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NON-INTEGRABILITY OF HÉNON-HEILES SYSTEM AND A THEOREM OF ZIGLIN

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1. Introduction.

This paper concerns the integrability of Hamiltonian systems with two degrees of freedom

$$\dot{q}_k = H_{p_k}, \quad \dot{p}_k = -H_{q_k} \quad (k=1, 2), \tag{1.1}$$

where the dot indicates the differentiation with respect to time variable t. We assume that the Hamiltonian H is of the form

$$H = H(q, p) = \frac{1}{2}|p|^2 + V(q); \quad |p|^2 = p_1^2 + p_2^2, \tag{1.2}$$

where $V(q)$ is a polynomial of q_1 and q_2. We consider this system in the complex domain. A single-valued function $F(q, p)$ is called an *integral* of (1.1) if it is constant along any solution curve $(q(t), p(t))$ of (1.1). This implies that $(d/dt)F(q(t), p(t))=0$, which leads to the identity

$$\sum_{k=1}^{2}(F_{q_k}H_{p_k} - F_{p_k}H_{q_k}) = 0. \tag{1.3}$$

In particular, the Hamiltonian H is an integral. In this paper, the system (1.1) is said to be *integrable* if there exists an *entire* integral F which is functionally independent of H.

From the viewpoint of dynamical systems, our interest is in the behavior of real solutions for real analytic Hamiltonian systems. However, in the majority of integrable problems of Hamiltonian mechanics, the known integrals can be extended to the complex domain. Therefore, it is natural to discuss the integrability of complex Hamiltonian systems in the above sense, that is, the existence of additional entire integrals other than the Hamiltonian. Moreover, a new aspect appears from considering solutions in complex time plane. It is the branching of solutions as functions of time variable t. In general, the solutions branch in finite or infinite manner by analytic continuation. In this paper, we discuss the integrability of (1.1) in connection with the branching of solutions.

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As for other various aspects of integrable systems, we refer to Kozlov [12].

In recent years, a direct method for testing the integrability has been developed [1, 3, 4, 6-8]. This method consists of requiring that the general solutions have the Painlevé property, i. e. have no movable singularities other than poles. It was first adopted by Kowalevski [10, 11] in the famous study of the motion of heavy solid body about a fixed point. Among recent researches, there have been many works dealing with the integrability of Hénon-Heiles Hamiltonian

$$H=\frac{1}{2}|p|^2+V(q)\,;\quad V(q)=\frac{1}{2}(aq_1^2+bq_2^2)+cq_1^2q_2+\frac{1}{3}dq_2^3, \tag{1.4}$$

where a, b, c, d are real constants. The original Hénon-Heiles Hamiltonian [9] corresponds to $a=b=c=1$ and $d=-1$. This direct method has been used to find parameter values a, b, c, d for which the system with (1.4) is integrable (see [1, 3, 7]). However, this method is practical rather than rigorous. On the other hand, Ziglin [16] has established rigorously a necessary condition for the integrability of Hamiltonian systems. Moreover, using his method Ziglin [17] has proved the non-integrability of the original Hénon-Heiles system. His method is based on considering a particular solution of (1.1) and its monodromy group whose definition will be given in Section 2.

The aim of this paper is to give a criterion for claiming rigorously the non-integrability of (1.1) with (1.2), especially with (1.4). Our arguments are based on a theorem of Ziglin [16, 17], and in the next section we review Ziglin's theorem. For the sake of completeness, we shall give its elementary proof in our setting. The main theorem (Theorem 2) is stated in Section 3. For using Ziglin's approach, it is needed to have a particular solution given in terms of elliptic functions of complex time. We consider a family of such periodic orbits. The main theorem gives a necessary condition for the integrability in connection with the behavior of their characteristic multipliers. It presents a typical situation where the integrability implies non-branching of solutions of variational equations. For the connection between integrability and non-branching of solutions, see [1, 3, 7, 16].

Our result can be applicable for Hamiltonians with non-homogeneous potentials rather than homogeneous ones. In Section 4, our result is applied to Hénon-Heiles Hamiltonians (1.4). In particular, for the case $a=b$ we prove that the system is integrable only if $c/d=0$, 1/6, 1/2 or 1 (Theorem 3). The cases $c/d=0$, 1/6 and 1 are well known integrable cases [3, 7]. In the case $c/d=1/2$, the system is seemed to be non-integrable [3], but we cannot have proved this rigorously.

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2. The reduced equation in normal variations and Ziglin's theorem.

The aim of this section is to give preliminary discussions for stating our main theorem, and to review Ziglin's theorem [16].

Let us consider a particular solution $z(t)=(q(t), p(t))$ of (1.1) which is not an equilibrium point. We consider $z(t)$ to be a complete analytic function of t, namely to be maximally analytically continued with respect to t. Then the phase curve $\Gamma=\{z(t)\}$ is a Riemann surface with local coordinate t. The variational equation of (1.1) along Γ is given by

$$\dot{\zeta}=JH_{zz}(z(t))\zeta . \tag{2.1}$$

Here $\zeta={}^t(\xi_1, \xi_2, \eta_1, \eta_2)$, J is the symplectic matrix

$$J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the identity matrix of degree two, and H_{zz} is the Hessian matrix of $H(q, p)$ given by

$$H_{zz}=\begin{pmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{pmatrix}.$$

Let us now denote $M=\boldsymbol{C}^4$. The variational equation (2.1) is defined on the tangent subbundle $T_\Gamma M$, which is obtained by restricting the base space of TM to Γ and whose coordinate system is given by (ζ, t). Our aim is to give an elementary proof of Ziglin's theorem in our setting. In the following, for any function $F(q, p)$ on M, F_z denotes the gradient vector of F, i.e., $F_z={}^t(F_q, F_p)$, and $\langle\, ,\rangle$ denotes $\langle w, w'\rangle=\sum_{j=1}^4 w_j w'_j$ for vectors $w, w'\in\boldsymbol{C}^4$ with entries w_j, w'_j $(j=1, \cdots, 4)$ respectively.

At first, we note that a 1-form dH is a time-dependent integral of (2.1). Indeed we have

$$\frac{d}{dt}dH(\zeta, t)=\frac{d}{dt}\langle H_z, \zeta\rangle=\langle H_{zz}JH_z, \zeta\rangle+\langle H_z, JH_{zz}\zeta\rangle=0 ,$$

where the argument of H_z and H_{zz} is $z(t)$. Therefore, dH is a non-constant time-dependent integral of (2.1). Next, according to Ziglin [16], we prove that more generally any integral of (1.1) induces a time-dependent integral of the variational equation (2.1). To this end, we consider the general system (1.1) without assuming (1.2).

Let $F(q, p)$ be an analytic function in a neighborhood of Γ. Suppose that at some point $z(t)\in\Gamma$ all the derivatives of F up to and including $(n-1)$-th order vanish, while at least one of its derivatives of n-th order is different from zero. This implies that the integer n is the smallest positive integer such that

$$D^r F(z(t))\neq 0 , \qquad |r|=n \tag{2.2}$$

for some multi-index $r=(r_1, r_2, r'_1, r'_2)$, where

$$D^r=\left(\frac{\partial}{\partial q_1}\right)^{r_1}\left(\frac{\partial}{\partial q_2}\right)^{r_2}\left(\frac{\partial}{\partial p_1}\right)^{r'_1}\left(\frac{\partial}{\partial p_2}\right)^{r'_2}; \quad |r|=\sum_{k=1}^{2}(r_k+r'_k).$$

Generally this integer n depends on the point $z(t)\in\Gamma$. However, we have the following lemma, whose assertion is used in [16] without proof.

