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SELF-HOMOTOPY EQUIVALENCES OF THE TOTAL SPACES OF A SPHERE BUNDLE OVER A SPHERE

Dedicated to Professor Minoru Nakaoka on his 60th birthday

BY SEIYA SASAO

§1. Introduction.

In this paper we study the group of homotopy classes of self-homotopy equivalences, $\mathcal{E}(X)$, for the total space of a S^m-bundle over S^n with the condition:

$$3<m+1<n<2m-2.$$

J. W. Rutter determined this group for the case of $m=3$ and $n=7$ in [3], and also some generalizations of Rutter's result are given in [4] and [6]. Moreover Y. Nomura computed $\mathcal{E}(X)$ for real and complex Stiefel manifolds in [5]. Then our purpose is to obtain a generalization of these results in a some sense. Let H be the natural representation:

$$H:\mathcal{E}(X)\longrightarrow \operatorname{Aut} H_*(X)$$

which is defined by $H(f)=f_*$ and we denote by $\mathcal{E}_+(X)$ the kernel of H. Then we have an exact sequence

$$\{1\}\longrightarrow \mathcal{E}_+(X)\longrightarrow \mathcal{E}(X)\underset{H}{\longrightarrow} \operatorname{Aut} H_*(X).$$

Hence it is almost sufficient for us to determine $\mathcal{E}_+(X)$ and H-image.

Let $q: X\to S^n$ be the S^m-bundle with the characteristic class $\xi\,(\in\pi_{n-1}(SO(m+1)))$. James-Whitehead showed in [2] that X has a CW-decomposition:

$$X=S^m\bigcup_{\beta} e^n\bigcup_{\alpha} e^{m+n},$$

where $\beta=p_*(\xi)$ for the usual projection $p: SO(m+1)\to S^m$.

Let $P_n^m(\beta)$ be the subgroup of $\pi_n(S^m)$,

$$\{x\,|\,[\iota_n, x]\in\beta\circ\pi_{m+n-1}(S^{n-1})\},$$

and we denote by η the generator of $\pi_{N+1}(S^N)$. We will prove

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THEOREM 1. *Suppose that* $[\iota_{m+1}, E\beta]\circ\eta\equiv 0 \bmod E\beta\circ\pi_{m+n+1}(S^n)$. *Then there exists an exact sequence*

$$\{0\} \longrightarrow H_\xi \longrightarrow \mathcal{E}_+(X) \longrightarrow G_\xi \longrightarrow \{0\}\,,$$

where

$$H_\xi = \pi_{m+n}(X)/[\iota_m, \pi_{n+1}(X)] \cup \{\pi_{m+1}(X)\circ J(\xi)\}$$

and

$$G_\xi = P_n^m(\beta)[\{\beta\circ\eta\} \subset \pi_n(S^m)/\{\beta\circ\eta\}\,.$$

Remark. For example, the assumption is always satisfied if $m=2 \bmod 4$ and $m\geqq 9$.

THEOREM 2. *Suppose* $2\beta=0$.

H-*image* $=Z_2\times Z_2$ *if* $2J(\xi)\equiv 0$ *and* $[\iota_{m+1}, E\beta]\equiv 0 \bmod E\beta\circ\pi_{m+n}(S^n)$

H-*image* $=Z_2$ *if either* $[\iota_{m+1}, E\beta]\equiv 0$, $2J(\xi)\not\equiv 0$ *or* $[\iota_{m+1}, E\beta]\not\equiv 0$,

$2J(\xi)\equiv 0 \bmod E\beta\circ\pi_{m+n}(S^n)$

H-*image* $=Z_2$ *if* $[\iota_{m+1}, E\beta]+2J(\xi)\equiv 0$ *and* $2J(\xi)\not\equiv 0 \bmod E\beta\circ\pi_{m+n}(S^n)$,

H-*image* $=\{0\}$ *otherwise.*

THEOREM 3. *Suppose that the order of* β *is odd. Then*

H-*image* $=Z_2$ *if* $[\iota_{m+1}, E\beta]+2J(\xi)\equiv 0 \bmod E\beta\circ\pi_{m+n}(S^n)$

and

H-*image* $=\{0\}$ *otherwise.*

Our method is based on Barcus-Barratt theory [1]. Let $A=S^m\cup_\beta e^n$ be the subcomplex of X and consider the fibring

$$r_A:(X^X, 1_X) \longrightarrow (X^A, i) \qquad (i=1_X\,|\,A)$$

defined by restricting maps on A. Then we have an exact sequence

$$\pi_1(X^A, i) \xrightarrow[\partial_{X,A}]{} \pi_0(r_A^{-1}(i), 1_X) \longrightarrow \pi_0(X^X, 1_X) \longrightarrow \pi_0(X^A, i)\,.$$

Using an identification of $\pi_0(r_A^{-1}(i), 1_X)$ with $\pi_{m+n}(X, x_0)$, the above sequence can be transformed into the exact sequence

$$\{0\} \longrightarrow G_{X,A} \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{E}(A)\,,$$

where $G_{X,A}$ denotes the group $i_*\pi_{m+n}(A)/\{i_*\pi_{m+n}(A)\cup\partial_{X,A}(\pi_1(X^A, i))\}$.

Since $\mathcal{E}(A)$ can be determined by Barcus-Barratt Theorem our work is to describe the group $G_{X,A}$ and the image $\mathcal{E}(X)\to\mathcal{E}(A)$. In § 2 the operation $\partial_{X,A}$ is investigated and § 3 $\partial_{X,A}$ is considered again from the view of Suspensionversion. § 4 contains some homotopy groups, and the image $\mathcal{E}(X)\to\mathcal{E}(A)$ is discussed in § 5. At last, in § 6, we give some examples.

§ 2. Barcus-Barratt Operation.

LEMMA 2.1. *$i_*(\pi_{m+n}(A))=\pi_{m+n}(X)$, $\pi_{m+n-1}(A)\cong Z\{\alpha\}+G(\beta)$ and the sequence*

$$\{0\} \longrightarrow i_{m*}\{\pi_{m+n-1}(S^m)\} \longrightarrow G(\beta) \longrightarrow \beta_*^{-1}(0) \longrightarrow \{0\}$$

is exact where $\beta_* : \pi_{m+n-2}(S^{n-1})\to\pi_{m+n-1}(S^m)$ is induced by β. Especialll y we have

$$G_{X,A}=\pi_{m+n}(X)/\partial_{X,A}\pi_1(X^A, i)\,.$$

Proof. The proof follows from the homotopy exact sequence and the homotopy excision theorem.

