A. MARTINEZ AND A. ROS
KODAI MATH. J.
7 (1984), 304—316

ON REAL HYPERSURFACES OF FINITE TYPE OF CP^m

BY A. MARTINEZ AND A. ROS

§1. Introduction.

Let M be a closed Riemannian manifold and Δ the Laplace-Beltrami operator of M acting on the smooth functions $C^\infty(M)$. It is well known that Δ is an elliptic operator with a discrete sequence of eigenvalues $0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty$. Let V_k be the eigenspace corresponding to the eigenvalue λ_k. Then V_k has finite dimension. Moreover the decomposition is orthogonal respect to the inner product

$$(f, g)=\int_M fgdV \tag{1.1}$$

and $\sum_k V_k$ is dense in $C^\infty(M)$.

Let $x : M\to E^m$ be an isometric immersion of M into the m-dimensional Euclidean space with coordinate functions x_i, that is, $x=(x_1, \cdots, x_m)$. Then for any $i=1, \cdots, m$, we have the decomposition

$$x_i=\sum_k (x_i)_k \qquad (L^2\text{-sense}). \tag{1.2}$$

As M is closed, V_0 consists of the constant functions on M and so, from (1.2) we can write

$$x_i-(x_i)_0=\sum_{k=p_i}^{q_i}(x_i)_k \tag{1.3}$$

where $q_i=\{\text{Sup}\, k\,|\,(x_i)_k\neq 0\}$ (respectively, $p_i=\{\text{Inf}\, k\,|\,(x_i)_k\neq 0\}$).

If $p=\text{Inf}_i\{p_i\}$ and $q=\text{Sup}_i\{q_i\}$ using (1.3) we obtain the following spectral decomposition (in a vector form)

$$x-x_0=\sum_{k=p}^{q} x_k \tag{1.4}$$

where $x_k : M\to E^m$ are smooth for any k, q is an integer or $q=\infty$, x_0 is a constant and $\Delta x_k=\lambda_k x_k$. x_0 is called center of gravity of M.

We shall say that the immersion x is of *finite type* if $q<\infty$. If not it will be called of *no finite type* [5].

Received December 2, 1983.

An immersion x of finite type will be called *Mono-order* (*Bi-order, Tri-order*, $\cdots$) if there exists only one (two, three, $\cdots$) of the x_k's that is (are) non null. If $p=q$, x is called of *order* p.

Considering the isometric immersion of the complex projective space CP^m in an Euclidean space $HM(m+1)$ given in [10], any submanifold of the complex projective space is isometrically immersed in $HM(m+1)$. In this paper we study the real hypersurfaces M of CP^m for which the immersion of M into $HM(m+1)$ is Mono-order or Bi-order.

In § 3 we classify the real hypersurfaces of CP^m for which the immersion in $HM(m+1)$ is Mono-order. We also give a bound of the first eigenvalue of their spectrum.

In § 4 we classify the minimal real hypersurfaces of CP^m for which the immersion in $HM(m+1)$ is Bi-order. We prove a spectral inequality envolving the first and second eigenvalues of the spectrum.

The manifolds are supposed to be connected and of real dimension $\geqq 2$ (if no other thing is mentioned).

For the necessary knowledge and notations of submanifold theory see [3, 4]. For the particular case of real hypersurfaces of CP^m see also [2, 6, 11] and for spectral geometry see [1].

§ 2. The complex projective space.

For details in this section see [8, 9, 10].

Let CP^m be the complex projective space obtained as a quotient space of the unit sphere $S^{2m+1}(1)=\{Z\in C^{m+1} \mid zz^*=z\bar{z}^t=1\}$ by identifying z with λz, $\lambda\in C$ and $|\lambda|=1$. Let g be the cannonical metric on CP^m, that is, the invariant metric such that the fibration $\Pi : S^{2m+1}(1)\to CP^m$ is a Riemannian submersion. It is known that CP^m with this metric is a complex-space-form of constant holomorphic sectional curvature 4 and its Riemannian curvature tensor is given by

$$\bar{R}(X, Y)Z=g(Y, Z)X-g(X, Z)Y+g(JY, Z)JX -g(JX, Z)JY+2g(X, JY)JZ \tag{2.1}$$

for any X, Y, Z in TCP^m.