LEMMA 1. *If $F(q, p)$ is an integral of* (1.1) *which is analytic in a neighborhood of Γ, then the smallest positive integer n in* (2.2) *is independent of $z(t)\in\Gamma$.*

Proof. Let us consider the identity (1.3), which is written as

$$\langle F_z, JH_z\rangle=0.$$

Without loss of generality, we assume that $F(z(t))=0$. By differentiating this identity with respect to z, we obtain

$$\langle F_{zz}, JH_z\rangle+\langle JH_{zz}, F_z\rangle=0.$$

This leads to a linear equation for F_z

$$\frac{d}{dt}F_z=-{}^t(JH_{zz}(z(t)))F_z.$$

The uniqueness of this equation implies that, if $F_z(z(t_0))=0$ for some $t_0\in\boldsymbol{C}$, we have $F_z(z(t))=0$ for any $t\in\boldsymbol{C}$. Therefore the assertion is proved when $n=1$. Furthermore we can prove this inductively when n is an arbitrary integer. Indeed, let us assume that $D^rF(z)=0$ along Γ for any $|r|\leqq n$. Then, similarly as above, we have a linear homogeneous equation for a vector with entries $D^rF(z)$ satisfying $|r|=n+1$. Hence we have proved that for any $z\in\Gamma$, either $D^rF(z)=0$ for any r with $|r|=n+1$, or $D^rF(z)\neq 0$ for some r with $|r|=n+1$. This completes the proof. Q. E. D.

For any $\zeta={}^t(\xi_1, \xi_2, \eta_1, \eta_2)\in\boldsymbol{C}^4(=T_{z(t)}M)$, let us introduce a differential operator

$$D_\zeta=\sum_{j=1}^{2}\left(\xi_j\frac{\partial}{\partial q_j}+\eta_j\frac{\partial}{\partial p_j}\right)=\left\langle\zeta, \frac{\partial}{\partial z}\right\rangle,$$

where $\partial/\partial z={}^t(\partial/\partial q_1, \partial/\partial q_2, \partial/\partial p_1, \partial/\partial p_2)$. From the above lemma, we can define a single-valued function $\Phi(\zeta, t)$ on $T_\Gamma M$ by

$$\Phi(\zeta, t)=D_\zeta^n F(z(t)), \tag{2.3}$$

where n is the smallest positive integer satisfying (2.2). This is a homogeneous polynomial of degree n in ζ. For this function $\Phi(\zeta, t)$, we have

LEMMA 2. *Let $F(q, p)$ be an integral of* (1.1) *which is analytic in a neighborhood of Γ. Then $\Phi(\zeta, t)$ is a time-dependent integral of the variational equation*

(2.1) *such that*

$$\Phi(\zeta, t)=\Phi(\zeta+\zeta_h, t), \qquad \zeta_h=\xi_h JH_z(z(t)) \tag{2.4}$$

for any scalar $\xi_h \in \boldsymbol{C}$.

Remark. The Hamiltonian vector field JH_z along Γ, i.e., $JH_z(z(t))$, satisfies the linear equation (2.1). Therefore, if $\zeta(t)$ is a solution of (2.1), then $\zeta(t)+\xi_h JH_z(z(t))$ is also a solution of (2.1) for any scalar ξ_h. This implies that the variational equation (2.1) is to be considered on the normal bundle $T_\Gamma M/T\Gamma \cong (T\Gamma)^\perp$. The identity (2.4) implies that $\Phi(\zeta, t)$ can be considered as a function on $T_\Gamma M/T\Gamma$.

Proof. To see that $\Phi(\zeta, t)$ is a time-dependent integral of (2.1), we prove that

$$\frac{d}{dt}\Phi(\zeta(t), t)=0$$

for any solution $\zeta=\zeta(t)$ of (2.1). If we introduce a differential operator

$$D_t=\left\langle JH_z, \frac{\partial}{\partial z}\right\rangle+\left\langle JH_{zz}\zeta, \frac{\partial}{\partial \zeta}\right\rangle,$$

this reads as

$$D_t\Phi(\zeta(t), t)=D_t D_\zeta^n F(z(t))=0, \tag{2.5}$$

where $\zeta=\zeta(t)$. Here we obtain the identity

$$D_\zeta D_t-D_t D_\zeta=\left\langle\left\langle\frac{\partial}{\partial z}(JH_{zz}\zeta), \frac{\partial}{\partial \zeta}\right\rangle, \zeta\right\rangle.$$

Since this does not contain the differentiation $\partial/\partial z$,

$$(D_\zeta D_t-D_t D_\zeta)D_\zeta^k F(z) \qquad (k=0, \cdots, n-1)$$

is a polynomial of ζ all of whose coefficients contain the derivatives $D^r F(z)$ with $|r|=k$ but do not contain those with $|r|\geqq k+1$, where $z\in M$ is arbitrary. Therefore we can see inductively that

$$D_t D_\zeta^n F(z)-D_\zeta^n D_t F(z) \tag{2.6}$$

contains the derivatives of F up to $(n-1)$-th order but do not contain those of n-th order. Since the positive integer n is the smallest one satisfying (2.2), this implies that (2.6) vanishes on the solution curve Γ. Here, if F is an integral of (1.1), then we have the identity $D_t F(z)=\langle JH_z, F_z\rangle=0$. Hence we have proved (2.5).

Next, to prove (2.4) we introduce a differential operator

$$D_H=\left\langle JH_z, \frac{\partial}{\partial z}\right\rangle.$$

We note that

$$\Phi(\zeta+\zeta_h,\ t)=D^n_{\zeta+\zeta_h}F(z(t))=(D_\zeta+\xi_h D_H)^n F(z(t))\,.$$

Here we obtain the identity

$$D_\zeta D_H-D_H D_\zeta=\left\langle\zeta,\ \left\langle JH_{zz},\ \frac{\partial}{\partial z}\right\rangle\right\rangle,$$

which defines the first order differential operator. Hence it follows that for any $z\in M$

$$(D_\zeta+\xi_h D_H)^n F(z)=\sum_{k=0}^{n}\binom{n}{k}\xi_h^k D_\zeta^{n-k}D_H^k F(z)+\cdots,$$

where the remainder terms contain the derivatives of $F(z)$ up to $(n-1)$-th order only. Similarly as above, this implies that the remainder terms vanish on the solution curve Γ. Then, noting that $D_H F(z)$ vanishes identically, we have

$$(D_\zeta+\xi_h D_H)^n F(z(t))=D_\zeta^n F(z(t))\,,$$

which leads to (2.4). This completes the proof. Q. E. D.

