^{k-1}h)(X, Y, X^{k-1}), (D^{2j-k-1}h)(X^{2j-k+1})\rangle \\ &= \langle (D^{k-1}h)(Y, X^{k}), (D^{2j-k-1}h)(X^{2j-k+1})\rangle .\end{aligned}$$

This completes the induction of (2.11). Especially for $Y=X$ and $k=j$ we have

$$\nu_{j+1} = \|(D^{j-1}h)(X^{j+1})\|^2 = (-1)^{j+1}\mu_{2j-1}. \tag{2.12}$$

Now let $X, Y \in U_xM$ we can choose $Z \in U_xM$ such that $\langle X, Z\rangle = 0$ and for some $\alpha \in [0, 2\pi]$, $Y = X\cos\alpha + Z\sin\alpha$. For $t \in [0, 2\pi]$ let $Y_t = X\cos t + Z\sin t$ then

$$\begin{aligned}&\frac{d}{dt}\langle (D^{j-1}h)(Y_t^{j+1}), (D^{j-1}h)(Y_t^{j+1})\rangle \\ &= 2\sum_{\gamma=0}^{j}\langle (D^{j-1}h)(Y_t^{\gamma}, -X\sin t + Z\cos t, Y_t^{j-\gamma}), (D^{j-1}h)(Y_t^{j+1})\rangle \\ &= 2(-1)^{j+1}\mu_{2j-1}(j+1)\langle Y_t, -X\sin t + Z\cos t\rangle = 0\end{aligned}$$

Hence $\|(D^{j-1}h)(Y_t^{j+1})\|^2$ is constant for $t \in [0, 2\pi]$, so

$$\|(D^{j-1}h)(X^{j+1})\| = \|(D^{j-1}h)(Y^{j+1})\| .$$

This proves μ_{2j-1} and ν_{j+1} are constant on U_xM.

Now for any $X, Y \in U_xM$ with $\langle X, Y\rangle = 0$ we have

$$\langle (D^{2j-1}h)(X^{2j+1}), h(X, Y)\rangle = \mu_{2j}\langle X, Y\rangle = 0$$

and

$$\begin{aligned}&\langle (D^{2j-2}h)(X^{2j}),\ (Dh)(X^2,\ Y)\rangle\\&=X\langle (D^{2j-2}h)(\tilde{X}^{2j}),\ h(\tilde{X},\ \tilde{Y})\rangle-\langle (D^{2j-1}h)(X^{2j+1}),\ h(X,\ Y)\rangle\\&=(X\mu_{2j-1})\langle X,\ Y\rangle=0\,.\end{aligned}$$

Suppose we have proved that for $1\leqq q\leqq 2j-2$, $0\leqq\gamma\leqq q$, $0\leqq s\leqq\gamma+1$

$$\langle (D^{2j-1-\gamma}h)(X^{2j-\gamma+1}),\ (D^{\gamma}h)(X^s,\ Y,\ X^{\gamma+1-s})\rangle=0\,, \tag{2.13}$$

then for $0\leqq s\leqq q+1$,

$$\begin{aligned}0&=\langle (D^{2j-1-q}h)(X^{2j-q+1}),\ (D^{q}h)(X^s,\ Y,\ X^{q+1-s})\rangle\\&=X\langle (D^{2j-2-q}h)(\tilde{X}^{2j-q}),\ (D^{q}h)(\tilde{X}^s,\ \tilde{Y},\ \tilde{X}^{q+1-s})\rangle\\&\quad-\langle (D^{2j-2-q}h)(X^{2j-q}),\ (D^{q+1}h)(X^{s+1},\ Y,\ X^{q+1-s})\rangle\\&=-\langle (D^{2j-2-q}h)(X^{2j-q}),\ (D^{q+1}h)(X^{s+1},\ Y,\ X^{q+1-s})\rangle\,.\end{aligned}$$

Again, since

$$\begin{aligned}&(D^{q+1}h)(X,\ Y,\ X^{q+1})-(D^{q+1}h)(Y,\ X^{q+2})\\&=R^{\perp}(X,\ Y)(D^{q-1}h)(X^{q+1})-\sum_{\gamma=0}^{q}(D^{q-1}h)(X^{\gamma},\ R(X,\ Y)X,\ X^{q-\gamma})\,,\end{aligned}$$

as before we can show that

$$\begin{aligned}&\langle (D^{2j-2-q}h)(X^{2j-q}),\ (D^{q+1}h)(Y,\ X^{q+2})\rangle\\&=\langle (D^{2j-2-q}h)(X^{2j-q}),\ (D^{q+1}h)(X,\ Y,\ X^{q+1})\rangle=0\,.\end{aligned}$$

This completes the induction of (2.13). Putting $\gamma=j$ we have

$$\langle (D^{j-1}h)(X^{j+1}),\ (D^{j}h)(Y,\ X^{j+1})\rangle=0\,,$$

hence we obtain

$$\begin{aligned}Y\mu_{2j-1}&=(-1)^{j+1}Y\langle (D^{j-1}h)(\tilde{X}^{j+1}),\ (D^{j-1}h)(X^{j+1})\rangle\\&=2(-1)^{j+1}\langle (D^{j-1}h)(X^{j+1}),\ (D^{j}h)(Y,\ X^{j+1})\rangle=0\,,\end{aligned}$$

it proves that μ_{2j-1} (and also ν_{j+1}) is a constant on M. It follows that $\mu_{2j}=0$. Thus by (2.10)-(2.13) we have (2.1). By inductive hypothesis $K_1, \cdots, K_{j-1}$ are constants and $(F_1), \cdots, (F_j)$ hold. As in Theorem 2.1 we have

$$\begin{aligned}K_j\sigma^{(j+1)}&=\tilde{\nabla}_X\sigma^{(j)}+K_{j-1}\sigma^{(j-1)}\\&=(K_1,\ \cdots,\ K_{j-1})^{-1}\sum a_{j+1,\,j+1-2l}(D^{j-1-2l}h)(X^{j+1-2l})\end{aligned}$$

where $a_{j+1,j+1-2l}$ are constants depending on $K_1, \cdots, K_{j-1}$. Since for $\gamma+s\leqq 2j-2$ $\langle (D^{\gamma}h)(X^{\gamma+2}), (D^{s}h)(X^{s+2})\rangle$ are all constants we see that K_j must be a constant. The theorem is proved.

COROLLARY 2.4. *An isometric immersion $f: M^n\to\overline{M}^{n+p}(c)$ is a helical immer-*

sion if and only if M has geodesic normal sections.

