T. INABA, T. NISHIMORI, M. TAKAMURA AND N. TSUCHIYA
KODAI MATH. J.
8 (1985), 112—119

OPEN MANIFOLDS WHICH ARE NON-REALIZABLE AS LEAVES

BY TAKASHI INABA, TOSHIYUKI NISHIMORI, MASASHI TAKAMURA AND NOBUO TSUCHIYA

§ 1. Introduction.

In these several years, many people have published their results on the qualitative theory of foliations, mainly of class C^2 and of codimension one. These results can be more or less interpreted as those on the realizability or non-realizability of manifolds as leaves having a certain qualitative property (e.g. Sondow [15], Nishimori [10, 11, 12], Cantwell-Conlon [1, 2, 3], Tsuchiya [17, 18, 19], Phillips-Sullivan [13], Inaba [7, 8], Takamura [16], and so on). From this point of view, one may naturally ask the existence of open manifolds which can never be realized as leaves of foliations of any closed manifold. In this paper, we answer this question:

THEOREM. *Let E be an arbitrary non-empty, compact, totally disconnected, metrizable space. Then for any $n \geqq 3$, there exists an n-dimensional open orientable manifold L whose endspace is homeomorphic to E and which cannot be realized as a leaf of a codimension one C^2 foliation of any closed manifold.*

Remark. (1) The endspace of an open manifold is known to be a compact, totally disconnected and metrizable space.

(2) Our result is in contrast with that of Cantwell-Conlon [3]. They showed that each compact, totally disconnected, metrizable space is homeomorphic to the endspace of a leaf of a codimension one foliation of a closed manifold.

§ 2. Preliminaries.

We begin with the following lemma, whose proof is easy and is omitted.

LEMMA 1. *Let E be any non-empty, compact, totally disconnected, metrizable space. Then there exists a countable, locally finite tree whose endspace is homeomorphic to E.*—

Let T be a tree as in Lemma 1. Denote by $\{v_k\}_{k \in N}$ the vertices of T and

Received June 1, 1984

$\{e_l\}_{l\in N}$ the edges of T, where $N=\{1, 2, 3, \cdots\}$. For each $k\in N$, we choose an n-dimensional closed orientable manifold L_k whose fundamental group is isomorphic to the finite cyclic group of order $2k+1$. We assign L_k to each vertex v_k of T. For each edge e_l of T with endpoints v_{k_1} and v_{k_2}, we perform a connected sum operation between L_{k_1} and L_{k_2}. We denote the resulting manifold by L. Clearly the endspace of L is homeomorphic to that of T, hence is homeomorphic to E.

In the sequel, we will show that L cannot occur as a leaf of any foliation. Here we note that it suffices to prove that L can never be realized as a leaf of a *transversely orientable* foliation of any closed *orientable* manifold. For, if L is realized as a leaf of a foliation $\mathfrak{F}$ of a closed manifold M, then L is covered by a leaf $\tilde{L}$ of a transversely orientable foliation $\tilde{\mathfrak{F}}$ of a closed orientable manifold $\tilde{M}$, where $\tilde{M}$ is a double or four-fold covering space of M. Since $\pi_1(L)$ is generated by elements of odd order, $\tilde{L}$ is diffeomorphic to L. Thus L is realized as a leaf of $\tilde{\mathfrak{F}}$.

For later use, we assign to each edge e_l of T a submanifold S_l of L diffeomorphic to an $(n-1)$-dimensional sphere, which is the junction sphere of the connected sum operation corresponding to e_l (see Figure 1).

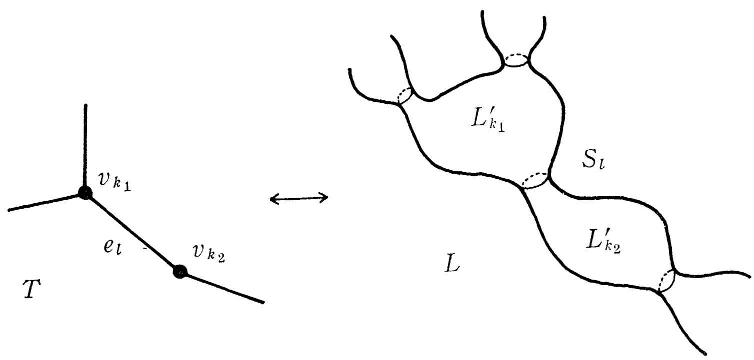

Figure 1.