Now we consider the Hamiltonian system (1.1) together with (1.2). The system is written as

$$\dot q_k=p_k\,,\quad p_k=-V_{q_k}\qquad(k=1,\ 2)\,. \tag{2.7}$$

In this paper, the particular solution $z(t)$ is essentially restricted to a special class such as $q_1(t)=p_1(t)=0$. Then we have

PROPOSITION 1. *Let* $\Gamma=\{z(t)=(q(t),\ p(t))\}$ *be a particular solution of* (1.1) *with* (1.2) *which is not an equilibrium point and satisfies* $q_1(t)=p_1(t)=0$. *Then the variational equation* (2.1) *is written as*

$$\ddot\xi_1+H_{q_1q_1}(z(t))\xi_1=0\,, \tag{2.8a}$$

$$\ddot\xi_2+H_{q_2q_2}(z(t))\xi_2=0\,, \tag{2.8b}$$

with $\eta_1=\dot\xi_1$, $\eta_2=\dot\xi_2$. *Moreover, equation* (2.8b) *admits a time-dependent integral* $dH(\zeta,\ t)=dH(\xi_2,\ \eta_2,\ t)$.

Proof. From the form (1.2) of H, it follows that $H_{pp}=I$ and $H_{qp}=H_{pq}=0$. Moreover, since $q_1=p_1=0$ along Γ, it follows from (2.7) that

$$\ddot p_1=-\frac{d}{dt}V_{q_1}(z(t))=-V_{q_1q_2}(z(t))\dot q_2=0\,.$$

This implies that $H_{q_1q_2}(z(t))=0$. To see this, it suffices to consider the system (2.7) locally and we can assume that the solution $z(t)$ is analytic in a domain of t-plane. Indeed, if $\dot q_2=0$ in the domain, then $p_2=0$ and Γ is an equilibrium point, which contradicts the assumption. Therefore there exists a neighborhood of t in which $\dot{q}_2 \neq 0$. Hence we have

$$H_{q_1q_2}(z(t)) = V_{q_1q_2}(0, q_2(t)) = 0$$

in the neighborhood of t. Since $V_{q_1q_2}(0, q_2)$ is a polynomial of q_2 alone, it follows that $V_{q_1q_2}(0, q_2)=0$ identically. Thus we have proved that $H_{q_1q_2}(0, q_2)=0$. Therefore we obtain (2.8a) and (2.8b) easily. It follows from $q_1=p_1=0$ that $dH(\zeta, t) = dH(\xi_2, \eta_2, t)$. Q.E.D.

Remark. In our main theorem (Theorem 2), we consider a family of particular solutions of (1.1) with (1.2) such that they are projected into a fixed complex line in q-space under the mapping $(q, p) \to q$ (see [A.2] in Section 3). Here a complex line in q-space is defined by $\mu_1 q_1 + \mu_2 q_2 = 0$ for some $(\mu_1, \mu_2) \in \boldsymbol{C}^2 \setminus \{0\}$. Then

$$(2.9) \qquad \begin{cases} q_1 = \dfrac{1}{\sqrt{\mu_1^2+\mu_2^2}}(\mu_1 x_1 - \mu_2 x_2), & q_2 = \dfrac{1}{\sqrt{\mu_1^2+\mu_2^2}}(\mu_2 x_1 + \mu_1 x_2), \\ p_1 = \dfrac{1}{\sqrt{\mu_1^2+\mu_2^2}}(\mu_1 y_1 - \mu_2 y_2), & p_2 = \dfrac{1}{\sqrt{\mu_1^2+\mu_2^2}}(\mu_2 y_1 + \mu_1 y_2) \end{cases}$$

defines a canonical transformation which takes the complex line into $x_1=0$ in x-space. Therefore, Proposition 1 can be applied to this situation.

The ξ_h in (2.4) can be considered as the tangential coordinate with respect to Γ. Let Γ be the particular solution given in Proposition 1, and we will use $\hat{\xi}_2$ in place of ξ_h. Then the corresponding normal coordinates are given by $(\xi_1, \eta_1, \hat{\eta}_2)$ which are determined by

$$(2.10) \qquad \zeta = \xi_1 \boldsymbol{e}_1 + \eta_1 \boldsymbol{f}_1 + \hat{\xi}_2 J H_z(z(t)) + \hat{\eta}_2 H_z(z(t)),$$

where $\boldsymbol{e}_1 = {}^t(1, 0, 0, 0)$, $\boldsymbol{f}_1 = {}^t(0, 0, 1, 0)$. Then we have

PROPOSITION 2. *Let $\Gamma = \{z(t)\}$ be a particular solution of* (1.1) *with* (1.2) *satisfying the same assumption as in Proposition* 1. *Assume that there exists an analytic integral $F(q, p)$ of* (1.1) *with* (1.2) *which induces the time-dependent integral $\Phi(\zeta, t)$ of* (2.1) *defined by* (2.3). *Then $\Phi(\zeta, t)$ is independent of $\hat{\xi}_2$, namely it is a polynomial of ξ_1, η_1 and $\hat{\eta}_2$. In particular, the integral $dH(\zeta, t)$ is given by*

$$(2.11) \qquad dH(\zeta, t) = [\{H_{q_2}(z(t))\}^2 + \{H_{p_2}(z(t))\}^2]\hat{\eta}_2.$$

Proof. In (2.4), put $\zeta = \xi_1 \boldsymbol{e}_1 + \eta_1 \boldsymbol{f}_1 + \hat{\eta}_2 H_z(z(t))$ and $\xi_h = \hat{\xi}_2$, then we can see that $\Phi(\zeta, t)$ is independent of $\hat{\xi}_2$. Hence $\Phi(\zeta, t)$ is a homogeneous polynomial of ξ_1, η_1 and $\hat{\eta}_2$. Moreover, (2.11) is obtained easily. Q.E.D.

Since $dH(\zeta, t)$ is an integral of (2.8b), we can solve (2.8b) for $\hat{\eta}_2$ explicitly and then also for the tangential coordinate $\hat{\xi}_2$.

Equation (2.8a) is called the *reduced equation in normal variations* (or simply *reduced equation*). In Ziglin [16, 17], it is essential to consider the monodromy group of the reduced equation, which is defined as follows.

Consider loops (closed paths) in Γ having a common base point z_0. Since Γ is parametrized by the time variable $t \in \boldsymbol{C}$, a loop in Γ corresponds to a path in t-plane. In what follows, the analytic continuation along a loop in Γ is considered as the analytic continuation along the path in t-plane. Let $\{\varphi(t), \phi(t)\}$ be a fundamental system of solutions of the reduced equation (2.8a). If $\tilde{\varphi}(t)$ and $\tilde{\phi}(t)$ denote the analytic continuation of $\varphi(t)$ and $\phi(t)$ along a loop $\gamma \subset \Gamma$ respectively, then $\{\tilde{\varphi}(t), \tilde{\phi}(t)\}$ also defines a fundamental system of solutions of (2.8a). Therefore there exists a 2×2 constant matrix $C(\gamma)$ such that

$$(\tilde{\varphi}(t), \tilde{\phi}(t))=(\varphi(t), \phi(t))C(\gamma) .$$

Here we note that equation (2.8a) is a Hamiltonian system with the Hamiltonian $H(\xi_1, \eta_1, t)=(1/2)(\eta_1^2+H_{q_1q_1}(z(t))\xi_1^2)$. Since $C(\gamma)$ is defined by the analytic continuation of the solution of (2.8a), it is symplectic, namely in this case $C(\gamma)\in \mathrm{SL}(2, \boldsymbol{C})$ (i.e., $\det C(\gamma)=1$). If we fix the base point z_0 and the fundamental system $\{\varphi(t), \phi(t)\}$, then this matrix $C(\gamma)$ depends only on the homotopy class $[\gamma]$ of γ. Hence the correspondence $\rho : [\gamma] \to C(\gamma)$ defines a group homomorphism $\rho : \pi_1(\Gamma, z_0) \to \mathrm{SL}(2, \boldsymbol{C})$, where $\pi_1(\Gamma, z_0)$ is the fundamental group of Γ. The image $G= \rho(\pi_1(\Gamma, z_0))$ is called the *monodromy group* of the reduced equation (2.8a), and its element is called the *monodromy matrix*. The following example gives the situation to be considered in Sections 3 and 4.