Let $HM(m)=\{B\in gl(m, C) \mid \bar{B}=B^t\}$ with metric

$$g(A, B)=\frac{1}{2}\text{trace}(AB) \qquad \text{for any} \quad A, B\in HM(m). \tag{2.2}$$

In [10], Sakamoto proves that the map $\tilde{\phi} : S^{2m+1}(1)\to HM(m+1)$ given by

$$\tilde{\phi}(z)=z^*z=\bar{z}^t z \qquad z\in S^{2m+1}(1) \tag{2.3}$$

induces an immersion $\phi : CP^m\to HM(m+1)$ satisfying

(A) $\phi(CP^m)=\{B\in HM(m+1) \mid B^2=B \text{ and trace } B=1\}$.

(B) ϕ is an equivariant full isometric imbedding into

$$H_1M(m+1)=\{B\in HM(m+1)\,|\,\text{trace } B=1\}.$$

In the following (if nothing is mentioned) we shall consider CP^m identified with $\psi(CP^m)$.

Under this identification [8, 9] the tangent and normal spaces at each point $B\in CP^m$ are given, respectively, by

$$(2.4)\qquad \begin{aligned} T_BCP^m&=\{X\in HM(m+1)\,|\,XB+BX=X\},\\ T_B^\perp CP^m&=\{Z\in HM(m+1)\,|\,ZB=BZ\}.\end{aligned}$$

For any Q in $HM(m+1)$, the component of Q in T_BCP^m is

$$(2.5)\qquad Q^T=QB+BQ-2BQB=QB+BQ-4g(B, Q)B$$

Moreover the complex structure J induced on CP^m by ψ is given by

$$(2.6)\qquad JX=\sqrt{-1}(I-2B)X$$

for any $X\in T_BCP^m$, I being the identity matrix of $HM(m+1)$.

We shall denote by D the Riemannian connection of $HM(m+1)$, by $\tilde{\nabla}$ the one induced on CP^m and $\tilde{\sigma}$, $\tilde{\nabla}^\perp$, $\tilde{A}$ and $\tilde{H}$, respectively, the second fundamental form, the normal connection, the Weingarten endomorphism and the mean curvature vector of CP^m in $HM(m+1)$. Now, analogously as the case of holomorphic sectional curvature 1, [8, 9] we have

$$(2.7)\qquad \tilde{\sigma}(X, Y)=(XY+YX)(I-2B),\quad \tilde{A}_ZX=(XZ-ZX)(I-2B),$$

$$(2.8)\qquad \tilde{H}_B=\frac{2}{m}(I-(m+1)B),$$

$$(2.9)\qquad \tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y),$$

for any X, Y in T_BCP^m and Z in $T_B^\perp CP^m$. From (2.9) we get

$$(2.10)\qquad \overline{\nabla}\tilde{\sigma}=0,$$

that is, the second fundamental form of the immersion is parallel, where $(\overline{\nabla}_X\tilde{\sigma})(Y, Z)=\tilde{\nabla}_X^\perp\tilde{\sigma}(Y, Z)-\tilde{\sigma}(\tilde{\nabla}_XY, Z)-\tilde{\sigma}(Y, \tilde{\nabla}_XZ)$ for any X, Y, $Z\in TCP^m$.

From the equation of Gauss, (2.1), (2.6) and (2.7) it also follows

$$(2.11)\qquad \begin{aligned} g(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W))&=2g(X, Y)g(V, W)+g(X, V)g(Y, W)\\ &\quad+g(X, W)g(Y, V)+g(JX, V)g(JY, W)+g(JX, W)g(JY, V),\end{aligned}$$

$$(2.12)\qquad g(\tilde{\sigma}(X, Y), I)=0,\quad g(\tilde{\sigma}(X, Y), B)=-g(X, Y),$$

for any X, Y, V, $W\in T_BCP^m$.

§ 3. Immersions of order k.

Let M be a connected real hypersurface of CP^m. We shall denote by ∇ the Riemannian connection induced on M by $\tilde{\nabla}$, by N a unit normal vector field to M in CP^m, by A, σ, H, respectively, the Weingarten endomorphism, the second fundamental form and the mean curvature vector of M in CP^m, and by (ϕ, f) the almost-contact structure on M, [11].