We will make use of the following algebraic lemma, which is a corollary to the Kurosh Subgroup Theorem (cf. Magnus-Karrass-Solitar [9, p. 245]).

LEMMA 2. *Suppose that $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are finitely presented groups, each of which is indecomposable with respect to the free product operation $*$. If $A_1*\cdots*A_p\cong B_1*\cdots*B_q$, then $p=q$ and $B_1, \cdots, B_p$ can be rearranged so that A_i is isomorphic to B_i for every i, $1\leqq i\leqq p$.*—

In the following, we denote by L'_k the manifold obtained from L_k by deleting finitely many open n-disks.

LEMMA 3. *For any $k \in \boldsymbol{N}$, there exist no embeddings from the disjoint union of two copies of L'_k into L.*

Proof. Suppose that there were an embedding

$$\phi : L'_k \cup L'_k \longrightarrow L .$$

Since the image of ϕ is compact, there is a positive integer N such that $\phi(L'_k \cup L'_k)$ is contained in $L_1 \sharp \cdots \sharp L_N$. Note that every embedded $(n-1)$ sphere in L separates L because the first Betti number of L vanishes. Thus we can find a compact manifold R such that

$$L_k \sharp L_k \sharp R = L_1 \sharp \cdots \sharp L_N .$$

Considering the fundamental groups of both hand sides, we have

$$\boldsymbol{Z}_{2k+1} * \boldsymbol{Z}_{2k+1} * G = \boldsymbol{Z}_3 * \cdots * \boldsymbol{Z}_{2N+1}$$

for some group G. This equality contradicts Lemma 2, for in the right hand side, the $\boldsymbol{Z}_{2k+1}$ factors appear at most once. □

LEMMA 4. *Let W be a codimension-zero noncompact submanifold of L such that W is a closed subset of L and that ∂W is compact. Then there exists $k \in \boldsymbol{N}$ such that W contains an L'_k in its interior.*

Proof. By the hypothesis, W is a neighborhood of some end of L. Therefore $W \cap L'_k \neq \emptyset$ for infinitely many k. On the other hand, $\partial W \cap L'_k \neq \emptyset$ for at most finitely many k since ∂W is compact. Hence there exists some $k \in \boldsymbol{N}$ such that L'_k is contained in $\operatorname{Int} W$. □

§ 3. Proof of Theorem.

We suppose that L is a leaf of a codimension one foliation $\mathcal{F}$ of some closed manifold M. As is noted in § 2, we may suppose that $\mathcal{F}$ is transversely orientable and M is orientable. We will lead us to a contradiction at the end of this section.

First of all we remark that the holonomy group of L is trivial. This follows from the fact that the fundamental group of L is generated by elements of finite order and $\mathcal{F}$ is of codimension one.

Claim I. *L is a proper leaf.*

Proof. Since L has no holonomy, a relative version of the Reeb stability theorem implies that for each k there is a neighborhood N of L'_k $(\subset L)$ in M such that $(N, \mathcal{F}|N)$ is diffeomorphic to the product foliation $(L'_k \times (-1, 1), \{L'_k \times \{t\}\}_{-1<t<1})$. If L is nonproper, then $L \cap N$ has infinitely many connected components, each of which is diffeomorphic to L'_k. This contradicts Lemma 3. □

Claim II. *L is not a totally proper leaf (i.e., a leaf whose closure consists of proper leaves).*