EXAMPLE. Assume that the function $Q(t)=H_{q_1q_1}(z(t))$ in (2.8a) is a nonconstant (non-trivial) elliptic function of t possessing only one singular point (pole) in a period parallelogram Ω. Then the phase curve Γ is identified as the real 2-dimensional punctured torus. Let (ω_1, ω_2) be a pair of basic periods of $Q(t)$ which determines the period parallelogram Ω. Then equation (2.8a) is so-called *Hill's equation* [13] with respect to each period ω_1 and ω_2. Then there exists a constant matrix g_1 and g_2 satisfying

$$(\varphi(t+\omega_k), \phi(t+\omega_k))=(\varphi(t), \phi(t))g_k \qquad (k=1, 2) .$$

The monodromy group is generated by these two matrices (linear transformations) g_1 and g_2. This matrix g_k $(k=1, 2)$ is also called the monodromy matrix with respect to the period ω_k $(k=1, 2)$. It is to be noted that the commutator $g_*= g_1g_2g_1^{-1}g_2^{-1}$ gives the monodromy matrix corresponding to the loop around the singular point.

Let us denote the monodromy group by G. The following lemma plays a fundamental role in Ziglin [16].

LEMMA 3. *Let* $\Gamma=\{z(t)=(q(t), p(t))\}$ *be a particular solution of* (1.1) *with* (1.2) *which is not an equilibrium point and satisfies* $q_1(t)=p_1(t)=0$. *If the system* (1.1) *has an integral which is analytic in a neighborhood of* Γ *and functionally independent of* H, *then there exists a homogeneous polynomial of* ξ_1 *and* η_1 *such that it is invariant under the action of the monodromy group* G.

Proof. Let $F(q, p)$ be an integral of (1.1) which is functionally independent of $H(q, p)$, and let $\Phi(\xi_1, \eta_1, \hat{\eta}_2, t)$ be an integral of (2.1) induced by $F(q, p)$. Since dH of the form (2.11) is an integral of (2.8b), we can eliminate $\hat{\eta}_2$ by using the relation $dH(\zeta, t)=\text{const}$. Therefore Φ is reduced to a polynomial of ξ_1 and η_1. This is an integral of (2.8a). However, this may be a constant function in general. Therefore, we need to take a suitable polynomial of H and F in place of F in the above discussions. Then we can obtain an integral of (2.8a) which is a non-constant polynomial of ξ_1 and η_1. This is possible because F is functionally independent of H. We omit the details (see [16]). In particular, this polynomial is invariant under the analytic continuation of the solutions of (2.8a) along a loop in Γ. Let $\Psi(\xi_1, \eta_1, t)=\sum_{k+l\leq n}\phi_{kl}(t)\xi_1^k\eta_1^l$ be the integral of (2.8a). Here we note that the coefficients $\phi_{kl}(t)$ are single-valued functions on the Riemann surface Γ. If we fix the base point $z_0\in\Gamma$ of the loop with its coordinate t_0, then $\Psi(\xi_1, \eta_1, t_0)$ gives a polynomial of ξ_1, η_1 which is invariant under the action of the monodromy group G. Since any $g\in G$ is a linear transformation, each homogeneous part of $\Psi(\xi_1, \eta_1, t_0)$ is invariant under the action of the monodromy group G, and therefore gives the desired polynomial. This completes the proof. Q. E. D.

To state Ziglin's theorem, we need the following definition.

DEFINITION. A transformation $g_0\in G$ is said to be *non-resonant* if any eigenvalue λ of g_0 satisfies that $\lambda^n\neq 1$ for any nonzero integer n.

Now, Ziglin's theorem is stated in our situation as follows:

THEOREM 1. (Ziglin [16]). *Suppose that there exists a particular solution* $\Gamma=\{z(t)\}$ *of* (1.1) *with* (1.2) *which is not an equilibrium point and satisfies* $q_1(t)=p_1(t)=0$. *Assume that the system* (1.1) *has an integral which is analytic in a neighborhood of* Γ *and functionally independent of* H. *Then, if there exists a non-resonant transformation* g_0 *in the monodromy group* G, *any transformation in* G *commutes or permutes the eigenspaces of* g_0.

COROLLARY. *Let* g_1 *and* g_2 *be elements of* G. *If* g_1 *is non-resonant, then the commutator* $g_*=g_1g_2g_1^{-1}g_2^{-1}$ *is equal either to the identity or to* g_1^2. *Similarly, if* g_2 *is non-resonant, then either* g_* *is the identity or* $g_*=g_2^{-2}$.

Remark. Let E_1 and E_2 denote the eigenspaces of g_0. Then, "commute" means that g transforms E_1 into E_1 and E_2 into E_2. On the other hand, "permute" means that g transforms E_1 into E_2 and E_2 into E_1.

Proof of Theorem 1. Assume that there exists a non-resonant transformation $g_0\in G$. Let $\Psi(\xi, \eta)$ be the homogeneous polynomial in Lemma 3, where we use ξ, η in place of ξ_1, η_1. Then there exists a symplectic base of $\boldsymbol{C}^2$ such that

$$g_0=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}.$$

Let $\Psi(\xi, \eta)=\sum_{k+l=N}\phi_{kl}\xi^k\eta^l$, where $\phi_{kl}\in \boldsymbol{C}$. Then the invariance of Ψ under G leads to

$$\sum_{k+l=N}\phi_{kl}\xi^k\eta^l=\sum_{k+l=N}\phi_{kl}(\lambda^k\mu^l)\xi^k\eta^l.$$

Here $\lambda\mu=1$ because $\det g_0=1$, and λ, μ are not roots of unity because g_0 is non-resonant. Therefore this implies that $\Psi(\xi, \eta)=\phi_{ss}(\xi\eta)^s$ for some positive integer s $(2s=N)$. If we set

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any $g\in G$, we have by the invariance of Ψ under g that

$$(a\xi+c\eta)^s(b\xi+d\eta)^s=(\xi\eta)^s.$$

Then it follows that $ab=cd=0$. Since $\det g=1$, we have $ad-bc=1$. Therefore we obtain the following two cases: (i) $b=c=0$ and $ad=1$, or (ii) $a=d=0$ and $bc=-1$. These cases satisfy the above equation, where s is an even integer for the case (ii). The transformation g commutes the eigenspaces of g_0 in the case (i), and on the other hand in the case (ii) g permutes those of g_0. This completes the proof. Q. E. D.

