1. Introduction

Among all submanifolds of an almost Hermitian manifold, there are two typical classes: one is the class of almost complex submanifolds, and the other is the class of totally real submanifolds. A Riemannian submanifold \( (M, \phi) \) (or briefly \( M \)) of an almost Hermitian manifold \( (\tilde{M}, \tilde{J}, \langle \cdot, \cdot \rangle) \) (or briefly \( \tilde{M} \)) is called an almost complex submanifold provided that

\[
J \phi((d \phi)_p(X)) \in (d \phi)_p(T_p(M))
\]

for any \( X \in T_p(M), \ p \in M \). The most typical example of nearly Kaehlerian manifolds is a \( 6 \)-dimensional sphere \( S^6 \). In fact, Fukami and Ishihara \([3]\) proved that there exists a nearly Kaehlerian structure on a \( 6 \)-dimensional sphere \( S^6 \) by making use of the properties of the Cayley division algebra. We shall call it the canonical nearly Kaehlerian structure on \( S^6 \). In this paper, we shall study almost complex submanifolds of a \( 6 \)-dimensional unit sphere \( S^6 \) with the canonical nearly Kaehlerian structure. First of all, Gray \([1]\) proved that with respect to the canonical nearly Kaehlerian structure, \( S^6 \) has no \( 4 \)-dimensional almost complex submanifolds. We shall prove the following Theorems and some related results. In the following Theorems, we assume that \( M=\langle M, \phi \rangle \) is an almost complex submanifold of \( S^6 \). Then it follows that \( \dim M=2 \). We denote by \( K \) the Gaussian curvature of \( M \).

**Theorem A.** If \( (M, \phi) \) is not totally geodesic, then the degree of \( \phi \) is 3.

**Theorem B.** If \( K \) is constant on \( M \), then \( K=1 \) or \( 1/6 \) or 0.

**Theorem C.** Assume that \( M \) is compact. If \( K>1/6 \) on \( M \), then \( K=1 \) on \( M \), and if \( 1/6 \leq K<1 \) on \( M \), then \( K=1/6 \) on \( M \).

In the last section of this paper, we shall give some examples of almost complex submanifolds of \( S^6 \) corresponding to the cases, \( K=1, \ 1/6 \) and 0 in Theorem B. We note that the result of Theorem B is a special case of the result obtained by Kenmotsu under more general situation (\([6]\)).

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2. Riemannian submanifolds

Let \( \tilde{M}, \langle , \rangle \) (or briefly \( \tilde{M} \)) be a Riemannian manifold and \( (M, \phi) \) (or briefly \( M \)) be a Riemannian submanifold of \( \tilde{M} \) with isometric immersion \( \phi \). Let \( \nabla \) (resp. \( \tilde{\nabla} \)) be the Riemannian connection on \( M \) (resp. \( \tilde{M} \)) and \( R \) (resp. \( \tilde{R} \)) be the curvature tensor of \( M \) (resp. \( \tilde{M} \)). We denote by \( \sigma \) the second fundamental form of \( M \) in \( \tilde{M} \). Since \( \phi \) is locally an imbedding, we may identify \( \tilde{p} \in \tilde{M} \) with \( \phi(p) \in M \) locally, and \( T_p(M) \) with the subspace \( \langle d\phi \rangle_p(T_p(M)) \) of \( T_{\phi(p)}(\tilde{M}) \). Then, the Gauss formula, Weingarten formula are given respectively by

\[
\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,
\]

\[

\tilde{\nabla}_X \xi = - A^\xi X + \nabla^\xi \xi, \quad X, Y \in \mathfrak{X}(M),
\]

where \( \xi \) is a local field of normal vector to \( M \) and \( - A^\xi X \) (resp. \( \nabla^\xi \xi \)) denotes the tangential part (resp. normal part) of \( \tilde{\nabla}_X \xi \).

The tangential part \( A^\xi X \) is related to the second fundamental form \( \sigma \) as follows:

\[
\langle \sigma(X, Y), \xi \rangle = \langle A^\xi X, Y \rangle, \quad X, Y \in \mathfrak{X}(M).
\]

We denote by \( R^\perp \) the curvature tensor of the normal connection, i.e., \( R^\perp(X, Y) = [\nabla^\xi, \nabla^Y] - \nabla^Y X \). Then, the Gauss, Codazzi and Ricci equations are given respectively by

\[
\langle R(X, Y)Z, Z' \rangle = \langle \tilde{R}(X, Y)Z, Z' \rangle + \langle \sigma(X, Z'), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, Z') \rangle,
\]

\[
(\tilde{R}(X, Y)Z)^\parallel = (\nabla^Y \sigma)(Y, Z) - (\nabla^X \sigma)(X, Z),
\]

\[
\langle \tilde{R}(X, Y)\xi, \eta \rangle = \langle R^\perp(X, Y)\xi, \eta \rangle - \langle [A^\eta, A^\xi] X, Y \rangle,
\]

for \( X, Y, Z, Z' \in \mathfrak{X}(M) \), where \( (\nabla^Y \sigma)(Y, Z) = \nabla^Y \sigma(Y, Z) - \sigma(\nabla^X Y, Z) - \sigma(Y, \nabla^X Z) \) and \( \xi, \eta \) are local fields of normal vectors to \( M \).