Let $r_{S^m}:(X^A, i)\to(X^{S^m}, \imath_m)$ be the fibring $(i_m=\imath|S^m : S^m\to X)$ and let $\Lambda_{A,X}$ be the fibre $r_{S^m}^{-1}(i_m)$, i. e.

$$\Lambda_{A,X}=\{f : A \longrightarrow X|f|S^m=\imath_m\}\,.$$

Consider the exact sequence

$$\pi_1(\Lambda_{A,X}, i) \longrightarrow \pi_1(X^A, i) \longrightarrow \pi_1(X^{S^m}, \imath_m) \longrightarrow \pi_0(\Lambda_{A,X}, i)$$

and identifications

$$\pi_1(X^{S^m}, i_m) \underset{d_1}{\longleftrightarrow} \pi_{m+1}(X, x_0) \quad \text{and} \quad \pi_1(\Lambda_{A,X}, i) \underset{d_2}{\longleftrightarrow} \pi_{n+1}(X, x_0)$$

given by

$$S^1\times S^m \underset{f}{\longrightarrow} X, \qquad d_1(f)=d(f, \imath_m\circ pr)$$

and

$$S^1\times A \underset{g}{\longrightarrow} X, \qquad d_2(g)=d(g, \imath\circ pr)\,,$$

where d denotes the separation elemen (see Appendix).

LEMMA 2.2. *By the composition*

$$\pi_{n+1}(X, x_0) \underset{d_2}{\longleftrightarrow} \pi_1(\Lambda_{A,X}, i) \longrightarrow \pi_1(X^A, i) \xrightarrow[\partial_{X,A}]{\qquad} \pi_{m+n}(X, x_0)$$

any element z is mapped to Whitehead product $[\iota_m, z]$.

For the proof we need some preparations. Let ϕ be a map $A\to A\vee S^n$ $(A=S^m\cup e^n\to(S^m\cup e^n)\vee S^n)$ which is obtained from shrinking the equator of e^n to a point.

LEMMA 2.3. $\phi_*(\alpha)=\alpha+[\iota_m, \iota_n]$ $(\in\pi_{m+n-1}(A\vee S^n))$

Proof. From the assumption on m, n we have the decomposition

$$\pi_{m+n-1}(A\vee S^n)=\pi_{m+n-1}(A)\oplus\pi_{m+n-1}(S^n)\oplus Z[\iota_m, \iota_n]\,.$$

Clearly the first factor of $\phi_*(\alpha)$ is α and the second factor is zero by the existence of the projection $X \to S^n$. Since the third factor is determined by the cohomology ring of X we may think that it is just $[\iota_m, \iota_n]$. These complete the proof.

Let us define three spaces X_i $(i=0, 1, 2)$ as follows:

$$X_0=(A \vee S^n) \bigcup_{\phi_*(\alpha)} e^{m+n}, \quad X_1=X \vee S^n \quad \text{and} \quad X_2=S^m \times S^n \cup (A \vee S^n).$$

Then three Barcus-Barratt operations are obtained from fibrings:

$$(X^{X_i}, v_i) \longrightarrow (X^{A \vee S^n}, i \vee (x_0)) \qquad (i=0, 1, 2),$$

where (x_0) denotes the constant map $S^n \to x_0 (\in X)$ and v_i is an appropriate extension of $i \vee (x_0)$ over X. We denote them by

$$\partial_i=\partial_{X_i, A \vee S^n}: \pi_1(X^{A \vee S^n}, i \vee (x_0)) \longrightarrow \pi_{m+n}(X, x_0), \qquad (i=0, 1, 2).$$

Now, applying the additive theorem of Barcus-Barratt we have

LEMMA 2.4. $\partial_0=\partial_1+\partial_2$

Since $\pi_1(X^{A \vee S^n}, i \vee (x_0))$ has a decomposition

$$\pi_1(X^{A \vee S^n}, i \vee (\pi_0))=\pi_1(X^A, i) \oplus \pi_1(X^{S^n}, (x_0))$$

we may regard $\pi_1(X^{S^n}, (x_0))$ as a subgroup of $\pi_1(X^{A \vee S^n}, i \vee (x_0))$.

LEMMA 2.5. *The restriction* $\partial_1 | \pi_1(X^{S^n}, (x_0))=0$.

Proof. It is sufficient from definitions to show that the image of the homomorphism

$$\pi_1(X^{X_1}, v_1) \longrightarrow \pi_1(X^{A \vee S^n}, i \vee (x_0))$$

contains $\pi_1(X^{S^n}, (x_0))$ for the map $v_1: X_1=X \vee S^n \to X$, $(1_X \vee (x_0))$, and then this means that any map: $S^1 \times (A \vee S^n) \to X$ is extendable over $S^1 \times (X \vee S^n)$ if $f | S^1 \times A = i \circ \mathrm{proj}_A$ and $f | * \times S^n=(x_0)$. Since the map $\tilde{f}: S^1 \times (X \vee S^n) \to X$ defined by

$$\tilde{f} | S^1 \times X=1_X \circ \mathrm{proj}_X \quad \text{and} \quad \tilde{f} | S^1 \times S^n=f | S^1 \times S^n$$

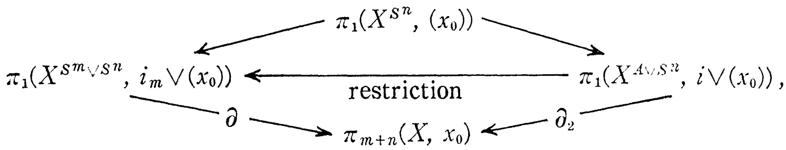
satisfies the conditions the proof is completed.

LEMMA 2.6. *The restriction* $\partial_2 | \pi_1(X^{S^n}, (x_0))$ *can be identified with the homomorphism*

$$\pi_1(X^{S^n}, (x_0))=\pi_{n+1}(X, x_0) \longrightarrow \pi_{m+n}(X, x_0)$$

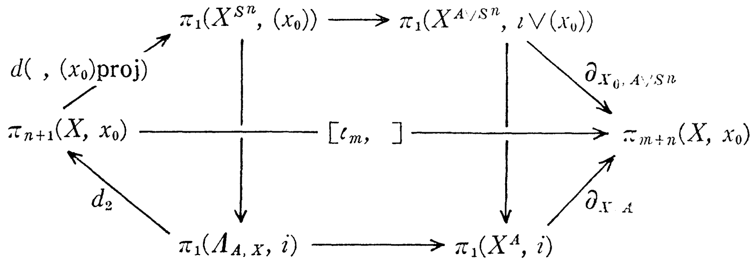
defined by Whitehead product $[\iota_m, \ ]$.