Proof of Corollary. If g_1 is non-resonant, let $g_0=g_1$ and $g=g_2$ in the above proof of Theorem 1. Then we can prove that $g_*=I$ (identity) in the case (i), and $g_*=g_1^2$ in the case (ii). The proof is similar when g_2 is non-resonant. This completes the proof. Q. E. D.

3. Main Theorem.

We are now in a position to state our main theorem. Let us consider the complex Hamiltonian system (1.1) with (1.2) under the following assumptions:

[A.1] There exists a family of non-trivial doubly periodic orbits Γ_h (i. e., elliptic functions of complex time) of (1.1), which depend analytically on a parameter h varying on (h_0, h_1).

[A.2] For any $h\in(h_0, h_1)$, Γ_h is projected into a fixed complex line in q-space under the mapping $(q, p)\to q$.

Here the complex line is defined by $\mu_1q_1+\mu_2q_2=0$. Then, since $\dot q_k=p_k$ from (2.7), it follows that $\mu_1p_1+\mu_2p_2=0$. Therefore, carrying out the canonical transformation (2.9), it transforms into $x_1=y_1=0$. Let $z_h(t)$ denote the coordinate of Γ_h. By Proposition 1, we obtain a family of reduced equations

$$\ddot\xi_1+Q_h(t)\xi_1=0\,;\quad Q_h(t)=H_{x_1x_1}(z_h(t)) \tag{3.1}$$

with $\eta_1=\dot\xi_1$. Under the assumptions [A.1] and [A.2], the coefficients $Q_h(t)$ are elliptic functions of t. Suppose that $Q_h(t)$ is non-constant, and let $(\omega_1(h), \omega_2(h))$ be a pair of basic periods which determines a period parallelogram. We assume

[A.3] For any $h \in (h_0, h_1)$, the coefficient $Q_h(t)$ in (3.1) has only one singular point (pole) in the period parallelogram. The eigenvalues of the monodromy matrix around it are independent of h.

Let $g_1(h)$, $g_2(h)$ denote the monodromy matrices corresponding to the period $\omega_1(h)$ and $\omega_2(h)$ respectively, and let $g_*(h)$ be the monodromy matrix around the singular point. Then our main theorem is stated as follows:

THEOREM 2. *Let the Hamiltonian system* (1.1) *with* (1.2) *satisfy* [A.1], [A.2] *and* [A.3]. *Assume that the system* (1.1) *has an integral which is analytic in a neighborhood of the family* $\{\Gamma_h\}$ *and functionally independent of* H. *Then either* $g_*(h)$ *is the identity for any* $h \in (h_0, h_1)$, *or the traces of both* $g_1(h)$ *and* $g_2(h)$ *are constant functions in* $h \in (h_0, h_1)$.

Remarks. (i) Let λ, μ be eigenvalues of g_k $(k=1, 2)$. Then, since $\lambda\mu=1$, the invariance of the trace of g_k is equivalent to that of eigenvalues of g_k.
(ii) This theorem shows that integrability implies non-branching of solutions of the reduced equations when the eigenvalues of both $g_1(h)$ and $g_2(h)$ are not constant.
(iii) In the above, h is considered as a real parameter. However, the same assertion as in Theorem 2 holds also when h is considered as a complex parameter.

Proof. Assume that the trace of $g_1(h)$ varies with h. Then there exists a dense subset S of (h_0, h_1) such that $g_1(h)$ is non-resonant for any $h \in S$. By the corollary to Theorem 1, it follows that $g_*(h)=I$ (identity) or $g_*(h)=g_1^2(h)$ for any $h \in S$. Suppose that $g_*(\hat{h}) \neq I$ holds for some $\hat{h} \in (h_0, h_1)$. Then, since the components of $g_*(h)$ are analytic functions of $h \in (h_0, h_1)$ because of [A.1], $g_*(h) \neq I$ holds in a neighborhood of $\hat{h}$. Hence we have $g_*(h)=g_1^2(h)$ for any $h \in S'$, where S' is the intersection of the neighborhood of $\hat{h}$ with S. Since the trace of $g_1(h)$ varies with h, this implies that the trace of $g_*(h)$ also varies with h. This contradicts the assumption [A.3]. Hence we have $g_*(h)=I$ for any $h \in (h_0, h_1)$. If we assume that the trace of $g_2(h)$ varies with h, we have the same conclusion by the similar way. This completes the proof. Q. E. D.

Generally speaking, the assumptions [A.1], [A.2] and [A.3] are satisfied if the potential $V(q_1, q_2)$ is a third- or fourth-degree polynomial. As an example, we apply Theorem 2 to the Hénon-Heiles system in the next section, where the parameter h corresponds to the energy value of Γ_h.

4. Application to Hénon-Heiles system.

In this section, we apply Theorem 2 to the Hénon-Heiles system. Our main purpose is to prove the following result.

THEOREM 3. *Assume that* $a=b$ $(\neq 0)$ *in the Hénon-Heiles Hamiltonian* (1.4).

Then the system (1.1) *has an entire integral which is functionally independent of* H *only if* $c/d=0$, $1/6$, 1 *or* $1/2$.

We prove this theorem in several steps. We begin without assuming $a=b$ in (1.4). The assumption $a=b$ will be essential only for the final step (v).

(i) Families of doubly periodic orbits.

Let us consider the Hénon-Heiles Hamiltonian (1.4) without assuming $a=b$. The corresponding Hamiltonian system is given by

$$\dot{q}_1=p_1\,,\quad \dot{p}_1=-aq_1-2cq_1q_2\,, \tag{4.1a}$$

$$\dot{q}_2=p_2\,,\quad \dot{p}_2=-bq_2-cq_1^2-dq_2^2\,. \tag{4.1b}$$

By setting $q_1=p_1=0$, this system is reduced to (4.1b) with $q_1=0$. Since the Hamiltonian H is an integral, the phase curve of this system is given by

$$\frac{1}{2}p_2^2+V(0,\,q_2)=h\,,$$

where h is the energy parameter. This leads to

$$\frac{dq_2}{dt}=\sqrt{2(h-V(0,\,q_2))}\,.$$

Let $\{\alpha_1,\,\alpha_2,\,\alpha_3\}=\{\alpha_i,\,\alpha_j,\,\alpha_l\}$ be a set of roots of the equation $V(0,\,q_2)=h$. Then it follows that

$$\int_{\alpha_i}^{q_2}\frac{dq_2}{\sqrt{(q_2-\alpha_i)(q_2-\alpha_j)(q_2-\alpha_l)}}=\int_0^t\sqrt{-\frac{2d}{3}}\,dt\,.$$

Here, setting $q_2=\alpha_i+(\alpha_j-\alpha_i)\xi^2$ we have

$$\int_{\alpha_i}^{q_2}\frac{dq_2}{\sqrt{(q_2-\alpha_i)(q_2-\alpha_j)(q_2-\alpha_l)}}=\frac{2}{\sqrt{\alpha_l-\alpha_i}}\int_0^{\xi}\frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}\,,$$

where

$$k^2=\frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_l}\,.$$

Hence we have

$$\int_0^{\xi}\frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}=\sqrt{\frac{d(\alpha_i-\alpha_l)}{6}}\,t\equiv\tau\,.$$