Proof. If every geodesic γ is a normal section, i.e. contained in a totally geodesic submanifold M_0 with $T_xM_0=E(x, X)$, $X=\dot{\gamma}$, then $X\in T_x(M_0)$, $\tilde{\nabla}_X X=h(X, X)\in T_xM_0$ and $\tilde{\nabla}_X h(X, X)\in T_xM_0, \cdots, \tilde{\nabla}_X(D^{i-2}h)(X^i)\in T_xM_0$ for all i. This means $A_{(D^{i-2}h)(X^i)}X\wedge X=0$ for all $X\in U_xM$. By theorem 2.3 the Frenet curvatures are all constants.

COROLLARY 2.5. *Let $f: M^n\to\bar{M}^{n+p}(c)$ be an isometric immersion with every geodesic being of order d. Then:*

(a) *If d is even and the Frenet curvatures $K_1, K_2, \cdots, K_{(d/2)-1}$ are constants then f is helical, $K_1, K_2, \cdots, K_{d-1}$ are constants.*

(b) *If d is odd and the Frenet curvatures $K_1, K_2, \cdots, K_{(d-3)/2}$ are constants and $K_{(d-1)/2}$ is constant on every unit sphere U_xM then f is helical.*

Proof. (a). If $K_1, K_2, \cdots, K_{(d/2)-1}$ are constants by Theorem 2.1 for $k\leqq d-1$, $X\in U_xM$, $A_{(D^{k-2}h)(X^k)}X\wedge X=0$. Using this fact we can easily show that the Frenet frames $\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(d)}$, where $\sigma=f\circ\gamma$, γ being the geodesic issued from x and tangent to X, are linear combinations of X, $h(X^2), \cdots, (D^{d-2}h)(X^d)$. Hence $\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(d)}\in E(x, X)$. By the theory of differential equations σ is contained in the totally geodesic submanifold M_0 having $E(x, X)$ as tangent space at x. Thus γ is a geodesic normal section and f is helical.

(b). By the remark after Theorem 2.1 and the same argument in (a) f is also helical.

Chen and Verheyen proved this corollary in the case $d=3, 4$. ([5]). Also see Nakagawa [8].

Next we consider some problems related to the order d.

THEOREM 2.6. *Let M be a compact submanifold in E^m having geodesic normal sections, then the order of M is even.*

Proof. By Corollary 4 in [5] the geodesics on M are closed curves. But these curves are helices and a helix of odd order in E^m cannot be closed. See D. Ferus and S. Schirrmacher [6].

THEOREM 2.7. *Let M be a spherical submanifold in E^{n+p} having geodesic normal sections then the order of M is even (in E^{n+p}).*

Proof. Suppose a geodesic γ of M is of odd order $2m+1$. Then there are constants $\gamma_0, \gamma_1, \cdots, \gamma_m, a_1, \cdots, a_m$ and orthogonal vectors $e_0, e_1, \cdots, e_{2m}\in E^{n+p}$ such that

$$\gamma(t)=\gamma_0 te_0+\sum_{i=1}^{m}\gamma_i[e_{2i-1}\cos a_jt+e_{2i}\sin a_jt].$$

Since M is contained in some sphere with center x,

$$(\gamma_0 t-x_0)^2+\sum_{i=1}^{m}[(\gamma_i \cos a_i t-x_{2i-1})^2+(\gamma_i \sin a_i t-x_{2i})^2]=R^2$$

where $x_k=\langle x, e_k\rangle$, R is a constant. But this implies that $\gamma_0=0$ which is a contradiction.

In order to classify helical submanifolds in spaces of constant curvature, an important problem is to determine the upper bound of the dimension of the ambient spaces. By Sakamoto [9] a helical submanifold M immersed into a sphere is also a helical submanifold of a Euclidean space. By the proposition 5.6 in Sakamoto [9] we know that if $M^n \subset E^m$ is a helical submanifold of order d then M^n is contained in the linear subspace

$$O_x^d=Sp\{X, (D^{k-2}h)(X_1, \cdots, X_k); X, X_1, \cdots, X_k \in T_x M, k=2, 3, \cdots, d\}.$$

We have

LEMMA 2.8. *Let $M^n \subset \tilde{M}^{n+p}(c)$ be an immersion. Then for $j \geqq 1$*

$$\dim O_x^j \leqq \binom{n+j}{j}-1.$$

Proof. For $j=1$ we have

$$O_x^1=Sp\{X, X\in T_x M\}=T_x M.$$

So $\dim O_x^1=n$. Suppose that we have $\dim O_x^{j-1} \leqq \binom{n+j-1}{j-1}-1$. Noticing that $O_x^j=Sp\{O_x^{j-1}, V\}$ where

$$V=Sp\{(D^{j-2}h)(X_1, \cdots, X_j); X_1, \cdots, X_j \in T_x M\},$$

if we can show that (where $e_1, \cdots, e_n$ are basis vectors for $T_x M$)

$$V \subset Sp\{(D^{j-2}h)(e_1^{k_1}, e_2^{k_2}, \cdots, e_n^{k_n}); k_1+k_2+\cdots+k_n=j; O_x^{j-1}\}, \tag{2.14}$$

since there are $\binom{n+j-1}{j}$ vectors in the set $\{(D^{j-2}h)(e_1^{k_1}, e_2^{k_2}, \cdots, e^{k_n})\}$, then we have $\dim O_x^j \leqq \binom{n+j-1}{j}+\binom{n+j-1}{j-1}-1=\binom{n+j}{j}-1$.

To prove (2.14) notice that

$$V=Sp\{(D^{j-2}h)(e_{i_1}, e_{i_2}, \cdots, e_{i_j}); i_1, i_2, \cdots, i_j=1, 2, \cdots, n\}.$$

We only need to show that for any γ, $1\leqq \gamma \leqq j-3$,

$$\begin{aligned}(D^{j-2}h)&(e_{i_1}, e_{i_2}, \cdots, e_{i_\gamma}, e_{i_{\gamma+1}}, \cdots, e_{i_j}))\\ &-(D^{j-2}h)(e_{i_1}, \cdots, e_{i_{\gamma+1}}, e_{i_\gamma}, \cdots, e_{i_j})\in O_x^{j-1}.\end{aligned} \tag{2.15}$$