Proof. Consider the commutative diagram



where ∂ on the left hand is the boundary operator derived from the fibring $X^{S^m \times S^n} \to X^{S^m \vee S^n}$. Then by Barcus-Barratt formula (p. 66 of [1]) the proof is completed.

Now, using the following diagram, the proof of lemma 2.2 is completed from lemma 2.3, 2.4, 2.5 and 2.6.



here we identify the space A with $A \vee S^n / S^n$ and the map $\iota : A \to X$ with the map $A \xrightarrow{\phi} A \vee S^n \xrightarrow{\iota \vee (x_0)} X$.

§ 3. Suspension of Barcus-Barratt Operation.

In this section our purpose is to describe the group $E\{\partial_{X, A}(X^A, i)\}$ as a subgroup of $\pi_{m+n+1}(EX, x_0)$ with other terms. First we consider the general case. For any spaces Y and K, the map $\Sigma : Y^K \to EY^{EK}$ which assigns each map $f : K \to Y$ to the map $Ef : EK \to EY$ induces the homomorphism

$$\Sigma^* : \pi_1(Y^K, u) \longrightarrow \pi_1(EY^{EK}, Eu), \qquad (u : K \to Y),$$

i.e. for $f : S^1 \times K \to Y$, $\Sigma^*(f)(s, (t, x)) = (t, f(s, x))$ $(x \in K)$.

Since, for a map $h : L \to K$, it holds

$$\Sigma^* h^*(f)(s, (t, y)) = (t, f(s, h(y))$$

and

$$(Eh)^* \Sigma^*(f)(s, (t, y)) = (\Sigma f)(s, (t, h(y)) = (t, f(s, h(y)) \qquad (y \in L)$$

we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(Y^K, u) & \xrightarrow[\ \Sigma^*\ ]{} & \pi_1(EY^{EK}, Eu) \\ \downarrow h^* & & \downarrow Eh^* \\ \pi_1(Y^L, u\,h) & \xrightarrow[\ \Sigma^*\ ]{} & \pi_1(EY^{EL}Eu\,Eh) \end{array} \tag{3.1}$$

Now, applying the diagram 3.1 to our case $Y=X$, $K=A$ and $h=\beta$, we have

LEMMA 3.2. *There exists a commutative diagram*

$$\begin{array}{ccccc} \pi_1(X^A, i) & \xrightarrow{\ \partial_{X,A}\ } & \pi_{m+n}(X, x_0) & \overset{E}{\searrow} & \\ \downarrow r_* & \overset{\Sigma^*}{\searrow} & \pi_1(EX^{EA}, Ei) & \xrightarrow{\ \partial_{EX,EA}\ } & \pi_{m+n+1}(EX, x_0) \\ Z_2=\pi_1(X^{S^m}, \iota_m) & & \downarrow r_* & \nearrow \partial_{EA,S^{m+1}} & \\ \downarrow & \overset{\Sigma^*}{\searrow} & & & \\ \{0\} & & \pi_1(EX^{S^{m+1}}, \iota_{m+1})=Z_2 & & \end{array}$$

In the above diagram if we identify $\pi_1(EX^{S^{m+1}}, \iota_{m+1})$ with $\pi_{m+2}(EX, x_0)$ we have

LEMMA 3.3. *$\partial_{EA,S^{m+1}}$ may be considered as the composition $\circ EJ(\xi)$, where ξ denotes the characteristic class of the bundle.*

Proof. We note that there exists a map: $T(\xi)=S^{m+1}\bigcup_{J(\xi)} e^{m+n+1}\rightarrow EX$ of degree ± 1. Then the proof completed by applying the sphere theorem of [1] to the diagram

$$\begin{array}{ccc} \pi_1(EX^{S^{m+1}}, \iota_{m+1}) & \xrightarrow{\ \partial_{EA,S^{m+1}}\ } & \pi_{m+n+1}(EX, x_0) \\ \uparrow \simeq & \nearrow \partial_{T(\xi),S^{m+1}} & \\ \pi_1(T(\xi)^{S^{m+1}}, \iota_{m+1}) & & \end{array}$$

which is obtained from using lemma 3.2.

LEMMA 3.4. $E\partial_{X,A}\{\pi_1(X^A, i)\}=\pi_{m+2}(EX)\circ EJ(\xi)$.

Proof. Consider the sequence associated with the fibring $r: X^A\rightarrow X^{S^m}$

$$\begin{array}{ccccccc} \pi_1(A_{X,A}, i) & \longrightarrow & \pi_1(X^A, i) & \xrightarrow{\ r_*\ } & \pi_1(X^{S^m}, \iota_m) & \xrightarrow{\ \partial_{A,S^m}\ } & \pi_n(X, x_0) \\ & & & & \mid\cong & & \Cup \\ & & & & Z_2=\pi_{m+1}(S^m) & \longrightarrow & \iota_m\circ\eta\circ E\beta \end{array}$$

Since $\partial: \pi_{n+1}(S^n)\rightarrow\pi_n(S^m)$ is given by $\partial(\eta)=\beta\circ\eta$ and we have $\beta\circ\eta=\eta\circ E\beta$, by the assumption $n \leqq 2m-2$ r_* is onto. Thus the proof follows from lemma 3.2 and 3.3.

§ 4. The suspension $\pi_k(X) \to \pi_{k+1}(EX)$

Now we are interested in the kernel of the suspension

$$E_k : \pi_k(X) \longrightarrow \pi_{k+1}(EX) \qquad (k=m+n,\ m+n-1).$$

Let ν be the attaching map for a cell of a *CW*-complex, then we denote by $\bar{\nu}$ the characteristic map for the cell. By the homotopy excision we know

LEMMA 4.1. *For $i=1, 2$ there exists a decomposion*

$$\pi_{k+i}(EX, S^{m+1}) \cong \overline{J(\xi)} \circ \pi_{k+i}(D^{m+n+1}, S^{m+n}) + \overline{E\beta} \circ \pi_{k+i}(D^{n+1}, S^n)$$
$$+ [\iota_{m+1}, \overline{E\beta} \circ \pi_{n+2}(D^{n+1}, S^n)]_r ,$$

where $[\ ,\ ]_r$ denotes relative Whitehead product.