This implies that ξ is Jacobi's elliptic function $sn(\tau,\,k)$, where k is called the modulus of $sn(\tau,\,k)$. Since $q_2=\alpha_i+(\alpha_j-\alpha_i)\xi^2$, we have thus families of doubly periodic orbits $\Gamma_h(\alpha_i,\,\alpha_j)$ on $H^{-1}(h)$ whose q-coordinates are given as follows:

$$\Gamma_h(\alpha_i,\,\alpha_j):\begin{cases} q_1(t,\,h)=0\,,\quad q_2(t,\,h)=\alpha_i+(\alpha_j-\alpha_i)sn^2\tau\,,\\ \tau=\beta t\,,\quad \beta=\sqrt{\dfrac{d(\alpha_i-\alpha_l)}{6}}\,,\quad k^2=\dfrac{\alpha_i-\alpha_j}{\alpha_i-\alpha_l}\,.\end{cases} \tag{4.2}$$

In the above, the roots α_i, α_j, α_l are chosen arbitrary from α_1, α_2, α_3 in such a way as $\alpha_i \neq \alpha_l$, and they are expressed as

$$\tag{4.3}\begin{aligned}\alpha_1 &= \frac{b}{2d}\Big(2\cos\frac{\theta}{3}-1\Big),\\ \alpha_2 &= -\frac{b}{2d}\Big(\sqrt{3}\sin\frac{\theta}{3}+\cos\frac{\theta}{3}+1\Big),\\ \alpha_3 &= \frac{b}{2d}\Big(\sqrt{3}\sin\frac{\theta}{3}-\cos\frac{\theta}{3}-1\Big),\end{aligned}$$

where

$$\cos\theta = \frac{12d^2}{b^3}h-1.$$

Here we note that, as h varies from 0 to $b^3/6d^2$, θ varies from π to 0. These families $\{\Gamma_h(\alpha_i, \alpha_j)\}$ satisfy [A.1] and [A.2].

The elliptic function $sn^2\tau$ has a pair of basic periods $(2K, 2K+2iK')$ with respect to $\tau=\beta t$, and it has only one pole of order 2 at $\tau=2K+iK'$ in the period parallelogram. Here $K=K(k)$ is the complete elliptic integral of the first kind and $K'=K(k')$ $(k'=\sqrt{1-k^2})$ is the complementary complete elliptic integral of first kind.

In particular when $a=b$, there exists families of doubly periodic orbits other than $\Gamma_h(\alpha_i, \alpha_j)$. We assume that $c\neq 0$, $d/c\neq 2, 3$ in addition to $a=b$. If we search for solutions moving on a complex line

$$q_1=\mu q_2, \quad p_1=\mu p_2,$$

then by the compatibility condition for (4.1a) and (4.1b) we must have

$$\tag{4.4}\mu=\pm\sqrt{2-\frac{d}{c}}.$$

The canonical transformation (2.9) with $\mu_2/\mu_1=\mu$ takes (1.4) into

$$\begin{aligned}H&=\frac{1}{2}(y_1^2+y_2^2)+\frac{a}{2}(x_1^2+x_2^2)+\frac{1}{\sqrt{1+\mu^2}}\Big\{\frac{c+d}{3}\mu x_1^3-(c-d)x_1^2x_2+\frac{2c}{3}x_2^3\Big\}\\ &\equiv\frac{1}{2}|y|^2+U(x),\end{aligned}$$

and the corresponding Hamiltonian system is given by

$$\dot{x}_k=y_k, \quad \dot{y}_k=-U_{x_k} \qquad (k=1, 2).$$

Similarly as above, by setting $x_1=y_1=0$ we obtain the desired orbits $\Lambda_h(\alpha_i, \alpha_j)$ on $H^{-1}(h)$ such that the x-coordinates are given as follows:

$$\tag{4.5}\Lambda_h(\alpha_i, \alpha_j): \begin{cases} x_1(t, h)=0, \quad x_2(t, h)=\alpha_i+(\alpha_j-\alpha_i)sn^2\tau,\\ \tau=\beta t, \quad \beta=\sqrt{\dfrac{e(\alpha_i-\alpha_l)}{6}}, \quad k^2=\dfrac{\alpha_i-\alpha_j}{\alpha_i-\alpha_l}.\end{cases}$$

Here $\{\alpha_1, \alpha_2, \alpha_3\}=\{\alpha_i, \alpha_j, \alpha_l\}$ is a set of roots of the equation $U(0, x_2)=h$, and they are expressed as (4.3) with replacing b and d by a and e respectively, and h is assumed to vary from 0 to $a^3/6e^2$, where

$$e=\frac{2c}{\sqrt{1+\mu^2}}=\frac{2c}{\sqrt{3-\frac{d}{c}}}. \tag{4.6}$$

(ii) The reduced equations in normal variations.

We consider the reduced equations in normal variations along the solutions stated above. In the following, let us use the time variable $\tau=\beta t$ in place of t. Then the reduced equations (3.1) are written as

$$\xi_1''+Q(\tau, h)\xi_1=0, \tag{4.7}$$

where

$$\begin{aligned} &Q(\tau, h)=\beta^{-2}\{a+2cq_2(t, h)\} && \text{for } \Gamma_h(\alpha_i, \alpha_j), \\ &Q(\tau, h)=\beta^{-2}\left\{a-\frac{2(c-d)}{\sqrt{1+\mu^2}}x_2(t, h)\right\} && \text{for } \Lambda_h(\alpha_i, \alpha_j), \end{aligned} \tag{4.8}$$

and ξ_1'' indicates $d^2\xi_1/d\tau^2$. For our purpose, we take $\Gamma_h(\alpha_2, \alpha_3)$ and $\Lambda_h(\alpha_2, \alpha_3)$. Then we have

$$\kappa\equiv k^2=\frac{2\tan\frac{\theta}{3}}{\sqrt{3}+\tan\frac{\theta}{3}},$$

and so we can represent $Q(\tau, h)$ as function of τ and κ, which will be denoted by $Q(\tau, \kappa)$. Here, as h varies from 0 to $b^3/6d^2$ (or $a^3/6e^2$), κ varies from 1 to 0. Indeed, the reduced equations along $\Gamma_h(\alpha_2, \alpha_3)$ and $\Lambda_h(\alpha_2, \alpha_3)$ are expressed as follows:

$$\begin{cases} \xi_1''+Q(\tau, \kappa)\xi_1=0; \\ Q(\tau, \kappa)=4\chi\sqrt{1-\kappa+\kappa^2}+4\gamma(1+\kappa-\sqrt{1-\kappa+\kappa^2})-12\gamma\kappa sn^2(\tau, \kappa), \end{cases} \tag{4.9}$$

where

$$\begin{aligned} &\chi=-\frac{a}{b}+2\frac{c}{d}, \quad \gamma=\frac{c}{d} && \text{for } \Gamma_h(\alpha_2, \alpha_3), \\ &\chi=\frac{d}{c}-2, \quad \gamma=\frac{1}{2}\left(\frac{d}{c}-1\right) && \text{for } \Lambda_h(\alpha_2, \alpha_3). \end{aligned} \tag{4.10}$$

Remark. For families other than $\Gamma_h(\alpha_2, \alpha_3)$ and $\Lambda_h(\alpha_2, \alpha_3)$, we obtain the corresponding reduced equations of the form (4.9) with changes of (4.10). They are not needed to prove Theorem 3 and so we omit them.