Extending $e_1, \cdots, e_n$ to vector fields $\tilde{e}_1, \cdots, \tilde{e}_n$ in a neighborhood of x such that at x, $\nabla_{e_i}\tilde{e}_k=0$ for all i, k,

$$
\begin{aligned}
&(D^{j-2}h)(e_{i_1}, e_{i_2}, \cdots, e_{i_\gamma}, e_{i_{\gamma+1}}, \cdots, e_{i_j}) \\
&\qquad -(D^{j-2}h)(e_{i_1}, \cdots, e_{i_{\gamma+1}}, e_{i_\gamma}, \cdots, e_{i_j}) \\
&=\nabla^\perp_{e_{i_1}}\nabla^\perp_{\tilde{e}_{i_2}}\cdots\nabla^\perp_{\tilde{e}_{i_{\gamma-1}}}[(D^{j-\gamma-1}h)(\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}}, \cdots, \tilde{e}_{i_j}) \\
&\qquad -(D^{j-\gamma-1}h)(\tilde{e}_{i_{\gamma+1}}, \tilde{e}_{i_\gamma}, \cdots, \tilde{e}_{i_j})] \\
&=\nabla^\perp_{e_{i_1}}\nabla^\perp_{\tilde{e}_{i_2}}\cdots\nabla^\perp_{\tilde{e}_{i_{\gamma-1}}}[R^\perp(\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}})(D^{j-\gamma-3}h)(\tilde{e}_{i_{\gamma+2}}, \cdots, \tilde{e}_{i_j}) \\
&\qquad -\sum_{s=\gamma+2}^{j}(D^{j-\gamma-3}h)(\tilde{e}_{i_{\gamma+2}}, \cdots, R(\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}})\tilde{e}_{i_s}, \cdots, \tilde{e}_{i_j})].
\end{aligned}
$$

But $R^\perp(\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}})(D^{l-\gamma-3}h)(\tilde{e}_{i_{\gamma+2}}, \cdots, \tilde{e}_{i_j})$ is a linear combination of $h(\tilde{e}_i, \tilde{e}_l)$, $1\leqq i, l\leqq n$, in fact if ξ is orthogonal to all $h(\tilde{e}_i, \tilde{e}_l)$, $1\leqq i, l\leqq n$, then

$$\langle A_\xi\tilde{e}_i, \tilde{e}_l\rangle=\langle\xi, h(\tilde{e}_i, \tilde{e}_l)\rangle=0,$$

hence $A_\xi=0$ and $\langle R^\perp(\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}})\eta, \xi\rangle=\langle[A_\eta, A_\xi]\tilde{e}_{i_\gamma}, \tilde{e}_{i_{\gamma+1}}\rangle=0$ for all $\eta\in N_xM$. Thus all terms in the last expression are in O_x^{j-1}. This proves (2.15).

Thus we have

THEOREM 2.9. *Let $M^n\subset E^m$ be a helical immersion of order d then M^n is contained in a linear subspace V of E^m with* $\dim V\leqq\binom{n+d}{d}-1$.

§3. Surface with geodesic normal sections.

Chen and Verheyen [5] studied surfaces with geodesic normal sections, they showed that in E^5 the only surfaces with geodesic normal sections are (i) a 2-plane E^2; (ii) an ordinary 2-sphere in a 3-plane; (iii) the Veronese surfaces in E^5. They also gave some partial results in E^6.

In this section we will prove the following theorems.

THEOREM 3.1. *Let M^2 be a surface with constant curvature immersed in E^7, then M^2 has geodesic normal sections if and only if M is contained in one of the followings.*

(i) *a 2-plane E^2;*
(ii) *an ordinary 2-sphere in a 3-plane;*
(iii) *the Veronese surface in a 5-plane;*
(iv) *the 3rd standard immersion of a 2-sphere $S^2\subset E^7$.*

THEOREM 3.2. *There is no surface M^2 helically immersed into E^m of order 3.*

First we prove the following lemma.

LEMMA 3.3. *Let $M^2\subset E^m$ be a helical immersion, $\{e_1, e_2\}$ are orthonormal vectors in T_xM, $x\in M$. $\beta=\|h(e_1, e_2)\|$. Then*

$$\langle(Dh)(e_1^3), h(e_1^2)\rangle=\langle(Dh)(e_1^3), h(e_1, e_2)\rangle=\langle(Dh)(e_1^2, e_2), h(e_1^2)\rangle=0, \tag{3.1}$$

(3.2) $$\langle (Dh)(e_1^3),\ h(e_2^2)\rangle = -3\beta e_1\beta\,,$$

(3.3) $$\langle (Dh)(e_1^2,\ e_2),\ h(e_1,\ e_2)\rangle = \beta e_1\beta\,,$$

(3.4) $$\langle (Dh)(e_1^2,\ e_2),\ h(e_2^2)\rangle = -\beta e_2\beta\,.$$

Proof. (3.1) is proved in theorem 2.1. Using Lemma (1.1) to $\langle (Dh)(e_1^3),\ h(e_1^2)\rangle =0$ we have

(3.5) $$\langle (Dh)(e_1^3),\ h(e_2^2)\rangle + 6\langle (Dh)(e_1^2,\ e_2),\ h(e_1,\ e_2)\rangle + 3\langle (Dh)(e_1,\ e_2^2),\ h(e_1^2)\rangle = 0$$

and

(3.6) $$\langle (Dh)(e_1^3),\ h(e_2^2)\rangle + \langle (Dh)(e_2^2,\ e_1),\ h(e_1^2)\rangle = e_1\langle h(\tilde{e}_1^2),\ h(\tilde{e}_2^2)\rangle = e_1(K_1^2 - 2\beta^2) = -4\beta e_1\beta\,,$$

where $\tilde{e}_1$, $\tilde{e}_2$ denote the vector fields adapted to e_1, e_2 and

(3.7) $$\langle (Dh)(e_1^2,\ e_2),\ h(e_1,\ e_2)\rangle = \frac{1}{2}e_1\|h(\tilde{e}_1,\ \tilde{e}_2)\|^2 = \beta e_1\beta\,.$$

Combining (3.5)-(3.7) we get (3.2)-(3.4).