It is known from § 2 that CP^m can be imbedded in $HM(m+1)$. So, any submanifold of CP^m is isometrically immersed in $HM(m+1)$. In particular, $x=\psi\circ i : M\to HM(m+1)$ is an isometric immersion of M in $HM(m+1)$. We shall denote by $\bar{H}$ its mean curvature vector in $HM(m+1)$ and by $\bar{H}^\perp$ its component in $T^\perp CP^m$. Then we obtain the following result

THEOREM 3.1. *Let M be a real hypersurface of CP^m $(m\geqq 2)$. Then M is minimal in some hypersphere of $HM(m+1)$ if and only if M is locally congruent to the geodesic hypersphere $\Pi\left(S^1\left(\sqrt{\frac{1}{2m+2}}\right)\times S^{2m-1}\left(\sqrt{\frac{2m+1}{2m+2}}\right)\right)$, where Π is the usual fibration of CP^m.*

Proof. Let us suppose that M is minimal in a hypersphere of $HM(m+1)$ of center Q, which we can suppose diagonal (If it is not we can apply an isometry of type $B\mapsto PBP^{-1}$, $P\in U(m+1)=\{P\in GL(m+1, C)/P\bar{P}^t=I\}$). Thus

$$\bar{H}_B=a(B-Q)\,, \tag{3.1}$$

for some non-null real number a, and

$$Q=\begin{pmatrix} q_1 & & & & & \\ & \ddots (m_1 & & & & \\ & & q_1 & & & \\ & & & \ddots & & \\ & & & & q_r & \\ & & & & & \ddots (m_r \\ & & & & & q_r \end{pmatrix}, \qquad q_i\neq q_j \ (i\neq j)$$

From (2.4), $B\in T_B^\perp CP^m$. Thus from (2.12), multiplying scalarly (3.1) by B, we have

$$g(B, Q)=\alpha \qquad \text{for any} \quad B\in M \tag{3.2}$$

where α is a constant, $\alpha=\dfrac{2+a}{2a}$.

From (3.2) Q is normal to M in $HM(m+1)$, and so, putting $g(Q, N)=\lambda$, we have $X(\lambda)=g(Q, \tilde{\sigma}(X, N))$ for any $X\in TM$. Consequently, from (2.11) and (2.12), multiplying scalarly (3.1) by $\tilde{\sigma}(X, N)$, we get

$$X(\lambda)=0 \tag{3.3}$$

that is, λ is constant.

From (2.5), the component of Q in $T_B CP^m$ is $Q^T = BQ + QB - 4g(B, Q)B$, then

$$\lambda^2 = g(Q^T, Q^T) = g(Q^T, Q) = g(QB + BQ - 4g(B, Q)B, Q)$$
$$= 2g(Q^2, B) - 4g(B, Q)^2.$$

Hence

(3.4) $$g(B, Q^2) = \beta = \frac{\lambda^2 + 4\alpha^2}{2} = \text{constant}$$

for any $B \in M$.

As $g(B, I) = \frac{1}{2}$ for any $B \in CP^m$, M being a real hypersurface and ψ a full imbedding into $H_1M(m+1)$, from (3.2) and (3.3) we get that Q, Q^2 and I are linearly dependent vectors, that is, there exist θ_1, θ_2, θ_3 real number such that

(3.5) $$\theta_1 Q^2 + \theta_2 Q + \theta_3 I = 0.$$

Consequently

(3.6) $$Q = \begin{pmatrix} \lambda_1 & & & & & \\ & \cdot (m_1 & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \cdot (m_2 & \\ & & & & & \lambda_2 \end{pmatrix}, \quad \text{for some} \quad \lambda_1, \lambda_2 \in R$$

Note from (3.1) that trace $Q = 1$. Then $m_1\lambda_1 + m_2\lambda_2 = 1$.

If $\lambda_1 = \lambda_2$, i.e. $Q = \frac{1}{m+1} I$, then from (3.1) it follows that M is minimal in CP^m. But it is known (Theorem 2.8, of [8]) that there exist no minimal real hypersurfaces in CP^m $(m \geqq 2)$ which are minimal in some hypersphere of $HM(m+1)$. So $\lambda_1 \neq \lambda_2$ and the points of M satisfy the equation

(3.7) $$\text{trace } QB = 2\alpha = \text{constant} \qquad \text{for any} \quad B \in M.$$

Let $B = \psi(z) = z^* z$, with $\|z\|^2 = zz^* = 1$, then (3.7) can be written in the form

(3.8) $$|z_0|^2 + |z_1|^2 + \cdots + |z_{m_1}|^2 = r = \text{constant}$$

with $z = (z_0, \cdots, z_{m_1}, \cdots, z_m)$.