Consider the following ladder:

$$\begin{array}{ccccccccc}
\pi_{k+1}(S^n) = \pi_{k+1}(X, S^m) & \xrightarrow{\partial} & \pi_k(S^m) & \xrightarrow{i_{m*}} & \pi_k(X) & \xrightarrow{j_*} & \pi_k(X, S^m) \\
\downarrow & & \downarrow & & \downarrow E_k & & \downarrow \\
\pi_{k+2}(EX, S^{m+1}) & \xrightarrow{\partial} & \pi_{k+1}(S^{m+1}) & \xrightarrow{i_*} & \pi_{k+1}(EX) & \xrightarrow{j_*} & \pi_{k+1}(EX, S^{m+1})
\end{array}$$

First we note that the homomorphism

$$\pi_{k+i}(X, S^m) \longrightarrow \pi_{k+i+1}(EX, S^{m+1}) \qquad (i=0, \pm 1)$$

is injective because we have a commutative diagram

$$\begin{array}{ccc}
\pi_{k+i}(X, S^m) & \xrightarrow[q_*]{\cong} & \pi_{k+i}(S^n) \\
\downarrow E & & \cong \downarrow E \\
\pi_{k+i+1}(EX, S^{m+1}) & \xrightarrow[Eq_*]{} & \pi_{k+i+1}(S^{m+1}) .
\end{array}$$

Hence we have

$$E_k^{-1}(0) = i_{m*}(E^{-1}\partial\pi_{k+1}(EX, S^{m+1})) \tag{4.2}$$

On the other hand, from lemma 4.1, we have

$$\partial\pi_{k+2}(EX, S^{m+1}) = J(\xi)\circ\pi_{k+1}(S^{m+1}) \cup E\beta\circ\pi_{k+1}(S^n) \cup [\iota_{m+1}, E\beta\pi_{k-m-1}(S^n)] \tag{4.3}$$

LEMMA 4.4. *For $x \in \pi_s(S^{n-1})$ $(s \leqq 2m-2)$, $J(\xi)\circ E^{m+1}x$ is contained in the E-image if and only if $\beta\circ x=0$.*

Proof. Take Hopf invariant of the element, i. e.

$$H(J(\xi)\circ E^{m-1}x)=\pm HJ(\xi)\circ E^{m+1}x=\pm E^{m+1}\beta E^{m+1}x=\pm E^{m+1}(\beta\circ x)\,.$$

Then the proof follows from $s\leqq 2m-2$.

Now, suppose that $\beta\circ x=0$. Then there exists $\sigma_x\in\pi_{s+1}(X)$ such that $q_*(\sigma_x)=Ex$. Lemma 4.4 is more exactly stated as follows:

LEMMA 4.5. *There exists an element* $\xi_X\in\pi_s(SO(m))$ *satisfying*

(1) $EJ(\xi_X)=J(\xi)\circ E^{m+1}x$

(2) $i_{m*}(J(\xi_X))=[\iota_m,\sigma_x]$

Proof. Let ξ' be the induced bundle over S^{s+1} by the map Ex. Since $p_*(\xi')=p_*(\xi)\circ x=\beta\circ x=0$ there exists an element ξ_X of $\pi_s(SO(m))$ which is mapped to ξ by the inclusion $SO(m)\rightarrow SO(m+1)$. Then we have

$$EJ(\xi_X)=-J(\xi')=-J(\xi\circ x)=\pm J(\xi)\circ E^{m+1}x\,.$$

Next, consider the commutative diagram

$$\begin{array}{ccccc} & \nearrow^{i_m} & X & \xrightarrow{\ q\ } & S^n \\ S^m & & \uparrow & & \uparrow Ex \\ & \searrow_{i_m} & Y & \xrightarrow{\ q'\ } & S^{s+1} \end{array},$$

then By [2] we have, in $\pi_{s-m}(Y)$,

$$i_{m*}(J(\xi_X))+[\iota_m,\iota_{s+1}]=0$$

for a cross-section ι_{s+1} of q'. Clearly this shows (2).

Now, we know that there exists an element w_ξ of $\pi_{m+n-1}(S^m)$ such that

if $2\beta=0$ then $Ew_\xi=[\iota_{m+1},\iota_{m+1}]\circ E^{m+1}\beta$

if m is odd[7] and $2\beta=0$ then $Ew_\xi=J(2\xi)\pm[\iota_{m+1},\iota_{m+1}]\circ E\beta$.

Then from (4.2), (4.3), and lemma 4.5 we obtain

LEMMA 4.6. $E^{-1}_{m-n}(0)=[\iota_m,\pi_{n+1}(X)]\cup\{i_{m*}(w_\xi\circ\eta)\}$

$E^{-1}_{m-n-1}(0)=[\iota_m,\pi_n(X)]\cup\{i_{m*}(w_\xi)\}$.

LEMMA 4.7. *Suppose that* $[\iota_{m+1},E\beta]\circ\eta\equiv 0 \bmod E\beta\circ\pi_{m+n+1}(S^n)$. *Then we have*

$$\partial_{X,A}\pi_1(X^A,i)=\{[\iota_m,\pi_{n+1}(X)]\}\cup\{\pi_{m+1}(S^m)\circ J(\xi)\}\,.$$

Proof. By lemma 3.4 there exists an element γ_ξ of $\pi_1(X^A,i)$ satisfying

(1) $E\partial_{X,A}(\gamma_\xi)=i_{m+2*}(\eta)\circ EJ(\xi)$

(2) γ_ξ is mapped to the generator of $\pi_1(X^{S^m},i_m)=Z_2$ by r_*.

Since $\pi_1(X^A, i)$ is the sum of $\{\gamma_\xi\}$ and the image $\pi_1(\Lambda_{X,A}, i)\to\pi_1(X^A, i)$ the proof is completed by lemma 2.6 and 4.6.

§ 5. Proof of theorems.

Recall the sequence in § 1

$$0 \longrightarrow G_{X,A} \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{E}(A)\,,$$

and imbed this one in a diagram as follows:

$$\begin{array}{ccccccc}
 & & & & AutH_*(X) & \xrightarrow{\ r_*\ } & AutH_*(A) \\
 & & & & H_X\uparrow & & \uparrow H_A \\
\{0\} & \longrightarrow & G_{X,A} & \longrightarrow & \mathcal{E}(X) & \longrightarrow & \mathcal{E}(A) \\
 & & \downarrow & & \uparrow & & \uparrow \\
\{0\} & \longrightarrow & H_\xi & \longrightarrow & \mathcal{E}_+(X) & \longrightarrow & G_\xi\subset\mathcal{E}_+(A) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \{1\} & & \{1\}
\end{array}$$

Then, if $[\iota_{m+1}, E\beta\circ\eta]\in E\beta\circ\pi_{m+n+1}(S^n)$, we have from lemma 2.1 and 4.7

LEMMA 5.1. $H_\xi=\pi_{m+n}(X)/\{[\iota_m, \pi_{n+1}(X)]\}\cup\{\pi_{m+1}(X)\circ J(\xi)\}$

Next, consider the exact sequence

$$\pi_n(S^m) \xrightarrow[t]{\qquad} \mathcal{E}(A) \xrightarrow[d]{\qquad} Z_2\times Z_2$$

which is defined by

$$t(f): A \xrightarrow[\phi]{\qquad} A\vee S^n \xrightarrow[1\vee f]{\qquad} A\vee S^m \xrightarrow[1\vee \imath_m]{\qquad} A \qquad (f\in\pi_n(S^m))$$

and $d(h)$=(degree on e^m of h, degree on e^n of h).