Now we prove theorem 3.1. Let M^2 be a surface with constant Gauss curvature K, helically immersed in E^7. If the immersion is of order 1 or 2, by theorem 2.8, M^2 is contained in a 5-dimensional linear subspace of E^7, thus by the result of Chen and Verheyen M^2 is of case (i), (ii) or (iii). Suppose the immersion f is of order at least 3 then $K_1,\ K_2>0$. Using the notations in [5], i.e. $\alpha=\|H\|$, $\xi_3=(1/\alpha)H$, H being the mean curvature vector, $\xi_4=1/2\beta(h(e_1^2)-h(e_2^2))$, $\xi_5=1/\beta h(e_1,\ e_2)$, by lemma 3.3 $(Dh)(e_1^3)$ is orthogonal to ξ_3, ξ_4, ξ_5 since β is a constant. We may assume that $\alpha\beta\neq 0$ since the case $\alpha\beta=0$ has been discussed in [5]. But $\|(Dh)(e_1^3)\|=K_1K_2$ so we may assume $(Dh)(e_1^3)=K_1K_2\xi_6$ then ξ_6 is a unit vector orthogonal to ξ_3, ξ_4 and ξ_5. Thus we can find a unit vector ξ_7 such that ξ_3, ξ_4, ξ_5, ξ_6 and ξ_7 form an orthonormal basis for N_xM. Since $(Dh)(e_1^2,\ e_2)$, $(Dh)(e_1,\ e_2^2)$ and $(Dh)(e_2^3)$ are all orthogonal to ξ_3, ξ_4, ξ_5 and $\|(Dh)(e_2^3)\| =K_1K_2$, $\langle (Dh)(e_1^3),\ (Dh)(e_1^2,\ e_2)\rangle=0$, $\langle (Dh)(e_2^3),\ (Dh)(e_1,\ e_2^2)\rangle=0$. We may assume that there are $\theta\in[0,\ 2\pi]$ and real numbers a, b such that

(3.8) $$(Dh)(e_1^2,\ e_2) = K_1K_2 a\xi_7\,,$$

(3.9) $$(Dh)(e_1,\ e_2^2) = K_1K_2 b(\cos\theta\xi_6 - \sin\theta\xi_7)$$

(3.10) $$(Dh)(e_2^3) = K_1K_2(\sin\theta\xi_6 + \cos\theta\xi_7)\,.$$

Using lemma 1.1 to $\langle (Dh)(e^3),\ (Dh)(e^3)\rangle = K_1^2K_2^2(\langle e,\ e\rangle)^3$ for all $e\in T_xM$,

(3.11) $$2\langle (Dh)(e_1^3),\ (Dh)(e_1,\ e_2^2)\rangle + 3\langle (Dh)(e_1^2,\ e_2),\ (Dh)(e_1^2,\ e_2)\rangle = K_1^2K_2^2\,,$$

(3.12) $$\langle (Dh)(e_1^3),\ (Dh)(e_2^3)\rangle + 9\langle (Dh)(e_1^2,\ e_2),\ (Dh)(e_1,\ e_2^2)\rangle = 0\,,$$

(3.13) $$2\langle (Dh)(e_1^2,\ e_2),\ (Dh)(e_2^3)\rangle + 3\langle (Dh)(e_1,\ e_2^2),\ (Dh)(e_1,\ e_2^2)\rangle = K_1^2K_2^2\,.$$

Combining all the equations (3.8)-(3.13) we have four solutions:

Case 1. $b=-1$, $a=-1$, $\theta=0$; Case 2. $b=1/3$, $a=1/3$, $\theta=0$; Case 3. $b=a=1$, $\theta=\pi$; Case 4. $b=a=-1/3$, $\theta=\pi$. If we replace ξ_7 by $-\xi_7$ then Case 3 reduces to Case 1 and Case 4 reduces to Case 2. Thus we have basically two possible cases:

$$\text{(3.14)} \qquad \text{Case 1:} \quad (Dh)(e_1^3)=K_1K_2\xi_6\,, \quad (Dh)(e_1^2, e_2)=-K_1K_2\xi_7\,,$$
$$(Dh)(e_1, e_2^2)=-K_1K_2\xi_6\,, \quad (Dh)(e_2^3)=K_1K_2\xi_7\,.$$

$$\text{(3.15)} \qquad \text{Case 2:} \quad (Dh)(e_1^3)=K_1K_2\xi_6\,, \quad (Dh)(e_1^2, e_2)=\left(\frac{1}{3}\right)K_1K_2\xi_7\,,$$
$$(Dh)(e_1, e_2^2)=\left(\frac{1}{3}\right)K_1K_2\xi_6\,, \quad (Dh)(e_2^3)=K_1K_2\xi_7\,.$$

We first consider case 2. Choose $\{e_1, e_2\}$ to be orthonormal vector fields. Then $\{e_1, e_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7\}$ is a moving frame of E^7. Let ω_1^2 be the connection form. Then

$$\begin{aligned}\nabla^\perp_{e_1}\xi_3&=\nabla^\perp_{e_1}\frac{1}{2\alpha}(h(e_1^2)+h(e_2^2))\\ &=\frac{1}{2\alpha}[(Dh)(e_1^3)+(Dh)(e_1, e_2^2)+2\omega_1^2(e_1)h(e_1, e_2)+2\omega_2^1(e_1)h(e_1, e_2)]\\ &=\frac{1}{2\alpha}\left[K_1K_2\xi_6+\frac{1}{3}K_1K_2\xi_6\right]=\frac{2}{3\alpha}K_1K_2\xi_6\,,\end{aligned}$$

$$\nabla^\perp_{e_2}\xi_3=\nabla^\perp_{e_2}\frac{1}{2\alpha}[h(e_1^2)+h(e_2^2)]=\frac{2}{3\alpha}K_1K_2\xi_7\,.$$

Thus we have

$$\text{(3.16)} \qquad \nabla^\perp\xi_3=\frac{2}{3\alpha}K_1K_2(\omega^1\xi_6+\omega^2\xi_7)\,.$$

Similarly we have

$$\text{(3.17)} \qquad \nabla^\perp\xi_4=2\omega_1^2\xi_5+\frac{K_1K_2}{3\beta}(\omega^1\xi_6-\omega^2\xi_7)\,,$$

$$\text{(3.18)} \qquad \nabla^\perp\xi_5=2\omega_2^1\xi_4+\frac{K_1K_2}{3\beta}(\omega^2\xi_6+\omega^1\xi_7)\,.$$