Consequently, from (3.8) M will be locally congruent to a hypersurface of the type $M_{p,q}(r_1, r_2) = \Pi(S^p(\sqrt{r_1}) \times S^q(\sqrt{r_2}))$ with $r_1 + r_2 = 1$, $p + q = 2m$.

In the following we see which $M_{p,q}(r_1, r_2)$ are minimal in a hypersphere of $HM(m+1)$.

From (2.3)

(3.9) $$\tilde{\psi}(z, w) = \begin{pmatrix} z^* \\ w^* \end{pmatrix}(z, w) = \left(\begin{array}{c|c} \bar{z}_i z_j & \bar{z}_i w_j \\ \hline \bar{w}_j z_i & \bar{w}_i w_j \end{array}\right),$$

where $(z, w)\in S^p(\sqrt{r_1})\times S^q(\sqrt{r_2})\subset S^{2m+1}(1)$.

Finally, from (3.9), the properties of Δ and the fact that the fibres of $\Pi : S^p(\sqrt{r_1})\times S^q(\sqrt{r_2})\to M_{p,q}(r_1, r_2)$ are totally geodesic it follows

$$\Delta x=-(2m-1)\bar{H}=\left(\begin{array}{c|c} \dfrac{2(p+1)}{r_1}\bar{z}_i z_j-4r_1 I & \left(\dfrac{p}{r_1}+\dfrac{q}{r_2}\right)\bar{z}_i w_j \\ \hline \left(\dfrac{p}{r_1}+\dfrac{q}{r_2}\right)\bar{w}_j z_i & \dfrac{2(q+1)}{r_2}\bar{w}_i w_j-4r_2 I \end{array}\right), \tag{3.10}$$

if $p, q>1$, and

$$\Delta x=-(2m-1)\bar{H}=\left(\begin{array}{c|c} 0 & \left(\dfrac{1}{r_1}+\dfrac{2m-1}{r_2}\right)\bar{z}_0 w_i \\ \hline \left(\dfrac{1}{r_1}+\dfrac{2m-1}{r_2}\right)\bar{w}_i z_0 & \dfrac{2(2m)}{r_2}\bar{w}_i w_j-4r_2 I \end{array}\right), \tag{3.11}$$

if $p=1$, where x is the immersion of $M_{p,q}(r_1, r_2)$ in $HM(m+1)$ induced by ϕ.

Thus from (3.1), (3.10) and (3.11) we can conclude that $M_{p,q}(r_1, r_2)$ is minimal in a hypersphere of $HM(m+1)$ if and only if $p=1$, $q=2m-1$, $r_1=\dfrac{1}{2(m+1)}$, $r_2=\dfrac{2m+1}{2m+2}$, which concludes the proof.

From Theorem 3.1 and the definition of Mono-order it follows

COROLLARY 3.2. *Let M be a closed real hypersurface of CP^m $(m\geqq 2)$. Then the isometric immersion $x : M\to HM(m+1)$ is Mono-order if and only if M is congruent to the geodesic hypersphere*

$$M_{1,2m-1}\left(\frac{1}{2m+2}, \frac{2m+1}{2m+2}\right)=\Pi\left(S^1\left(\sqrt{\frac{1}{2m+2}}\right)\times S^{2m-1}\left(\sqrt{\frac{2m+1}{2m+2}}\right)\right).$$

The following result is known

THEOREM A [7]. *Let M^n be an n-dimensional closed Riemannian manifold and $x : M^n\to E^m$ an isometric immersion of M into the Euclidean space. Then*

$$\frac{\lambda_1}{n}\mathrm{vol}(M)\leqq\int_M \|H\|^2 dV,$$

and the equality holds if and only if M is an order 1 submanifold of E^m, H being the mean curvature vector of the immersion and λ_1 the first spectral eigenvalue.