Clearly d is equivalent to the representation H and moreover the kernel of t is determined by the sphere theorem of [1] as follows:

$$t^{-1}(1_X)=\{\eta\circ E\beta\}=\{\beta\circ\eta\}\,.$$

Since the definition of t and lemma 2.3 imply

$$t(f)_*(\alpha)=\alpha+[\iota_m, f] \qquad (X=A\bigcup_\alpha e^{m+n})$$

the element $t(f)$ is contained in the image $\varepsilon(X)\to\varepsilon(A)$ if and only if $[\iota_m, f]\in\partial\pi_{m+n}(S^n)=\beta\circ\pi_{m+n-1}(S^{n-1})$.

Thus, noting $rH_X=H_Ar$, we have

LEMMA 5.2. $G_\xi=P^m_n(\beta)/\{\beta\circ\eta\}$ *if* $[\iota_{m+1}, E\beta\circ\eta]\in E\beta\circ\pi_{m+n+1}(S^n)$

Now we proced to study of the representation H_X. First we note

LEMMA 5.3. *The kernel* $(q|A)_*:\pi_{m+n-1}(A)\to\pi_{m+n-1}(S^n)$ *is generated by* α *and the* i_{m*}*-image* $(i_m:S^m\to A)$.

Proof. This is easily obtained from the diagram $(k=m+n-1)$

$$\begin{array}{ccccc} \pi_k(S^m) & \xrightarrow[i_{m*}]{} & \pi_k(A) & \xleftarrow{\ \partial\ } & \pi_{k+1}(X,A)\simeq Z_2 \\ & & \downarrow & \searrow{\scriptstyle (q|A)_*} & \\ & & \pi_k(X) & \xrightarrow[q_*]{} & \pi_k(S^n) \end{array} .$$

Let f be a map: $A\to A$ satisfying

$$f_*(e^m)=ae^m \quad\text{and}\quad f_*(e^n)=be^n$$

which we call a map of type (a, b) and denote by f_a^b. Then the following lemma is easy.

LEMMA 5.4. *There exists a map of type* (a, b) *if and only if* $(b-a)\beta=0$.

Let g_a^b be another map. Clearly there exists a map $g:S^n\to S^m$ by which g_a^b is represented as the composition of maps

$$g_a^b=(f_a^b\vee g)\circ\phi : A \xrightarrow[\phi]{} A\vee S^n \xrightarrow[f_a^b\vee g]{} A\vee S^m \longrightarrow A$$

Now we are interested in the element $f_{a*}^b(\alpha)$. Then lemma 2.3 gives

$$g_{a*}^b(\alpha)=f_{a*}^b(\alpha)+a[\iota_m, g] .$$

On the other hand, since we have

$$(q|A)_*f_{a*}^b(\alpha)=(b\iota_n)_*(q|A)_*(\alpha)=0$$

lemma 5.3 gives, for some $\sigma_a^b\in\pi_{m+n-1}(S^m)$,

$$f_{a*}^b(\alpha)=ab\alpha+i_{m*}(\sigma_a^b) .$$

Thus we have from these lemmas

LEMMA. 5.5. *There exists a map* $f: X\to X$ *whose restriction* $f|A$ *is of type* (a, b) *if and only if there exists a map* f_a^b *such that*

$$f_{a*}^b(\alpha)=ab\alpha+i_{m*}(\sigma_a^b) , \qquad \sigma_a^b\in a[\iota_m, \pi_n(S^m)]\cup\beta\circ\pi_{m+n-1}(S^{n-1}) .$$

Especially if $a=\pm1$ *the condition is equivalent to* $E\phi_a^b\in E\beta\circ\pi_{m+n}(S^n)$.

Next, for the reason of our dimensional assumption, the space A is desuspendable, so there exists a co-H-map $\nu:A\to A\vee A$ and the addition of two maps is defined as usal. Then we want to get some formula on $(f_a^b+f_c^d)_*(\alpha)$. For the purpose we must investigate the group $\pi_k(A\vee A)$ for $k=m+n-1$. First, by the well-known decomposition of this group it holds

$$\nu_*(\alpha)=\alpha+\alpha+\chi \qquad (\chi\in\partial\pi_{k+1}(A\times A,\ A\vee A))\,.$$

Next- since the order of β is finite there exists a map $\tau:S^n\to A$ of degree $o(\beta)$ and we have the element $[\iota_m^1,\ \tau^2]$ of $\pi_k(A\vee A)$ where each upper index denotes the order of A imbedded in $A\vee A$ and $o(\beta)$ is the order of the element. Let $Q:A=S^m\cup e^n\to S^n=A/S^m$ be the collapsing map, then for maps $Q\vee 1_A:A\vee A\to S^n\vee A$ and $1_A\vee Q:A\vee A\to A\vee S^n$ we have

LEMMA 5.6. $(1_A\vee Q)_*(\chi)=[\iota_m^1,\ \iota_n]$, $(Q\vee 1_A)_*(\chi)=(-1)^{mn}[\iota_n,\ \iota_m^2]$,

$(1_A\vee Q)_*([\iota_m^1,\ \tau^2])=0(\beta)[\iota_m^1,\ \iota_n]$ *and* $(Q\vee 1_A)_*([\iota^1,\ \tau_m^2])=0(\beta)[\iota_n,\ \iota_m^2]$.