The Ricci equation (1.4) can be rewritten as following:

$$\begin{aligned}R^\perp(X, Y)\xi_x&=\nabla^\perp_X\nabla^\perp_Y\xi_x-\nabla^\perp_Y\nabla^\perp_X\xi_x-\nabla^\perp_{[X,Y]}\xi_x\\ &=\nabla^\perp_X(\sum_y w_x^y(Y)\xi_y)-\nabla^\perp_Y(\sum_y \omega_x^y(X)\xi_y)-\sum_y \omega_x^y([X, Y])\xi_y\\ &=\sum_y (X\omega_x^y(Y))\xi_y+\sum_y \omega_x^y(Y)\omega_y^z(X)\xi_z-\sum_y (Y\omega_x^y(X))\xi_y\end{aligned}$$

$$-\sum_{y,z}\omega_x^y(X)\omega_y^z(Y)\xi_z-\sum_y\omega_x^y([X,Y])\xi_y$$
$$=\sum_y[(X\omega_x^y(Y))-(Y\omega_x^y(X))-\omega_x^y([X,Y])$$
$$+\sum_z(\omega_z^y(X)\omega_x^z(Y)-\omega_z^y(Y)\omega_x^z(X))]\xi_y$$
$$=2\sum_y(d\omega_x^y(X,Y)+\sum_z(\omega_z^y\wedge\omega_x^z)(X,Y))\xi_y\,.$$

So we can write

$$d\omega_x^y+\sum_z\omega_z^y\wedge\omega_x^z=\frac{1}{2}[A_{\xi_x},A_{\xi_y}] \tag{3.19}$$

where $[A_{\xi_x}, A_{\xi_y}]$ denotes a 2-form, having $\langle[A_{\xi_x}, A_{\xi_y}](X), Y\rangle$ as its value at X, Y. Let $x=6$, $y=7$ then

$$d\omega_6^7+\omega_3^7\wedge\omega_6^3+\omega_4^7\wedge\omega_6^4+\omega_5^7\wedge\omega_6^5=\frac{1}{2}[A_{\xi_6}, A_{\xi_7}]=0\,. \tag{3.20}$$

On the other hand

$$\langle\nabla_{e_1}^{\perp}\xi_6,\xi_7\rangle=\left\langle\nabla_{e_1}^{\perp}\left(\frac{1}{K_1K_2}(Dh)(e_1^3)\right),\frac{3}{K_1K_2}(Dh)(e_1^2,e_2)\right\rangle$$
$$=\frac{3}{K_1^2K_2^2}\langle(D^2h)(e_1^4)+3\omega_1^2(e_1)(Dh)(e_1^2,e_2),(Dh)(e_1^2,e_2)\rangle$$
$$=\frac{9}{K_1^2K_2^2}\cdot\frac{K_1^2K_2^2}{9}\cdot\omega_1^2(e_1)=\omega_1^2(e_1)\,,$$

$$\langle\nabla_{e_2}^{\perp}\xi_6,\xi_7\rangle=-\langle\nabla_{e_2}^{\perp}\xi_7,\xi_6\rangle=-\left\langle\nabla_{e_2}^{\perp}\left(\frac{1}{K_1K_2}(Dh)(e_2^3)\right),\frac{3}{K_1K_2}(Dh)(e_1,e_2^2)\right\rangle$$
$$=-\omega_2^1(e_2)=\omega_1^2(e_2)\,.$$

Thus $\omega_6^7=\omega_1^2$ and (3.16)-(3.20) gives

$$d\omega_1^2+\frac{2K_1K_2}{3\alpha}\omega^2\wedge\left(-\frac{2}{3\alpha}K_1K_2\omega^1\right)+\left(\frac{K_1K_2}{\beta}\omega^2\right)\wedge\left(-\frac{K_1K_2}{3\beta}\omega^1\right)$$
$$+\left(\frac{K_1K_2}{3\beta}\omega^1\right)\wedge\left(-\frac{K_1K_2}{3\beta}\omega^2\right)$$
$$=\left(K+\frac{2K_1^2K_2^2}{9\beta^2}-\frac{4K_1^2K_2^2}{9\alpha^2}\right)\omega^2\wedge\omega^1=\left(1+\frac{2K_1^2K_2^2}{9\alpha^2\beta^2}\right)K\omega^2\wedge\omega^1=0$$

Thus we have $K=0$. Let $x=4$, $y=5$ then

$$d\omega_4^5+\omega_3^5\wedge\omega_4^3+\omega_6^5\wedge\omega_4^6+\omega_7^5\wedge\omega_4^7$$
$$=2d\omega_1^2+\left(-\frac{K_1K_2}{3\beta}\omega^2\right)\wedge\left(\frac{K_1K_2}{3\beta}\omega^1\right)+\left(-\frac{K_1K_2}{3\beta}\omega^1\right)\wedge\left(-\frac{K_1K_2}{3\beta}\omega^2\right)$$
$$=\left(2K-\frac{2K_1K_2}{9\beta^2}\right)\omega^2\wedge\omega^1=-\frac{2K_1K_2}{9\beta^2}\omega^2\wedge\omega^1.$$

On the other hand $[A_{\xi_4}, A_{\xi_5}]=4\beta^2\omega^2\wedge\omega^1$, this is a contradiction. Thus case 2 is impossible.

Next we consider case 1. Similar computation as in case 2 we have

$$\nabla^{\perp}\xi_3=0, \tag{3.21}$$

$$\nabla^{\perp}\xi_4=2\omega_1^2\xi_5+\frac{K_1K_2}{\beta}(\omega^1\xi_6-\omega^2\xi_7), \tag{3.22}$$

$$\nabla^{\perp}\xi_5=2\omega_2^1\xi_4-\frac{K_1K_2}{\beta}(\omega^2\xi_6+\omega^1\xi_7), \tag{3.23}$$

and

$$\begin{aligned}\langle\nabla^{\perp}_{e_1}\xi_6, \xi_7\rangle&=\left\langle\nabla^{\perp}_{e_1}\left(\frac{1}{K_1K_2}(Dh)(e_1^3)\right), \frac{-1}{K_1K_2}(Dh)(e_1^2, e_2)\right\rangle\\&=-\frac{1}{K_1^2K_2^2}\langle(D^2h)(e_1^4)+3\omega_1^2(e_1)(Dh)(e_1^2, e_2), (Dh)(e_1^2, e_2)\rangle\\&=-3\omega_1^2(e_1).\end{aligned}$$

Similarly,

$$\langle\nabla^{\perp}_{e_2}\xi_6, \xi_7\rangle=-3\omega_1^2(e_2).$$

Thus we have

$$\nabla^{\perp}\xi_6=-\frac{K_1K_2}{\beta}\omega^1\xi_4+\frac{K_1K_2}{\beta}\omega^2\xi_5-3\omega_1^2\xi_7 \tag{3.24}$$

$$\nabla^{\perp}\xi_7=\frac{K_1K_2}{\beta}\omega^2\xi_4+\frac{K_1K_2}{\beta}\omega^1\xi_5-3\omega_2^1\xi_6. \tag{3.25}$$