Using this result, (2.11) and (2.12) it follows

COROLLARY 3.3. *Let M be a closed real hypersurface of CP^m. Then*

$$\lambda_1\leqq\frac{2m-1}{\mathrm{vol}(M)}\int_M \|H\|^2 dV+\frac{4(2m^2-1)}{2m-1}, \tag{3.12}$$

where H is the mean curvature vector of M in CP^m. Moreover, the equality in (3.12) holds if and only if M is congruent to the geodesic hypersphere

$$M_{1,2m-1}\left(\frac{1}{2m+2}, \frac{2m+1}{2m+2}\right).$$

Remark. From Theorem 2.8 of [8], if $H=0$, the equality in (3.12) never occurs.

§ 4. Bi-order Immersions.

Along this section M will be a minimal real hypersurface of CP^m and we shall denote by $\bar{H}$ the mean curvature vector of M in $HM(m+1)$. Then as M is minimal in CP^m, from (2.8) it follows

$$\bar{H}_B = H_B^{\perp} = \frac{4}{2m-1}(I-(m+1)B) - \frac{1}{2m-1}\tilde{\sigma}(N, N). \tag{4.1}$$

PROPOSITION 4.1. *Let M be a minimal real hypersurface of CP^m. Then*

$$\Delta\bar{H}(B) = \frac{4}{2m-1}JAJN + \frac{8(2m+1)}{2m-1}(I-(m+1)B) - \frac{2(2m+2+\|\sigma\|^2)}{2m-1}\tilde{\sigma}(N, N) + \frac{2}{2m-1}\sum_j \tilde{\sigma}(AE_j, AE_j), \tag{4.2}$$

where N is a unit normal vector field to M in CP^m and $\{E_1, \cdots, E_{2m-1}\}$ is an orthonormal basis of TM.

Proof. Let $\{E_1, \cdots, E_{2m-1}\}$ be an orthonormal basis in TM such that $(\nabla_{E_i}E_j)_B = 0$ for any $i, j=1, \cdots, 2m-1$. Then from (2.10), (2.11) and (4.1),

$$\begin{aligned}(d\bar{H})(E_j) &= -\frac{4(m+1)}{2m-1}E_j + \frac{2}{2m-1}\tilde{\sigma}(AE_j, N) + \frac{1}{2m-1}\tilde{A}_{\tilde{\sigma}(N,N)}E_j \\ &= -\frac{2(2m+1)}{2m-1}E_j + \frac{2}{2m-1}\tilde{\sigma}(AE_j, N) + \frac{2}{2m-1}g(JN, E_j)JN.\end{aligned} \tag{4.3}$$

Now from (4.3) and having in mind that $(\nabla_{E_i}E_j)_B=0$ it follows

$$\begin{aligned}\Delta\bar{H}(B) &= -\sum_j D_{E_j}D_{E_j}\bar{H} = \sum_j D_{E_j}\Big(\frac{2(2m+1)}{2m-1}E_j - \frac{2}{2m-1}\tilde{\sigma}(AE_j, N) \\ &\quad - \frac{2}{2m-1}g(JN, E_j)JN\Big) = \frac{2(2m+1)}{2m-1}\bar{H} + \frac{2}{2m-1}\sum_j g(\phi AE_j, E_j)JN \\ &\quad - \frac{2}{2m-1}\tilde{\sigma}(JN, JN) + \frac{2}{2m-1}JAJN + \frac{2}{2m-1}\tilde{A}_{\tilde{\sigma}(AE_j,N)}E_j \\ &\quad - \frac{2}{2m-1}\sum_j \tilde{\sigma}(\sigma(E_j, AE_j), N) - \frac{2}{2m-1}\sum_j \tilde{\sigma}((\bar{\nabla}_{E_j}A)E_j, N)\end{aligned}$$

$$+\frac{2}{2m-1}\sum_j \tilde{\sigma}(AE_j, AE_j).$$

From the above expression, (2.10), (2.11) and the fact that M is minimal in CP^m, we conclude

$$\Delta\bar{H}(B)=\frac{8(2m+1)}{2m-1}(I-(m+1)B)-\frac{2(2m+1)}{2m-1}\tilde{\sigma}(N, N)-\frac{2}{2m-1}\tilde{\sigma}(N, N)$$
$$+\frac{2}{2m-1}JAJN+\frac{2}{2m-1}JAJN-\frac{2}{2m-1}\|\sigma\|^2\tilde{\sigma}(N, N)$$
$$+\frac{2}{2m-1}\sum_j \tilde{\sigma}(AE_j, AE_j)-\frac{2}{2m-1}\sum_j \tilde{\sigma}((\bar{\nabla}_{E_j}A)E_j, N). \tag{4.4}$$