Proof. Suppose that L is totally proper. Then the qualitative theory of codimension one C^2 foliations (e.g., Tsuchiya [17] and Cantwell-Conlon [2]) shows that all the ends of L are periodic (for the definition of periodic ends, see Inaba [7]) and that some ends of L are isolated. Let W be a periodic neighborhood of an isolated end of L. Then W is described as an infinite repetition of a compact manifold P:

$$W=P\cup P\cup P\cup \cdots .$$

Now we have a contradiction as follows. By Lemma 4, $\mathrm{Int}W$ contains L_k' for some k. Since L_k' is compact, there is a positive integer N such that L_k' is contained in the first N repetition $P\cup \cdots \cup P$. Then

$$W=(P\cup \cdots \cup P)\cup(P\cup \cdots \cup P)\cup \cdots$$

contains infinitely many pairwise disjoint copies of L_k'. This contradicts Lemma 3. □

By Claims I and II, the left possibility is that L is a proper leaf whose closure contains nonproper leaves. Therefore the proof of Theorem is completed if the following claim is proved.

Claim III. *L is not a proper leaf whose closure contains nonproper leaves.*

Proof. Suppose that L is a proper leaf and contains nonproper leaves in its closure $\bar{L}$. Let U be the connected component of $M-(\bar{L}-L)$ which contains L. By the definition of U, the leaf L is closed in U. Let $\hat{U}$ be the Dippolito completion of U (Dippolito [4]). $\hat{U}$ is a manifold with boundary and has a canonical foliation induced from $\mathcal{F}$. Let

$$\hat{U}=K\cup A_1\cup\cdots\cup A_r$$

be Dippolito's nucleus-arm decomposition. By definition, K is compact and $\partial_{tr}K=\partial_{tr}A_1\cup \cdots \cup\partial_{tr}A_r$, where $\partial_{tr}X$ means the set of points of ∂X where $\mathcal{F}|X$ is transverse to ∂X. Furthermore for each i, $1\leqq i\leqq r$, A_i is the total space of a foliated interval bundle $p_i: A_i\to B_i$, where the base space B_i is connected and noncompact, and $p_i^{-1}(\partial B_i)=\partial_{tr}A_i$. We call K a *nucleus* and each A_i an *arm* of $\hat{U}$. For each i, choose a point $b_i\in\partial B_i$ and let $I_i=p_i^{-1}(b_i)$ (see Figure 2).

Subclaim 1. *There exists at least one arm which intersects L.*

Proof. If L is contained entirely in K, then $\bar{L}-L\subset\partial_{tan}K=\partial K\cap\partial\hat{U}$. Hence every leaf contained in $\bar{L}-L$ must be compact. This contradicts the hypothesis that $\bar{L}$ contains nonproper leaves. □

Since L is closed in U, it follows that for each arm A_i intersecting L the

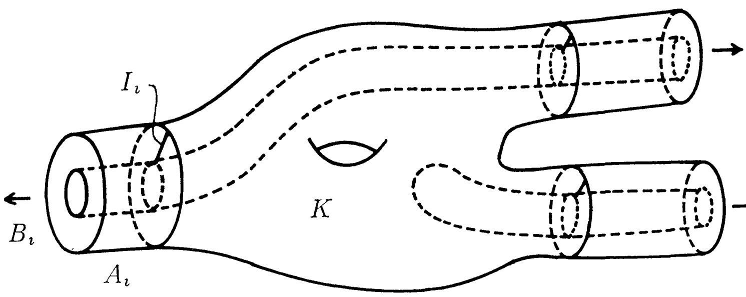

Figure 2.

intersection of the fibre I_ι and each connected component of $L\cap A_\iota$ is either of the following:

a) Countable points such that each point is isolated and that the only accumulation points are the two endpoints of I_ι.

b) A single point.

Therefore the following two cases may occur.

Case 1. There are an arm A_ι and a connected component W of $L\cap A_\iota$ such that the intersection of W and I_ι is of type a) above.

Case 2. For each arm A_ι intersecting L and for each connected component W of $L\cap A_\iota$, the intersection of W and I_ι is of type b) above.

Subclaim 2. Case 1 *cannot occur*.