Putting $x=4$, $y=5$ in (3.19)

$$\begin{aligned}&d\omega_4^5+\omega_6^5\wedge\omega_4^6+\omega_7^5\wedge\omega_4^7\\&=2d\omega_1^2+\frac{K_1K_2}{\beta}\omega^2\wedge\frac{K_1K_2}{\beta}\omega^1+\frac{K_1K_2}{\beta}\omega^1\wedge\left(-\frac{K_1K_2}{\beta}\omega^2\right)\\&=\left(2K+\frac{2K_1^2K_2^2}{\beta^2}\right)\omega^2\wedge\omega^1=4\beta^2\omega^2\wedge\omega^1.\end{aligned}$$

Thus we have

$$K+\frac{K_1^2K_2^2}{\beta^2}=2\beta^2. \tag{3.26}$$

Putting $x=6$, $y=7$ in (3.19)

$$\begin{aligned}&d\omega_6^7+\omega_4^7\wedge\omega_6^4+\omega_5^7\wedge\omega_6^5\\&=-3d\omega_1^2+\left(-\frac{K_1K_2}{\beta}\omega^2\right)\wedge\left(-\frac{K_1K_2}{\beta}\omega^1\right)+\left(-\frac{K_1K_2}{\beta}\omega^1\right)\wedge\left(\frac{K_1K_2}{\beta}\omega^2\right)\\&=\left(-3K+\frac{2K_1^2K_2^2}{\beta^2}\right)\omega^2\wedge\omega^1=0.\end{aligned}$$

Thus we have $-3K+(2K_1^2K_2^2)/\beta^2=0$. Taking account of (3.26) we have

$$K_1^2=\frac{17}{2}K,\quad \beta^2=\frac{5}{2}K,\quad K_2^2=\frac{15}{34}K. \tag{3.27}$$

Let γ be the geodesic issued from $x\in M$ and $\sigma=f\circ\gamma$ as its image in E^7 Choosing e_1 such that $e_1=\dot\sigma$ along γ,

$$\tilde\nabla_{e_1}\dot\sigma=h(e_1,e_1)=\alpha\xi_3+\beta\xi_4=K_1\sigma^{(2)}.$$

So

$$\sigma^{(2)}=\frac{1}{K_1}(\alpha\xi_3+\beta\xi_4),$$

$$\tilde\nabla_{e_1}\sigma^{(2)}=-A_{\sigma^{(2)}}e_1+\frac{1}{K_1}(Dh)(e_1^3)=-K_1e_1+K_2\xi_6,\quad \text{so}\quad \sigma^{(3)}=\xi_6,$$

$$\tilde\nabla_{e_1}\sigma^{(3)}=\tilde\nabla e_1\xi_6=\frac{-K_1K_2}{\beta}\xi_4=-K_2\sigma^{(2)}+K_3\sigma^{(4)}.$$

Thus we have $K_3\sigma^{(4)}=-((K_1K_2)/\beta)\xi_4+(K_2/K_1)(\alpha\xi_3+\beta\xi_4)=(K_2\alpha/K_1\beta)(\beta\xi_3-\alpha\xi_4)$ and

$$K_3=\frac{K_2\alpha}{\beta}=\frac{3\sqrt{2}}{17}\sqrt{K} \tag{3.28}$$

and $\tilde\nabla_{e_1}\sigma^{(4)}=-K_3\xi_6=-K_3\sigma^{(3)}$. Thus γ is a helix of order 4. But e_1 can be chosen as any unit vector in T_xM, this means f is of order 4.

Now if we regard $\{\xi_3, \xi_4, \xi_5, \xi_6, \xi_7\}$ as a 5-plane bundle E on M^2, then (3.21)-(3.25) define a connection on E, equipped with the second fundamental form h and associated second fundamental tensor A. It is easy to check that they satisfy the equations of Gauss, Ricci, and Codazzi. Thus by the fundamental theorem of submanifold [2] we can conclude that there is an immersion $M^2\to E^7$ with normal bundle E, and up to a motion, this immersion is unique.

We can also write this immersion explicitly. Let $e_1, e_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7$ be the frame at x, γ_e be the geodesic issued from x, having tangent vector $e=e_1\cos\theta+e_2\sin\theta$, $0\leqq\theta<2\pi$, $\sigma_e=f\circ\gamma_e$. Then

$$\begin{aligned}
&\sigma_e^{(1)}(0)=e=e_1\cos\theta+e_2\sin\theta,\\
&\sigma_e^{(2)}(0)=(1/K_1)h(e,e)=(1/K_1)(\alpha\xi_3+\beta\cos 2\theta\xi_4+\beta\sin 2\theta\xi_5),\\
&\sigma_e^{(3)}(0)=(1/K_1K_2)(Dh)(e^3)=\xi_6\cos 3\theta-\xi_7\sin 3\theta,\\
&\sigma_4^{(4)}(0)=(1/K_3)(\tilde\nabla_e\sigma_e^{(3)}+K_2\sigma_2^{(2)})=(1/K_1)(\beta\xi_3-\alpha\cos 2\theta\xi_4-\alpha\sin 2\theta\xi_5).
\end{aligned} \tag{3.29}$$

Since $\sigma_e^{(1)}$, $\sigma_e^{(2)}$, $\sigma_e^{(3)}$ and $\sigma_e^{(4)}$ satisfy the Frenet equations and the initial condition (3.29), by solving these equations we get the helical immersion of the sphere $S^2\subset E^7$, which is the 3-rd standard immersion of $S^2\subset E^7$:

$$f(\theta, v)=(R/16)(\sin v+5\sin 3v)(e_1\cos\theta+e_2\sin\theta)$$
$$-(R\sqrt{6}/48)(3\cos v+5\cos 3v)\xi_3$$
$$+(R\sqrt{10}/16)(\cos v-\cos 3v)(\xi_4\cos 2\theta+\xi_5\sin 2\theta)$$
$$-(R\sqrt{15}/16)(\sin v-1/3\sin 3v)(\xi_6\cos 3\theta-\xi_7\sin 3\theta),$$

where $R=1/\sqrt{K}$ is the radius of S^2 and (θ, v) is the spherical coordinate on S^2. Thus Theorem 3.1 is proved.

Now we turn to theorem 3.2.