From the equation of Codazzi of M in CP^m it is easy to see that

$$\sum_j \tilde{\sigma}((\bar{\nabla}_{E_j}A)E_j, N)=0. \tag{4.5}$$

Consequently, from (4.4) we have

$$\Delta\bar{H}(B)=\frac{4}{2m-1}JAJN+\frac{8(2m+1)}{2m-1}(I-(m+1)B)$$
$$-\frac{2(2m+2+\|\sigma\|^2)}{2m-1}\tilde{\sigma}(N, N)+\frac{2}{2m-1}\sum_j \tilde{\sigma}(AE_j, AE_j),$$

which concludes the proof.

LEMMA 4.2. *Let M be a minimal hypersurface of CP^m. Then*

i) $g(B, B)=\frac{1}{2}$:

ii) $g(B, \bar{H})=-1$,

iii) $g(B, \Delta\bar{H})=\frac{4(1-2m^2)}{2m-1}$,

iv) $g(\bar{H}, \bar{H})=\frac{4(2m^2-1)}{(2m-1)^2}$,

v) $g(\Delta\bar{H}, \bar{H})=\frac{8(m+1)(4m^2-2m-1)+4\|\sigma\|^2-4\|AJN\|^2}{(2m-1)^2}$.

Proof. It follows easily from (2.11), (2.12) and (4.2).

DEFINITION 4.3. Let $x : M^n \to E^m$ be an isometric immersion of a closed Riemannian manifold into the Euclidean space with mean curvature vector H. x is called of *order* $\{k_1, k_2\}$, [9], if it is of the form

$$x-x_0=x_{k_1}+x_{k_2} \tag{4.6}$$

for some k_1, k_2.

It is easy to see that x is of order $\{k_1, k_2\}$ if and only if

$$\Delta H = aH + b(x - x_0) \tag{4.7}$$

for some $a, b \in R$; [9].

Note that $k_1 \neq k_2$ if and only if x is Bi-order. Moreover $a=0$ if and only if x is Mono-order.

As M cannot be Mono-order in $HM(m+1)$ (Theorem 2.8 of [8]) it follows that the immersion $x: M \to HM(m+1)$ is of order $\{k_1, k_2\}$ if and only if (4.7) holds with $a, b \neq 0$.

In the following we study the minimal real hypersurfaces of CP^m ($m \geqq 2$) satisfying

$$\Delta \bar{H}(B) = a\bar{H} + b(B - Q) \qquad \text{for any} \quad B \in M \tag{*}$$

$a, b \in R$, $a, b \neq 0$, Q being a constant, for which we need to prove

LEMMA 4.4. *Let M^n be a complex submanifold of CP^m of complex dimension n. If for any unit normal vector to M^n in CP^m, ξ, the Weingarten endomorphism, A_ξ, has at most four principal curvatures, which are constants on M^n, then M^n has parallel second fundamental form in CP^m.*

Proof. Let ξ be a unit normal vector field to M in CP^m such that $(\nabla^\perp \xi)(B) = 0$ for some fixed point $B \in M$, where $\nabla^\perp$ is the normal connection on M.

As M is a complex submanifold, the eigenvalues of A_ξ are λ, μ, $-\lambda$, $-\mu$, for some $\lambda, \mu \in R$. Let V_λ and V_μ be the distributions of the eigenspaces of A_ξ corresponding to the eigenvalues λ and μ respectively. If $\{E_1, \cdots, E_p\}$, $\{E_{p+1}, \cdots, E_n\}$ are local basis of orthonormal vector fields of V_λ and V_μ, respectively, then $\{JE_1, \cdots, JE_p\}$, $\{JE_{p+1}, \cdots, JE_n\}$ are local basis of orthonormal vector fields of the distributions $V_{-\lambda}$ and $V_{-\mu}$, respectively.