Proof. Let $\Phi_\iota : \pi_1(B_\iota, b_\iota)\to\mathrm{Diff}^2_+ I_\iota$ be the total holonomy of $\mathcal{F}|A_\iota$. In this situation Hector's theorem on convergence of holonomy (Hector [6], see also Inaba [7]) says that there exists a sequence $\{\alpha_n\}$ of elements of $\pi_1(B_\iota, b_i)$ such that

1) $\{\alpha_n\}$ generates $\pi_1(B_\iota, b_i)$, and
2) $\Phi_i(\alpha_n)$ converges to the identity map uniformly on I_ι.

Now let W be a connected component of $L\cap A_\iota$ such that $W\cap I_\iota$ is of type a). Then by Hector's theorem we see that there is a positive integer N such that if $n\geqq N$, then

$$\Phi_\iota(\alpha_n)|W\cap I_\iota=\text{the identity}.$$

In other words, there is a compact subset Q of B_ι such that each connected component of $W\cap p_i^{-1}(B_i-\mathrm{Int}\,Q)$ is diffeomorphic to $B_i-\mathrm{Int}\,Q$. Since $W\cap I_\iota$ is of type a), the number of such connected components is infinite. By Lemma 4, every connected component of $W\cap p_i^{-1}(B_i-\mathrm{Int}\,Q)$ contains L'_k for some $k\in\boldsymbol{N}$. Then we have an embedding

$$\phi : L'_k\cup L'_k \longrightarrow W\cap p_i^{-1}(B_\iota-\mathrm{Int}\,Q)\subset W\subset L\,.$$

This contradicts Lemma 3. □

Subclaim 3. Case 2 *cannot occur*.

Proof. Suppose that for each arm A_ι and for each connected component W of $L\cap A_\iota$, the intersection $W\cap I_\iota$ is of type b). Then W is diffeomorphic to B_ι. By Lemma 4, taking A_ι smaller if necessary, we may assume that

$$W\cap\partial A_\iota=S_l$$

for some l. (Recall that S_l is one of the junction spheres defined in § 2.) Since $\pi_1(B_\iota)$, which is isomorphic to $\pi_1(W)$, is generated by elements of finite order, the total holonomy of A_ι is trivial. That is, $(A_\iota, \mathcal{F}|A_\iota)$ is isomorphic to the product foliation $(W\times I, \{W\times\{t\}\}_{t\in I})$.

Note that $L\cap A_\iota=W$, for if $L\cap A_\iota$ has components other than W, we have a contradiction by Lemma 3.

Next we consider $L\cap K$. The following two cases may occur.

Case 2.1. The closure of $L\cap K$ in $\hat{U}$ contains a compact leaf in $\partial\hat{U}$.

Case 2.2. $L\cap K$ is compact.

In Case 2.1, the same arguments as in Claim II can be applied, since $L\cap K$ is a totally proper leaf of $\mathcal{F}|K$ and L has a periodic isolated end whose limit set is the compact leaf. Therefore Case 2.1 cannot occur. Now Case 2.2 is the only case left to us.

Suppose that Case 2.2 occurs. Each connected component of $L\cap K$ is a compact leaf of $\mathcal{F}|K$ whose fundamental group is generated by elements of finite order. Then by the relative version of the Reeb Global Stability Theorem, we see that $(K, \mathcal{F}|K)$ is isomorphic to the product foliation $(C\times I, \{C\times\{t\}\}_{t\in I})$. Combining the above arguments together, one obtains that $(\hat{U}, \mathcal{F}|\hat{U})$ is isomorphic to the product foliation $(L\times I, \{L\times\{t\}\}_{t\in I})$. Let F be one of the leaves in $\hat{\imath}(\partial\hat{U})$, where $\hat{\imath}: \hat{U}\to M$ is the natural immersion induced from the inclusion map $i: U\to M$. Then F is diffeomorphic to L, and furthermore F is contained in $\bar{L}$. (This fact follows from the definition of U.) Since L is not asymptotic to F in U, the leaf L must be asymptotic to F from the outside of U (see Figure 3).