Proof. The third and fourth are clear and the others follows from the diagram

$$\begin{array}{ccc}
\pi_k(A) & \xrightarrow[\nu_*]{} & \pi_k(A\vee A)=\pi_k(A)+\pi_k(A)+\partial\pi_{k+1}(A\times A,\ A\vee A)\\
 & & \downarrow (1_A\vee Q)_* \qquad \downarrow 1_A \qquad \downarrow Q_*\\
 & & \pi_k(A\vee S^n)=\pi_k(A)+\pi_k(S^n)+\partial\pi_{k+1}(A\times S^n,\ A\vee S^n)
\end{array}$$

LEMMA 5.7. *There exists an isomorphism*

$$\pi_k(A\vee A)=\pi_k(A)+\pi_k(A)+Z\{\chi\}+Z[\tau^1,\ \iota_m^2]+[\iota_m^1,\ \iota_m^2]\circ\pi_k(S^{2m-1})$$

Proof. Noting the assumption $m+1<n<2m-1$, consider the following diagram which is naturally obtained:

$$\begin{array}{ccccccc}
\pi_k(A\vee S^n) & \longleftarrow & \pi_k(A\vee A) & \longrightarrow & \pi_k(S^n\vee A) & & \\
\uparrow\partial & & \uparrow\partial & & \uparrow\partial & & \\
\pi_{k+1}(A\times S^n,\ A\vee S^n) & \longleftarrow & \pi_{k+1}(A\times A,\ A\vee A) & \longrightarrow & \pi_{k+1}(S^n\times A,\ S^n\vee A) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Z\simeq\pi_{k+1}(A\divideontimes S^n) & \longleftarrow & \pi_{k+1}(A\divideontimes A) & \longrightarrow & \pi_{k+1}(S^n\divideontimes A)\simeq Z & & \\
\nwarrow & & \downarrow \quad \nwarrow & & \uparrow \quad \nwarrow & & \\
 & & \pi_{k+1}(A\divideontimes A,\ A\divideontimes S^n) & & \pi_{k+1}(A\divideontimes S^n) & \longrightarrow & \pi_{k+1}(A\divideontimes S^m,\ S^m\divideontimes S^m)\\
 & & & & \uparrow & & \downarrow\partial\\
 & & & & \pi_{k+1}(S^m\divideontimes S^m) & & \pi_k(S^m\divideontimes S^m)
\end{array}$$

, where $\divideontimes$ denotes the reduced join operator.

Then the proof follows from lemma 5.6.

Now, consider the map $f_a^b \vee f_c^d : A \vee A \to A \vee A$, then we prove

LEMMA 5.8. $(f_a^b \vee f_c^d)_*([\tau^1, \iota_m^2]) \equiv bc[\tau^1, \iota_m^2] \mod [\iota_m^1 \circ \pi_n(S^m), \iota_m^2]$

and

$$(f_a^b \vee f_c^d)_*(\chi) \equiv ad\{\chi\} + \{(-1)^{mn}(bc-ad)/0(\beta)\}[\tau^1, \iota_m^2] \mod [\iota_m^1, \iota_m^2] \circ \pi_k(S^{2m-1})$$

Remark. $bc-ad$ is divisible by $o(\beta)$ because we have $b-c \equiv 0 \equiv d-a \mod o(\beta)$ (lemma 5.4).

Proof. By lemma 5.7 we can put

$$(f_a^b \vee f_c^d)_*(\chi) \equiv r\{\chi\} + s[\tau^1, \iota_m^2] \mod [\iota_m^1, \iota_m^2] \circ \pi_k(S^{2m-1})$$

for some integers r and s. Then from lemma 5.6 it follows that $r=ad$ and $o(\beta)s=(-1)^{mn}(bc-r)=(-1)^{mn}(bc-ad)$. Hence the proof is completed.

Let μ be the folding map $A \vee A \to A$. We note that there exists an element $\lambda \in \pi_{n-1}(SO(m))$ such that

$$o(\beta)\alpha = i_{m*}(J(\lambda)) + [\iota_m, \tau]$$

where $i_*(\lambda) = o(\beta)\xi$ for $i: SO(m) \to SO(m+1)$.

LEMMA 5.9. $\mu_*(\chi) = 2\alpha + i_{m*}(\sigma_2^2)$ *and* $\mu_*([\tau^1, \iota_m^2]) = (-1)^{mn}\{o(\beta)\alpha - i_{m*}(J(\lambda))\}$

Proof. By definition $\nu_*(\alpha) = \alpha + \alpha + \chi$, then we have

$$\mu_*\nu_*(\alpha) = (2 \cdot 1_A)_*(\alpha) \quad \text{i.e.} \quad 4\alpha + i_{m*}(\sigma_2^2) = 2\alpha + \mu_*(\chi).$$

Since $\mu_*([\tau^1, \iota_m^2]) = [\tau, \iota_m]$ the second follows from the above note.

LEMMA 5.10. $\sigma_{a+c}^{b+d} \equiv \sigma_a^b + \sigma_c^d + ad(\sigma_2^2) + \{(ad-bc)/o(\beta)\}J(\lambda)$

$$\mod [\iota_m, \pi_n(S^m)] \cup \{\beta \circ \pi_{m+n-1}(S^{n-1})\}$$

Proof. Apply above lemmas to the identity

$$(f_{a+c}^{b+d})_*(\alpha) = (f_a^b + f_c^d)_*(\alpha) = \mu_*(f_a^b \vee f_c^d)_*\nu_*(\alpha).$$

Then the proof easily follows.

Now, let $\hat{\sigma}_a^b$ be the suspension of σ_a^b, then lemma 5.10 gives rise

$$\hat{\sigma}_{a+c}^{b+d} \equiv \hat{\sigma}_a^b + \hat{\sigma}_c^d + ad(\hat{\sigma}_2^2) - (ab-cd)J(\xi) \mod E\beta \circ \pi_{m+n}(S^n).$$

LEMMA 5.11. $\hat{\sigma}_a^b = \{a(a-1)/2\}(\hat{\sigma}_{-1}^{-1}) + a(b-a)J(\xi) \mod E\beta \circ \pi_{m+n}(S^n)$

Proof. By lemma 5.4 $b=a+ko(\beta)$ for some integer k. Hence we have

$$\hat{\sigma}_a^b = \hat{\sigma}_{0+a}^{ko(\beta)+a} = \hat{\sigma}_0^{ko(\beta)} + \hat{\sigma}_a^a + ko(\beta)J(\xi) = \hat{\sigma}_a^a + a(b-a)J(\xi) \mod E\beta \cdot \pi_{m+n}(S^n)$$

On the other hand, lemma 5.10 implies $\hat{\sigma}^{a+1}_{a+1} \equiv \hat{\sigma}^{a}_{a} + a(\hat{\sigma}^{2}_{2})$, i.e. we have

$$\hat{\sigma}^{a}_{a} = \{a(a-1)/2\}(\hat{\sigma}^{-1}_{-1}) .$$

Thus the proof is completed.