(iii) Eigenvalues of the commutator $g_*(h)$.

In the reduced equations (4.9), the coefficient $Q(\tau, \kappa)$ has a pair of basic periods $(2K(\kappa), 2K(\kappa)+2iK'(\kappa))$ which determines a period parallelogram. $Q(\tau, \kappa)$ has only one pole of order 2 at $\tau=2K+iK'$ in the period parallelogram. Let $g_1(\kappa)$ and $g_2(\kappa)$ denote the monodromy matrices corresponding to the period $2K(\kappa)$ and $2K(\kappa)+2iK'(\kappa)$ respectively. Then the monodromy group $G(\kappa)$ is generated by $g_1(\kappa)$ and $g_2(\kappa)$. Here, since the correspondence of κ to h is one-to-one, we used the notation such as $g_1(\kappa)$ in place of $g_1(h)$, etc.

It is important that the eigenvalues of the commutator

$$g_*(\kappa)=g_1(\kappa)g_2(\kappa)g_1^{-1}(\kappa)g_2^{-1}(\kappa)$$

are given explicitly because $\tau=2K+iK'$ is a regular singular point for (4.9) (see [5]). Indeed, if we consider the Laurent expansion of $Q(\tau, \kappa)$ at $\tau=2K+iK'$, the coefficient of $(\tau-2K-iK')^{-2}$ is $-12\gamma\kappa\kappa^{-1}=-12\gamma$. Therefore the indicial equation of (4.9) at $2K+iK'$ is

$$\sigma(\sigma-1)-12\gamma=0\,.$$

Hence we have

PROPOSITION 3. *The eigenvalues* λ *of the monodromy matrix* $g_*(\kappa)$ *for* (4.9) *are independent of* κ *and given by*

$$\lambda=\exp(2\pi i\sigma)\,; \quad \sigma=\frac{1}{2}(1\pm\sqrt{1+48\gamma})\,. \tag{4.11}$$

Thus all the conditions of Theorem 2 have been proved to be satisfied.

(iv) Dependence of $\operatorname{tr} g_1(\kappa)$ on κ.

If the potential $V(q)$ is a homogeneous polynomial, the eigenvalues of $g_1(\kappa)$ and $g_2(\kappa)$ can be expressed explicitly in general (see [14, 15]). On the other hand, if $V(q)$ is non-homogeneous, we cannot know the explicit representations of the eigenvalues of $g_1(\kappa)$ nor $g_2(\kappa)$. However, we have only to know the variance of the eigenvalues of $g_1(\kappa)$ or $g_2(\kappa)$ with κ. Indeed we can give a sufficient condition for $\operatorname{tr} g_1(\kappa)$ to vary with κ. It is $(d^2/d\kappa^2)\operatorname{tr} g_1(\kappa)|_{\kappa=0}\neq 0$ in the following proposition.

PROPOSITION 4. *Let* $\operatorname{tr} g_1(\kappa)$ *denote the trace of the monodromy matrix* $g_1(\kappa)$ *for* (4.9). *Then we have*

$$\operatorname{tr} g_1(\kappa)|_{\kappa=0}=2\cos(2\pi\sqrt{\chi})\,, \quad \frac{d}{d\kappa}\operatorname{tr} g_1(\kappa)|_{\kappa=0}=0\,, \tag{4.12}$$

$$\frac{d^2}{d\kappa^2}\operatorname{tr} g_1(\kappa)|_{\kappa=0}=\begin{cases} \dfrac{-\pi^2\sin 2\pi\sqrt{\chi}}{2\pi\sqrt{\chi}}\left(\dfrac{9\gamma^2}{1-4\chi}+\dfrac{15}{4}\chi-\dfrac{9}{2}\gamma\right) & \left(\chi\neq 0, \dfrac{1}{4}\right), \\ -9\pi^2\gamma\left(\gamma-\dfrac{1}{2}\right) & (\chi=0)\,, \\ -\dfrac{9\pi^2}{2}\gamma^2 & \left(\chi=\dfrac{1}{4}\right). \end{cases} \tag{4.13}$$

Proof. For the convenience of discussions, let us carry out a change of time-scale from τ to u by $\tau=2K(\kappa)u$. Then instead of (4.9), we consider the linear equation

$$\frac{d^2\xi_1}{du^2}+P(u,\kappa)\xi_1=0\,;\quad P(u,\kappa)=4K^2(\kappa)Q(\tau,\kappa)\,, \tag{4.14}$$

where $Q(\tau,\kappa)$ is given in (4.9). The coefficient $P(u,\kappa)$ has a period 1 in u. The monodromy matrix $g_1(\kappa)$ is given by that of (4.14) corresponding to the period 1.

Now we note that there exists a fundamental system of solutions $\{\varphi(u,\kappa), \psi(u,\kappa)\}$ of (4.14) such that

$$\begin{cases} \varphi(0,\kappa)=1\,, & \psi(0,\kappa)=0\,,\\ \dot\varphi(0,\kappa)=0\,, & \dot\psi(0,\kappa)=1 \end{cases} \tag{4.15}$$

for any $\kappa\in[0,1]$. Here and in what follows the dot indicates the differentiation with respect to u. Then we have

$$g_1(\kappa)=\begin{pmatrix}\varphi(1,\kappa) & \psi(1,\kappa)\\ \dot\varphi(1,\kappa) & \dot\psi(1,\kappa)\end{pmatrix},$$

$$\operatorname{tr} g_1(\kappa)=\varphi(1,\kappa)+\dot\psi(1,\kappa)\,. \tag{4.16}$$

Since $P(u,\kappa)$ is analytic in κ at $\kappa=0$, the solutions φ and ψ are also analytic in κ at $\kappa=0$. Let $\varphi(u,\kappa)$, $\psi(u,\kappa)$ and $P(u,\kappa)$ have the following Taylor expansions at $\kappa=0$:

$$\varphi(u,\kappa)=\varphi_0(u)+\varphi_1(u)\kappa+\varphi_2(u)\kappa^2+\cdots,$$

$$\psi(u,\kappa)=\psi_0(u)+\psi_1(u)\kappa+\psi_2(u)\kappa^2+\cdots,$$

$$P(u,\kappa)=P_0(u)+P_1(u)\kappa+P_2(u)\kappa^2+\cdots.$$

Here the expansions of the form

$$\sqrt{1-\kappa+\kappa^2}=1-\frac{1}{2}\kappa+\frac{3}{8}\kappa^2+\cdots,$$

$$sn(\tau,\kappa)=\sin(\pi u)+\frac{1}{4}\kappa\sin(\pi u)\cos^2(\pi u)+\cdots,$$

$$K(\kappa)=\frac{\pi}{2}\left(1+\frac{1}{4}\kappa+\frac{9}{64}\kappa^2+\cdots\right)$$

hold [2], and then we have

$$P_0(u)=4\pi^2\chi\,,$$

$$P_1(u)=6\pi^2\gamma\cos(2\pi u)\,, \tag{4.17}$$

$$P_2(u)=\pi^2\left\{\frac{15}{8}\chi-\frac{9}{4}\gamma+3\gamma\cos(2\pi u)+\frac{3}{4}\gamma\cos(4\pi u)\right\}.$$

Equation (4.14) implies

$$\begin{aligned}&\ddot{\varphi}_0+P_0(u)\varphi_0=0\,,\quad \ddot{\varphi}_n+P_0(u)\varphi_n+\sum_{k=1}^{n}P_k(u)\varphi_{n-k}=0\,,\\ &\ddot{\phi}_0+P_0(u)\phi_0=0\,,\quad \ddot{\phi}_n+P_0(u)\phi_n+\sum_{k=1}^{n}P_k(u)\phi_{n-k}=0\end{aligned} \tag{4.18}$$

for $n=1, 2, \cdots$.