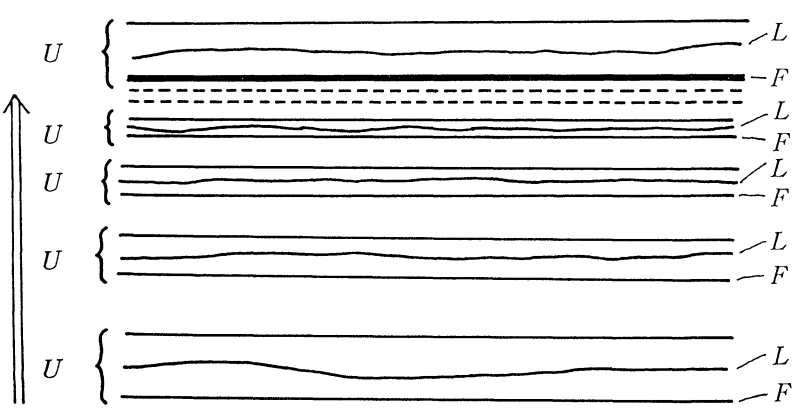

Figure 3.

This implies that F is nonproper. Using the arguments in the proof of Claim I for F instead of L, we have a contradiction. This completes the proof of Subclaim 3, hence that of Claim III, furthermore that of Theorem. □

Remark. It is not known whether there is an open 2-manifold which cannot be realized as a leaf of C^2 foliation of a closed 3-manifold.

REFERENCES

[1] CANTWELL, J. AND L. CONLON, Leaf prescriptions for closed 3-manifolds, Trans. Amer. Math. Soc., **236** (1978), 239-261.
[2] CANTWELL, J. AND L. CONLON, Poincaré-Bendixson theory for leaves of codimension one, Trans. Amer. Math. Soc., **262** (1981), 181-209.
[3] CANTWELL, J. AND L. CONLON, Endsets of leaves, Topology **21** (1982), 333-352.
[4] DIPPOLITO, P., Codimension one foliations of closed manifolds, Ann. of Math., **107** (1978), 403-453.
[5] HAEFLIGER, A., Variétés feuilletées, Ann. Scuola Norm. Pisa **16** (1962), 367-397.
[6] HECTOR, G., Architecture des feuilletages de class C^2, Astérisque 107-108 (1983), 243-258.
[7] INABA, T., Reeb stability for noncompact leaves, Topology **22** (1983), 105-118.
[8] INABA, T., A sufficient condition for the C^2 Reeb stability of noncompact leaves of codimension one foliations, to appear.
[9] MAGNUS, W., A. KARRASS AND D. SOLITAR, Combinatorial Group Theory, 1966, Interscience.
[10] NISHIMORI, T., Isolated ends of open leaves of codimension one foliations, Quart. J. Math. Oxford, **26** (1975), 159-167.
[11] NISHIMORI, T., Ends of leaves of codimension-one foliations, Tôhoku Math. J., **31** (1979), 1-22.
[12] NISHIMORI, T., Average Euler characteristic of leaves of codimension one foliations, to appear.
[13] PHILLIPS, A. AND D. SULLIVAN, Geometry of leaves, Topology **20** (1981), 209-218.
[14] REEB, G., Sur certaines propriétés topologiques des variétés feuilletées, Actual. Sci. Ind. No. **1183**, Hermann, Paris, 1962.
[15] SONDOW, J., When is a manifold a leaf of some foliation?, Bull. Amer. Math. Soc., **81** (1975), 622-624.
[16] TAKAMURA, M., in preparation.
[17] TSUCHIYA, N., Leaves of finite depth, Japan. J. Math., **6** (1980), 343-364.
[18] TSUCHIYA, N., The Nishimori decomposition of codimension one foliations and the Godbillon-Vey classes, Tôhoku Math. J., **34** (1982), 343-365.
[19] TSUCHIYA, N., On decompositions and approximations of foliated manifolds, to appear.

COLLEGE OF ARTS AND SCIENCES
CHIBA UNIVERSITY
YAYOICHO, CHIBA, 260 JAPAN

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, 060 Japan

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, 060 Japan

and

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, 152 Japan

Added in proof. We know that E. Ghys has obtained a similar result.