Let $X \in TM$ and $i, j = 1, \cdots, p$. Then as λ is constant

$$\begin{aligned} 0 = X(g(A_\xi E_i, E_j)) &= g((\bar{\nabla}_X A)_\xi E_i, E_j) + g(A_\xi \nabla_X E_i, E_j) \\ &+ g(A_\xi E_i, \nabla_X E_j) = g((\bar{\nabla}_X A)_\xi E_i, E_j) + \lambda(g(\nabla_X E_i, E_j) \\ &+ g(E_i, \nabla_X E_j)) = g((\bar{\nabla}_X A)_\xi E_i, E_j)\,. \end{aligned}$$

Hence, from the commutativity properties of $\bar{\nabla} A$ and J, we have

$$g((\bar{\nabla}_X A)_\xi Y, Z) = 0\,, \tag{4.8}$$

for all $X \in T_B M$, $Y, Z \in V_\lambda(B) \oplus V_{-\lambda}(B)$.

In the same way

$$g((\bar{\nabla}_X A)_\xi Y, Z) = 0\,, \tag{4.9}$$

for all $X \in T_B M$, $Y, Z \in V_\mu(B) \oplus V_{-\mu}(B)$.

Finally if $X, Y, Z \in T_B M$, taking orthogonal projection on $V_\lambda(B) \oplus V_{-\lambda}(B)$ and $V_\mu(B) \oplus V_{-\mu}(B)$, from (4.8), (4.9) and Codazzi equation, we conclude the Lemma.

THEOREM 4.5. *Let M be a minimal real hypersurface of CP^m ($m\geqq 2$). Then M satisfies condition (*) if and only if one of the following cases holds:*

i) *M is locally congruent to the geodesic hypersphere*

$$M_{1,2m-1}\left(\frac{1}{2m-1}, \frac{2m-2}{2m-1}\right).$$

ii) *M is locally congruent to the hypersurface $M_{m,m}\left(\frac{1}{2}, \frac{1}{2}\right)$ and in this case m is even.*

Proof. Take M verifying (*). We can suppose that Q is diagonal. Put

$$Q=\begin{pmatrix} q_1 & & & & & \\ & \ddots (m_1 & & & & \\ & & q_1 & & & \\ & & & \ddots & & \\ & & & & q_k & \\ & & & & & \ddots (m_k \\ & & & & & q_k \end{pmatrix}.$$

From (2.11), (2.12) and lemma 4.2, multiplying scalarly (*) by B, we obtain

$$g(B, Q)=a_1, \qquad \text{for any} \quad B\in M, \tag{4.10}$$

where a_1 is a real constant. So Q is normal to M in $HM(m+1)$. Hence

$$Xg(Q, N)=g(Q, \tilde{\sigma}(X, N)), \qquad \text{for all} \quad X\in TM. \tag{4.11}$$

From (2.11), (2.12) and (4.2), multiplying scalarly (*) by $\tilde{\sigma}(X, N)$, we have

$$g(Q, \tilde{\sigma}(X, N))=\frac{-4}{b(2m-1)}g(A^2JN, \phi X). \tag{4.12}$$

Moreover, as Q is normal to M in $HM(m+1)$, from (*) and (4.2) it is easy to prove that $AJN=\mu JN$, for some real constant μ. Hence, from (4.11), (4.12) and by a similar reasoning to the one used in the proof of theorem 3.1, we have

$$g(B, Q^2)=a_2, \qquad \text{for any} \quad B\in M, \tag{4.13}$$

for some real constant a_2, and

$$Q=\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots (m_1 & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots (m_2 & \\ & & & & & \lambda_2 \end{pmatrix}, \tag{4.14}$$

where $\lambda_1, \lambda_2\in R$, $m_1\lambda_1+m_2\lambda_2=1$.

Case i), $\lambda_1=\lambda_2$, i.e. $Q=\frac{1}{m+1}I$. Then from (*) we have

$$\Delta\bar{H}(B)=a\bar{H}+b\left(B-\frac{1}{m+1}I\right) \qquad \text{for any } B\in M,\ a,\ b\in R,\ a,\ b\neq 0. \tag{4.15}$$

By equaling the tangent components of (4.15) we obtain, from (4.2)

$$AJN=0\,. \tag{4.16}$$

From (4.16) and using the Codazzi equation for the immersion of M in CP^m we have

$$g(A\phi AX,\ Y)=g(\phi X,\ Y)\,, \qquad \text{for any} \quad X,\ Y\in TM. \tag{4.17}$$