LEMMA 3.4. *Let* $f: M^2\subset E^m$ *be a helical immersion of order* 3. $\{e_1, e_2\}$ *is an orthonormal basis for* T_xM, $x\in M$. *Then*

$$(D^2h)(e_1^4)=-K_2^2h(e_1^2), \tag{3.30}$$

$$(D^2h)(e_1^3, e_2)=(1/2)(K_1^2-K_2^2-4\beta^2)h(e_1, e_2), \tag{3.31}$$

$$(D^2h)(e_2, e_1^3)=(-1/2)(3K_1^2+K_2^2-12\beta^2)h(e_1, e_2), \tag{3.32}$$

$$(D^2h)(e_1^2, e_2^2)=((-1/6)K_2^2-(1/2)K_1^2+2\beta^2)h(e_1^2) +((-1/6)K_2^2+(1/2)K_1^2-2\beta^2)h(e_2^2). \tag{3.33}$$

Proof. Let γ be a geodesic issued from x with tangent vector e and the Frenet frame for $\sigma=f\circ\gamma$ be $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$. By theorem 2.1, $\sigma^{(1)}=e$, $\sigma^{(2)}=(1/K_1)h(e^2)$, $\sigma^{(3)}=(K_1K_2)^{-1}(Dh)(e^3)$. And the Frenet formula gives $\tilde\nabla_e\sigma^{(3)}=-K_2\sigma^{(2)}$, i.e.

$$(K_1K_2)^{-1}(D^2h)(e^4)=(-K_2\sqrt{K_1})h(e^2),$$

or

$$(D^2h)(e^4)=-K_2^2h(e^2).$$

Since this is true for all unit vectors $e\in U_x(M)$ by lemma 1.1 we have

$$3(D^2h)(e_1^3, e_2)+(D^2h)(e_2, e_1^3)=-2K_2^2h(e_1, e_2), \tag{3.34}$$

$$(D^2h)(e_1^2, e_2^2)+(D^2h)(e_2^2, e_1^2)=-\left(\frac{1}{3}\right)K_2^2(h(e_1^2)+h(e_2^2)). \tag{3.35}$$

By the Ricci identity

$$(D^2h)(e_1^3, e_2)-(D^2h)(e_2, e_1^3)=R^\perp(e_1, e_2)h(e_1^2)-2h(R(e_1, e_2)e_1, e_1).$$

Using (1.3) and Proposition 13 in [5] there is an adapted orthonormal frame $\{e_1, e_2, \xi_3, \cdots, \xi_m\}$ for which we can find that

$$R^\perp(e_1, e_2)\xi_i=0 \text{ if } i\neq 4, 5;\; R^\perp(e_1, e_2)\xi_4=-2\beta^2\xi_5;\; R^\perp(e_1, e_2)\xi_5=2\beta^2\xi_4, \tag{3.36}$$

since $R(e_1, e_2)e_1=-Ke_2=(-K_1^2+3\beta^2)e_2$, where K is the Gauss curvature of M^2. Thus we have

$$(D^2h)(e_1^3, e_2)-(D^2h)(e_2, e_1^3)=2(K_1^2-4\beta^2)h(e_1, e_2)\,. \tag{3.37}$$

By (3.34) and (3.37) we get (3.31) and (3.32).

Similarly we have

$$\begin{aligned}(D^2h)(e_1^2, e_2^2)&-(D^2h)(e_2^2, e_1^2)\\ &=R^\perp(e_1, e_2)h(e_1, e_2)-h(R(e_1, e_2)e_1, e_2)-h(R(e_1, e_2)e_2, e_1)\\ &=\beta^2[h(e_1^2)-h(e_2^2)]+(K_1^2-3\beta^2)h(e_2^2)-(K_1^2-3\beta^2)h(e_1^2)\\ &=-(K_1^2-4\beta^2)[h(e_1^2)-h(e_2^2)]\,.\end{aligned}$$

Taking into account of (3.35) we get (3.33).

LEMMA 3.5. *Let $f: M^2 \to E^m$ be a helical immersion of order* 3. *Then M^2 has constant Gauss curvature.*

Proof. Suppose M^2 is not of constant Gauss curvature then β is not a constant. Since M^2 is connected there exists $x \in M^2$ such that $\beta \neq 0$, $d\beta \neq 0$ in a neighborhood U of x. Choose a unit vector field e_1 in U such that $d\beta(e_1)=e_1\beta=0$ and a unit vector field e_2 in U orthogonal to e_1. Then by Lemma 3.3

$$\langle (Dh)(e_1^3), h(e_2^2)\rangle=0\,.$$

Differentiating along the direction of e_2 we have

$$\begin{aligned}&\langle (D^2h)(e_2, e_1^3), h(e_2^2)\rangle+\langle (Dh)(e_1^3), (Dh)(e_2^3)\rangle\\ &\quad+3\omega_1^2(e_2)\langle (Dh)(e_1^2, e_2), h(e_2^2)\rangle+2\omega_2^1(e_2)\langle (Dh)(e_1^3), h(e_1, e_2)\rangle=0\,.\end{aligned}$$

By lemma 3.3 and 3.4 we get

$$\langle (Dh)(e_1^3), (Dh)(e_2^3)\rangle=-3\omega_1^2(e_2)\langle (Dh)(e_1^2, e_2)\,;\; h(e_2^2)\rangle=3\omega_1^2(e_2)\beta e_2\beta \tag{3.38}$$

Also by Lemma 3.3

$$\langle (Dh)(e_1, e_2^2), h(e_1^2)\rangle=0\,.$$

Differentiating along the direction of e_2 we have

$$\begin{aligned}&\langle (Dh)(e_1, e_2^2), (Dh)(e_1^2, e_2)\rangle+\langle (D^2h)(e_2^3, e_1), h(e_1^2)\rangle+\omega_1^2(e_2)\langle (Dh)(e_2^3), h(e_1^2)\rangle\\ &\quad+2\omega_2^1(e_2)\langle (Dh)(e_1^2, e_2), h(e_1^2)\rangle+2\omega_1^2(e_2)\langle (Dh)(e_1, e_2^2), h(e_1, e_2)\rangle=0\,.\end{aligned}$$

Hence we have

$$\langle (Dh)(e_1, e_2^2), (Dh)(e_1^2, e_2)\rangle+\omega_1^2(e_2)(-3\beta e_2\beta)+2\omega_1^2(e_2)(\beta e_2\beta)=0\,,$$

that is

$$\langle (Dh)(e_1, e_2^2), (Dh)(e_1^2, e_2)\rangle=\omega_1^2(e_2)\beta e_2\beta\,. \tag{3.39}$$

Combining (3.38), (3.39) and (3.12) we find

(3.40) $$\langle (Dh)(e_1^3),\ (Dh)(e_2^3)\rangle = \langle (Dh)(e_1,\ e_2^2),\ (Dh)(e_1^2,\ e_2)\rangle = 0\,,$$

(3.41) $$\omega_1^2(e_2)\beta e_2\beta = 0\,.$$

(3.41) is true for all points in U. But $\beta e_2\beta \neq 0$ in U, thus $\omega_1^2(e_2)=0$ in U.