Since lemma 5.11 shows that it is important for our purpose to determine $\hat{\sigma}^{-1}_{-1}$, so here we recall the definition of σ^{-1}_{-1}, which is given by

$$(-1_A)_*(\alpha) = \alpha + i_{m*}(\sigma^{-1}_{-1}) .$$

Then, applying the suspension operator, we have

$$(-1_{EA})_*(E\alpha) = E\alpha + i_{m+1*}(\hat{\sigma}^{-1}_{-1}) , \quad \text{i.e.} \quad i_{m+1*}(\hat{\sigma}^{-1}_{-1}) = -2E\alpha .$$

On the other hand, since we may regard the mapping cone of the projection $q: X \to S^n$ as the Thom space of the vector bundle characterized by ξ we can put

$$E\alpha = \imath_{m+1*}(\lambda_\xi J(\xi)) , \qquad \lambda_\xi = 1 \quad \text{or} \quad -1 .$$

Hence, using $\imath^{-1}_{m+1*}(0) = \{[\iota_{m+1}, E\beta]\} \cup E\beta \circ \pi_{m+n}(S^n)$, we know that

$$\hat{\sigma}^{-1}_{-1} \equiv -2\lambda_\xi J(\xi) + c_\xi[\iota_{m+1}, E\beta] \mod E\beta \circ \pi_{m+n}(S^n) \tag{5.12}$$

for some integer c_ξ. For example, if ξ has a cross-section then we may take $\lambda_\xi = -1$ ([2]), but, in general, it is not easy to determine λ_ξ.

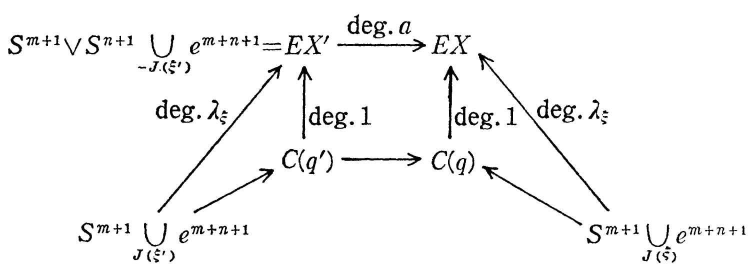
LEMMA 5.13. $o(\beta)(1+\lambda_\xi)J(\xi) = 0$

Proof. Consider the following diagrams $(a = o(\beta))$:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ S^n & \xrightarrow[\text{deg. } a]{} & S^n \end{array}$$

and

$(\xi' = a\xi)$



Then we can obtain

$$a \equiv a\lambda_\xi\lambda_{\xi'} \mod o(J(\xi)) \quad \text{and} \quad -\lambda_{\xi'}J(\xi') = J(\xi') , \quad \text{i.e.} \quad -\lambda_{\xi'}aJ(\xi) = aJ(\xi) .$$

Clearly these give the proof. Now we prove

LEMMA 5.14 *In* (5.12) *we can take*

(1) $\lambda_\xi=-1$ *and* $c_\xi=1$ *if* $2\beta=0$
(2) $c_\xi=-\lambda_\xi$ *or* $-\lambda_\xi+o(\beta)/2$ *otherwise*

Proof. (1) the case: $2\beta=0$.
By lemma 5.13 and 5.11 we have

$2J(\xi)=-2\lambda_\xi J(\xi)$ and $\hat{\sigma}_2^0\equiv\hat{\sigma}_{-1}^{-1}-4J(\xi)$. On the other hand, $f_{2*}^0(\alpha)=\sigma_2^0$ implies that

$$\hat{\sigma}_2^0=(Ef_2^0)_*(E\alpha)=2\lambda_\xi J(\xi)+[\iota_{m+1},\ \iota_{m+1}]HJ(\xi)=2\lambda_\xi J(\xi)+[\iota_{m+1},\ E\beta]$$

Hence we obtain

$$\hat{\sigma}_{-1}^{-1}\equiv 4J(\xi)+2\lambda_\xi J(\xi)+[\iota_{m+1},\ E\beta]\equiv 2J(\xi)+[\iota_{m+1},\ E\beta].$$

(2) the other case. Note that this occurs only in the case of m=odd.
Take Hopf-invariant on the both side of (5.12), then we have, from the formula $H(J(\xi))=-E^{m+1}(\xi)$ and $H([\iota_{m+1},\ E\beta])=2E^{m+1}\beta$,

$$2\lambda_\xi E^{m+1}+2c_\xi E^{m+1}\beta=0.$$

Then, in our dimensional restriction, this means $2(\lambda_\xi+c_\xi)=0$ and then the proof is completed.

Now the proof of theorem 2 and 3 are completed by the following lemma which is a conseqeuence of lemma 5.11 and 5.14.

LEMMA 5.15. *If* $2\beta=0$ *we have*

$$\hat{\sigma}_{-1}^{-1}\equiv 2J(\xi)+[\iota_{m+1},\ E\beta]$$

$$\hat{\sigma}^{-1}_{1}\equiv -2J(\xi)$$

$$\hat{\sigma}_{-1}^{1}\equiv[\iota_{m+1},\ E\beta] \quad \mathrm{mod}\ E\beta\circ\pi_{m+n}(S^n)$$

and if the order of β *is odd*

$$-\lambda_\xi\hat{\sigma}_{-1}^{-1}\equiv 2J(\xi)+[\iota_{m+1},\ E\beta].$$

Remark. Since the second case of lemma 5.15 can be shown to be true in the case $o(\beta)=2\cdot$odd Theorem 3 also holds in this case.

§ 6. Some Examples

(I) The case of having a cross section.

$$H_\xi=\pi_{m+n}(S^m)/\{\eta\circ J(\xi)\cup[\iota_m,\ \pi_{n+1}(S^m)]\}+\pi_{m+n}(S^n).$$

$$G_\xi=\{x\,|\,x\in\pi_n(S^m),\ [\iota_m,\ x]=0\}$$

$$\mathcal{E}(X) \longrightarrow Z_2 \quad \text{is onto if} \quad 2J(\xi)\neq 0\,,$$

and

$$\mathcal{E}(X) \longrightarrow Z_2\times Z_2 \quad \text{is onto if} \quad 2J(\xi)=0\,.$$

(II) Complex Stiefel manifolds $W_{n,2}$ $(n\geqq 5)$.