On the other hand, multiplying scalarly by $\tilde{\sigma}(X, Y)$ in (4.15), we get, from (2.11),

$$\begin{aligned} &4g(AX,\ AY)+4g(A\phi X,\ A\phi Y)\\ &=\{2a(2m+1)-(2m-1)b-8(2m+1)(m+1)\}\,g(X,\ Y)\\ &\quad+4\{2m+2+\|\sigma\|^2-2a\}\,g(X,\ JN)g(Y,\ JN)\,, \end{aligned} \tag{4.18}$$

for any X, $Y\in TM$. In particular, if $X\in TM$ and $g(X, JN)=0$, from (4.16) and (4.18) we have

$$A^2X-JA^2JX=\lambda X\,, \tag{4.19}$$

where λ is a real constant. From (4.17) and (4.19) we obtain

$$A^4X-\lambda A^2X+X=0\,, \tag{4.20}$$

for any $X\in TM$ with $g(X, JN)=0$.

Consequently, from (4.16) and (4.20) we conclude that

$$A=\left(\begin{array}{ccccccccc|c} \alpha &&&&&&&&& \\ &\ddots&&&&&&&& \\ &&\alpha&&&&&&& \\ &&&\beta&&&&&& \\ &&&&\ddots&&&&& \\ &&&&&\beta&&&& \\ &&&&&&-\alpha&&& \\ &&&&&&&\ddots&& \\ &&&&&&&&-\alpha& \\ &&&&&&&&&-\beta \\ &&&&&&&&&\ddots \\ &&&&&&&&&-\beta \\ \hline &&&&&&&&& 0 \end{array}\right). \tag{4.21}$$

respect to certain orthonormal basis $\{X_1, \cdots, X_{m-1}, JX_1, \cdots, JX_{m-1}, JN\}$ of TM, where $\alpha^2+\beta^2=\lambda$ and $\alpha^2\beta^2=1$. Then if $\alpha=\beta$, from (4.21) we have that m is even and M is locally congruent to $M_{m,m}\left(\frac{1}{2}, \frac{1}{2}\right)$ (see [11]). If $\alpha\neq\beta$, as α, β are constants on M and JN is principal with principal curvature 0, from Proposition 3.1 and Theorem 1 in [2], and taking into account Lemma 4.4, we see that M is locally congruent to a minimal tube of radius $\frac{\Pi}{4}$ on the complex quadric, embedded as a complex hypersurface of CP^m. But it is known that there are not tubes of the above radius on the complex quadric. So this last possibility cannot occurs.

Case ii) $\lambda_1 \neq \lambda_2$. As in the Theorem 3.1, it follows that M is locally congruent to a hypersurface of the type $M_{p,q}(r_1, r_2)$. From (3.10) and (3.11) we see that the only minimal submanifold of the above type with $Q \neq \frac{1}{m+1} I$, which is Bi-order in $HM(m+1)$ is $M_{1,2m-1}\left(\frac{1}{2m-1}, \frac{2m-2}{2m-1}\right)$.

Note that $M_{1,2m-1}\left(\frac{1}{2m-1}, \frac{2m-2}{2m-1}\right)$ and $M_{m,m}\left(\frac{1}{2}, \frac{1}{2}\right)$ are submanifolds of order {1, 2} in $HM(m+1)$. This concludes the proof.

The following result is due to B.Y. Chen and A. Ros, independently, see [5] and [9].

THEOREM B. *Let $x : M^n \to E^m$ be an isometric immersion of a compact manifold into the Euclidean space. Then*

$$\int_M (n^2 g(\Delta H, H) - n^2(\lambda_1 + \lambda_2) g(H, H) - n\lambda_1\lambda_2 g(x, H)) dV \geqq 0$$

and the equality holds if and only if x is of order {1, 2}.

As an application of Theorem B, using Lemma 4.2 we obtain

COROLLARY 4.6. *Let M be a compact minimal real hypersurface immersed in CP^m. Then*

$$(8(m+1)(4m^2-2m-1) - 4(2m^2-1)(\lambda_1+\lambda_2) + (2m-1)\lambda_1\lambda_2)\mathrm{vol}(M)$$
$$\geqq 4\int_M (\|AJN\|^2 - \|\sigma\|^2) dV$$

and the equality holds if and only if M is either i) *or* ii) *in Theorem* 4.5.

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DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
GRANADA (SPAIN)