Again, since $(Dh)(e_1^3)$ is orthogonal to $h(e_1^2)$, $h(e_1, e_2)$ and $h(e_2^2)$, so by (3.36), we have

$$R^{\perp}(e_1,\ e_2)(Dh)(e_1^3)=0\,.$$

By (3.30)-(3.33), we have

$$\begin{aligned}
\nabla^{\perp}_{e_1}\nabla^{\perp}_{e_2}(Dh)(e_1^3) &= \nabla^{\perp}_{e_1}(D^2h)(e_2,\ e_1^3)\\
&= -\frac{1}{2}(3K_1^2+K_2^2-12\beta^2)\nabla^{\perp}_{e_1}h(e_1,\ e_2)\\
&= -\frac{1}{2}(3K_1^2+K_2^2-12\beta^2)[(Dh)(e_1^2,\ e_2)+\omega_1^2(e_1)h(e_2^2)+\omega_2^1(e_1)h(e_1^2)]\,,
\end{aligned}$$

$$\begin{aligned}
\nabla^{\perp}_{e_2}\nabla^{\perp}_{e_1}(Dh)(e_1^3) &= \nabla^{\perp}_{e_2}[(D^2h)(e_1^4)+3\omega_1^2(e_1)(Dh)(e_1^2,\ e_2)]\\
&= \nabla^{\perp}_{e_2}[-K_2^2h(e_1^2)+3\omega_1^2(e_1)(Dh)(e_1^2,\ e_2)]\\
&= -K_2^2(Dh)(e_1^2,\ e_2)+3[e_2\omega_1^2(e_1)](Dh)(e_1^2,\ e_2)+3\omega_1^2(e_1)(D^2h)(e_2^2,\ e_1^2)\,,
\end{aligned}$$

$$\nabla^{\perp}_{[e_1,\, e_2]}(Dh)(e_1^3)=\omega_2^1(e_1)\nabla^{\perp}_{e_1}(Dh)(e_1^3)=\omega_2^1(e_1)[-K_2^2h(e_1^2)+3\omega_1^2(e_1)(Dh)(e_1^2,\ e_2)]\,.$$

Thus,

$$\begin{aligned}
R^{\perp}(e_1,\ e_2)(Dh)(e_1^3) &= \nabla^{\perp}_{e_1}\nabla^{\perp}_{e_2}(Dh)(e_1^3)-\nabla^{\perp}_{e_2}\nabla^{\perp}_{e_1}(Dh)(e_1^3)-\nabla^{\perp}_{[e_1,\, e_2]}(Dh)(e_1^3)\\
&= \left[-\frac{1}{2}(3K_1^2+K_2^2-12\beta^2)+K_2^2-3K\right](Dh)(e_1^2,\ e_2)\\
&= -\frac{1}{2}(9K_1^2-K_2^2-30\beta^2)(Dh)(e_1^2,\ e_2)\,.
\end{aligned}$$

Since β is not a constant, we have $(Dh)(e_1^2, e_2)=0$. On the other hand, by differentiating $\langle (Dh)(e_1^2, e_2), h(e_1^2)\rangle=0$ along the direction of e_2, we have

$$\langle (Dh)(e_1^2,\ e_2),\ (Dh)(e_1^2,\ e_2)\rangle+\langle (Dh)(e_2^2,\ e_1^2),\ h(e_1^2)\rangle=0\,,$$

hence

$$\begin{aligned}
\langle (Dh)(e_1^2,\ e_2),\ (Dh)(e_1^2,\ e_2)\rangle &= -\langle (Dh)(e_2^2,\ e_1^2),\ h(e_1^2)\rangle\\
&= \frac{1}{3}K_1^2K_2^2-\frac{1}{3}K_2^2\beta^2-K_1^2\beta^2+4\beta^4=0\,.
\end{aligned}$$

This shows that β is a constant, a contradiction. This proves Lemma 3.5.

Now we finish the proof of theorem 3.2. Let $M^2\subset E^m$ be a helical immersion of order 3. By lemma 3.5, M^2 has constant Gauss curvature. By theorem 2.9, we may assume that $m=9$. Let $\{e_1, e_2\}$ be orthonormal vector fields in some open subset $U\subset M$. Thus Lemma 3.3 shows that $(Dh)(e_1^3)$ is orthogonal to $h(e_1^2)$, $h(e_1, e_2)$ and $h(e_2^2)$, by (3.36), we have

$$R^{\perp}(e_1, e_2)(Dh)(e_1^3)=0.$$

The same computation as in the proof of lemma 3.5 gives

$$-\frac{1}{2}(9K_1^2-K_2^2-30\beta^2)(Dh)(e_1^2, e_2)=0. \tag{3.42}$$

But we also have

$$\langle (Dh)(e_1^2, e_2), (Dh)(e_1^2, e_2)\rangle=\frac{1}{3}K_1^2K_2^2-\frac{1}{3}K_2^2\beta^2-K_1^2\beta^2+4\beta^4, \tag{3.43}$$

and

$$\begin{aligned}\langle (Dh)(e_1^2, e_2), (Dh)(e_1^2, e_2)\rangle &=-\langle (D^2h)(e_1^3, e_2), h(e_1, e_2)\rangle \\ &=\frac{1}{2}(-K_1^2+K_2^2+4\beta^2)\beta^2.\end{aligned} \tag{3.44}$$

Comparing (3.43) and (3.44), we get

$$12\beta^4-(5K_2^2+3K_1^2)\beta^2+2K_1^2K_2^2=0 \tag{3.45}$$

If $(Dh)(e_1^2, e_2)=0$, by (3.44) we have $\beta=0$ or $-K_1^2+K_2^2+4\beta^2=0$, both are impossible by (3.45). If $(Dh)(e_1^2, e_2)\neq 0$, then $9K_1^2-K_2^2-30\beta^2=0$, this also contradicts with (3.45). Thus theorem 3.2 is proved.

Since a helical immersion $M^2\subset E^6$ has order no more than 3 ([5]), we have the following.

COROLLARY 3.6. *Let M^2 be a surface immersed into E^6. M^2 has geodesic normal sections if and only if M^2 is contained in one of the surfaces* (i), (ii) *and* (iii) *listed in theorem* 3.1.

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