Let ξ_n be the standard sphere bundle

$$S^{2n-3} \longrightarrow W_{n,2} \longrightarrow S^{2n-1}\,, \qquad \beta_n = n\eta\,.$$

Since $[\iota_{2n-2},\ \eta\circ\eta]=\eta\circ[\iota_{2n-1},\ \iota_{2n-1}]$ the assumption is satisfied. If n is even the case reduces to (I) and for odd n we have

if $n\equiv 1 \mod 4$, then $H_{\xi_n}=\pi_{4n-4}(W_{n,2})$, $G_{\xi_n}=\{0\}$ and $\mathcal{E}(W_{n,2})\longrightarrow Z_2$ is onto.

and

if $n\equiv 3 \mod 4$, then $H_{\xi_n}=\pi_{4n-4}(W_{n,2})/\iota_*\{[\iota_{2n-3},\ \pi_{2n}(S^{2n-3})]\}$,

$G_{\xi_n}=\{0\}$, $\varepsilon(W_{n,2})\longrightarrow Z_2$ is onto.

(III) Quaternion Stiefel manifolds $X_{n,2}$

Let τ_n be the standard bundle

$$S^{4n-5} \longrightarrow X_{n,2} \longrightarrow S^{4n-1}\,, \qquad \tau_n = n\nu\,.$$

Since $[\iota_{4n-4},\ \nu\circ\eta]=0$ the assumption is satisfied. Then if $n\geqq 3$ we have

$$H_{\tau_n}=\pi_{8n-6}(X_{n,2})/[\iota_{4n-5},\ \pi_{4n}(X_{n,2})]\cup\iota_*\{\eta\circ J(\tau_n)\}\,, \quad \text{and} \quad G_{\tau_n}=\{0\}\,.$$

The image $\varepsilon(X_{n,2})\rightarrow Z_2\times Z_2$ is more complicated, so we omit it.

APPENDIX: Separation elements

$$K\cup e^n \underset{g}{\overset{f}{\rightrightarrows}} X\,, \quad f\,|\,K=g\,|\,K \Rightarrow d(f,\ g)\in\pi_n(X)\,.$$

$$K\cup e^n_i = K\cup e^n\,, \qquad \hat{K}=e^n_1\cup K\cup e^n_2\,,$$

$$k:\hat{K}\longrightarrow K\cup e^n\,, \qquad k\,|\,K\cup e^m_i=\text{identity}.$$

1. The sequence: $0\rightarrow\pi_n(\hat{K})\rightarrow\pi_n(\hat{K},\ K)\times\pi_n(K\cup e^n)$ is exact.

For,

$$\begin{array}{ccccc} \pi_n(K) & \longrightarrow & \pi_n(\hat{K}) & \longrightarrow & \pi(K,\ \hat{K}) \\ & & k_*\big\downarrow\ \big\uparrow\iota_* & & \\ & & \pi_n(K\cup e^n) & & \end{array}$$

where $\iota: K\cup e^n\rightarrow e^n_1\cup K\subset\hat{K}$.

$$\xi: S^n=E^n_+\cup E^n_- \rightrightarrows D^n \underset{\chi_2}{\overset{\chi_1}{\rightrightarrows}} K\,.$$

$$\Rightarrow k_*(\xi)=0\,, \qquad \chi_1-\chi_2\in\pi_n(K,\ K)\,.$$

Then $d(f, g)=h_*(\xi)$, where $h:\hat{K}\to X$, $h|e_1^n\cup K=f$, $h|K\cup e_2^n=g$.

2. $$K\cup e^n \xrightarrow[H]{} e_1^n \underset{\lambda_1}{\cup} L \underset{\lambda_2}{\cup} e_2^n \overset{F}{\underset{G}{\rightrightarrows}} X,$$

$$H(K)\subset L,\ H(e^n)=e_1^n+e_2^n, \qquad F|L=G|L.$$

Then $d(FH, GH)=d(f_1, g_1)+d(f_2, g_2)$, where $f_i=F|e_i^n\cup L$ and $g_i=G|e_i^n\cup L$.

Proof.

$$\begin{array}{ccccccc} & \nearrow & H_1=H & \longrightarrow & e_{1,1}^n \cup\ \cup e_{2,1}^n & & L_1 \overset{F}{\searrow} \\ \hat{K}=e_1^n\cup K\cup e_2^n & \text{———} & \hat{H} & \longrightarrow & \hat{L}=\quad L & = & + \qquad X \\ & \searrow & H_2=H & \longrightarrow & e_{1,2}^n \cup\ \cup e_{2,2}^n & & L_2 \underset{G}{\nearrow} \end{array}$$

And consider the diagram:

$$\begin{array}{ccccc} 0 \text{——} & \pi_n(\hat{K}) & \longrightarrow & \pi_n(\hat{K}, K) \times \pi_n(K\cup e^n) & \\ & \xi^e \big\downarrow \hat{H}_* & & \big\downarrow \hat{H}_* \qquad \big\downarrow H_* & \\ 0 \longrightarrow & \pi_n(\hat{L}) & \longrightarrow & \pi_n(\hat{L}, L)\times\pi_n(e_1^n \underset{\lambda_1}{\cup} L \underset{\lambda_2}{\cup} e_2^n) & \\ & \big\uparrow j_{i*} & & \big\uparrow \qquad\qquad \big\uparrow & \\ 0 \longrightarrow & \pi_n(\hat{L}_i) & \longrightarrow & \pi_n(\hat{L}_i, L)\times\pi_n(L \underset{\lambda_i}{\cup} e_i^n), & \end{array}$$

where

$$\hat{L}_i=e_{i,1}^n \underset{\lambda_{i,1}}{\cup} L \underset{\lambda_{i,2}}{\cup} e_{i,2}^n \xrightarrow[j_i]{} \hat{L}.$$

Then

$$H_*(\xi) \longrightarrow (\lambda_{1,1}+\lambda_{2,1}-\lambda_{1,2}-\lambda_{2,2})\times 0$$

$$j_{1*}(\xi_1)+j_{2*}(\xi_2) \longrightarrow (\lambda_{1,1}-\lambda_{1,2}+\lambda_{2,1}-\lambda_{2,2})\times 0$$

Hence $\hat{H}_*(\xi)=j_{1*}(\xi_1)+j_{2*}(\xi_2)$ (from the injectivity). And then we have

$$\begin{aligned} d(FH, FG)&=(F\cup G)_*H_*(\xi) \\ &=(F\cup G)_*(j_{1*}(\xi_1))+(F\cup G)_*(j_{2*}(\xi_2)) \\ &=d(f_1, g_1)+d(f_2, g_2). \end{aligned}$$

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