In the following, our purpose is to solve (4.18) for $\varphi_n(u)$ and $\phi_n(u)$ for $n=0, 1, 2$ under the initial conditions (4.15), namely

$$\varphi_0(0)=1\,,\quad \dot{\varphi}_0(0)=0\,,\quad \phi_0(0)=0\,,\quad \dot{\phi}_0(0)=1\,,$$

$$\varphi_n(0)=\dot{\varphi}_n(0)=\phi_n(0)=\dot{\phi}_n(0)=0 \qquad (n=1, 2, \cdots)\,.$$

At first it follows from (4.18) with $n=0$ that

$$\begin{cases}\varphi_0(u)=\cos(\sqrt{P_0}u)\,,\quad \phi_0(u)=\dfrac{1}{\sqrt{P_0}}\sin(\sqrt{P_0}u) & (P_0\neq 0)\,,\\ \varphi_0(u)=1\,,\quad \phi_0(u)=u & (P_0=0)\,.\end{cases} \tag{4.19}$$

Here we note that $P_0=4\pi^2\chi$, and then we have

$$\operatorname{tr} g_1(\kappa)|_{\kappa=0}=\varphi_0(1)+\dot{\phi}_0(1)=2\cos(2\pi\sqrt{\chi})\,. \tag{4.20}$$

Next, by the method of variation of constants we can solve (4.18) for $n=1, 2, \cdots$ inductively as follows:

$$\varphi_n(u)=\int_0^u\phi_0(v-u)\sum_{k=1}^{n}P_k(v)\varphi_{n-k}(v)dv\,, \tag{4.21}$$

$$\phi_n(u)=\int_0^u\phi_0(v-u)\sum_{k=1}^{n}P_k(v)\phi_{n-k}(v)dv\,. \tag{4.22}$$

Then for $n=1$ we have

$$\begin{aligned}\varphi_1(u)+\dot{\phi}_1(u)&=\int_0^u P_1(v)\left\{\varphi_0(v)\phi_0(v-u)-\phi_0(v)\varphi_0(v-u)\right\}dv\\ &=-\phi_0(u)\int_0^u P_1(v)dv\,.\end{aligned}$$

Hence because of (4.17) we have

$$\frac{d}{d\kappa}\operatorname{tr} g_1(\kappa)|_{\kappa=0}=\varphi_1(1)+\dot{\phi}_1(1)=-\phi_0(1)\int_0^1 P_1(v)dv=0\,. \tag{4.23}$$

Similarly, for $n=2$ the formulas (4.21) and (4.22) give

$$\varphi_2(u)+\dot{\phi}_2(u)=\int_0^u \{P_1(v)R_1(v,\ u)+P_2(v)R_0(v,\ u)\}\,dv\,,$$

where

$$R_j(v,\ u)=\varphi_j(v)\phi_0(v-u)-\phi_j(v)\varphi_0(v-u) \qquad (j=0,\ 1)\,.$$

Here we have

$$R_0(v,\ u)=-\phi_0(u)\,,$$

$$R_1(v,\ u)=\int_0^v P_1(s)\phi_0(s-v)\{\varphi_0(s)\phi_0(v-u)-\phi_0(s)\varphi_0(v-u)\}\,ds\,.$$

Then, since $(d^2/d\kappa^2)\operatorname{tr} g_1(\kappa)|_{\kappa=0}=2(\varphi_2(1)+\dot{\phi}_2(1))$, we obtain (4.13) by a direct calculation using (4.17) and (4.19). This calculation is elementary and so we omit the details. Thus we have proved (4.12) and (4.13). Q. E. D.

(v) *Proof of Theorem* 3.

If $(d^2/d\kappa^2)\operatorname{tr} g_1(\kappa)|_{\kappa=0}\neq 0$, then the integrability implies $\lambda=1$ in Proposition 3. This gives a criterion for claiming the non-integrability of Hénon-Heiles system. As an example we prove Theorem 3.

Proof. Consider the families of doubly periodic orbits $\Gamma_h=\Gamma_h(\alpha_2,\ \alpha_3)$ and $\Lambda_h=\Lambda_h(\alpha_2,\ \alpha_3)$. It follows from $a=b$ that $\chi=2\gamma-1$ in (4.10). Then, from Proposition 4 it follows that

$$\frac{d^2}{d\kappa^2}\operatorname{tr} g_1(\kappa)|_{\kappa=0}=\begin{cases} \left\{\dfrac{-15\pi^2\sin(2\pi\sqrt{2\gamma-1})}{8\pi\sqrt{2\gamma-1}}\right\}\dfrac{(2\gamma-1)(2\gamma-5)}{(8\gamma-5)} & \left(\gamma\neq\dfrac{1}{2},\ \dfrac{5}{8}\right), \\ 0 & \left(\gamma=\dfrac{1}{2}\right), \\ -\dfrac{225\pi^2}{128} & \left(\gamma=\dfrac{5}{8}\right). \end{cases}$$

Assume that the system is integrable. We note that if $\gamma<1/2$, this quantity does not vanish and then $\lambda=(1/2)(1+\sqrt{1+48\gamma})$ must be an integer. Let us take the family $\{\Gamma_h\}$. Then, if $c/d<1/2$ it follows that $\sqrt{1+48c/d}$ is a positive odd integer. This implies that $c/d=1/6$ or 0. Next we take the family $\{\Lambda_h\}$. Then, if $(1/2)(d/c-1)<1/2$ it follows that $\sqrt{1+24(d/c-1)}$ is a positive odd integer. This implies that $c/d=1$ or 3/4 if $c/d>1/2$. Here, the case $c/d=3/4$ is not integrable one. Indeed, if $\gamma=3/4$ then we have $(d^2/d\kappa^2)\operatorname{tr} g_1(\kappa)|_{\kappa=0}\neq 0$ and $(1/2)(1+\sqrt{1+48\gamma})$ is not integer. Thus we have proved that $c/d=0$, 1/6, 1 or 1/2 if the system is integrable. Q. E. D.

Remark. If c/d is 0 or 1, the system is integrable. The case $c/d=1/6$ is also known as integrable one. On the other hand, the case $c/d=1/2$ i seemed to be non-integrable [3]. Our method cannot prove the non-integrability of this case.

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