In the sequel, the following convention for the notations will be used unless otherwise specified:

\[
X, Y, Z, \ldots, \in \mathfrak{X}(M), \quad U, V, W, \ldots, \in \mathfrak{X}(\tilde{M})
\]

and \( \mathfrak{X}(M) \) (resp. \( \mathfrak{X}(\tilde{M}) \)) denotes the set of all tangential vector fields to \( M \) (resp. \( \tilde{M} \)).

For the definition of the degree of the isometric immersion \( \phi \), we refer to [8].
3. 6-dimensional nearly Kaehlerian manifolds

In this section, for the sake of later uses, we shall recall some elementary formulas in a 6-dimensional nearly Kaehlerian manifold and furthermore the canonical nearly Kaehlerian structure on a 6-dimensional unit sphere $S^6$. Let $\tilde{M}$ be an almost Hermitian manifold with the almost Hermitian structure $(J, \langle \cdot, \cdot \rangle)$. We denote by $N$ the Nijenhuis tensor of $J$ and by $\tilde{\nabla}$ the Riemannian connection of $\tilde{M}$. It is known that the tensor field $N$ satisfies

\begin{equation}
N(JU, V) = N(U, JV) = -JN(U, V), \quad U, V \in \mathfrak{k}(\tilde{M}).
\end{equation}

Especially, if $\tilde{M}$ is a nearly Kaehlerian manifold (i.e., $(\tilde{\nabla}_U J)U = 0$, for any $U \in \mathfrak{k}(\tilde{M})$, then the tensor field $N$ is written in the following form (cf. [13]):

\begin{equation}
N(U, V) = -4J(\tilde{\nabla}_U J)V, \quad U, V \in \mathfrak{k}(\tilde{M}).
\end{equation}

From (3.2), we get

\begin{equation}
\langle N(U, V), W \rangle = -\langle N(U, W), V \rangle, \quad U, V, W \in \mathfrak{k}(\tilde{M}).
\end{equation}

An almost complex submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is called to be a $\sigma$-submanifold if the second fundamental form $\sigma$ is complex linear, i.e.,

\begin{equation}
\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y), \quad \text{for} \quad X, Y \in \mathfrak{k}(M),
\end{equation}

(cf. [12]). From (3.4), any $\sigma$-submanifold is necessarily minimal. Vanhecke [12] proved that if $\tilde{M}$ is a nearly Kaehlerian manifold, any almost complex submanifold is a $\sigma$-submanifold and is also a nearly Kaehlerian manifold. W now assume that $\tilde{M}$ is a 6-dimensional non-Kaehlerian, nearly Kaehlerian manifold. Then the followings hold in $\tilde{M}$ (cf. [7], [9]):

\begin{equation}
\tilde{\nabla}_U (\tilde{\nabla}_V J)W = -\frac{S}{30} (\langle U, V \rangle JW - \langle U, W \rangle JV + \langle JU, W \rangle V),
\end{equation}

\begin{equation}
\langle \tilde{\nabla}_U J \rangle (\tilde{\nabla}_V J)W = -\frac{S}{30} (\langle U, V \rangle JW - \langle U, W \rangle V

+ \langle JU, V \rangle JW - \langle JU, W \rangle JV),
\end{equation}

$U, V, W \in \mathfrak{k}(\tilde{M})$, where $S$ denotes the scalar curvature of $\tilde{M}$.

From (3.2), (3.5) and (3.6), we get

\begin{equation}
(\tilde{\nabla}_U N)(V, W) = \frac{2S}{15} (\langle JU, V \rangle JW - \langle JU, W \rangle JV + \langle JV, W \rangle JU),
\end{equation}

\begin{equation}
N(U, N(V, W)) = 16 (\tilde{\nabla}_U J)(\tilde{\nabla}_V J)W
\end{equation}
ALMOST COMPLEX SUBMANIFOLDS OF A 6-DIMENSIONAL SPHERE

\[ = -\frac{8S}{15} (\langle U, V \rangle W - \langle U, W \rangle V + \langle JU, V \rangle JW - \langle JU, W \rangle JV), \]

(3.9)

\[ = \frac{8S}{15} (\langle U, U' \rangle \langle V, V' \rangle - \langle U, V' \rangle \langle V, U' \rangle + \langle JU, U' \rangle \langle JV, V' \rangle - \langle JU, V' \rangle \langle JV, U' \rangle), \]

\( U, U', V, V', W \in \mathbb{R}^6 \).

We shall now recall the canonical nearly Kaehlerian structure on a 6-dimensional sphere \( S^6 \). Let \( C \) be the Cayley division algebra generated by \( \{ e_i = 1, e_i(l \leq i \leq 7) \} \) over real number field \( \mathbb{R} \) and \( C_+ \) be the subspace of \( C \) consisting of all purely imaginary Cayley numbers. We may identify \( C_+ \) with a 7-dimensional Euclidean space \( \mathbb{R}^7 \) with the canonical inner product \( (, ) \) (i.e., \( (e_i, e_j) = \delta_{ij}, 1 \leq i, j \leq 7 \)). The automorphism group of \( C \) is the compact simple Lie group \( G_2 \) and the inner product \( (, ) \) is invariant under the action of the group \( G_2 \). A vector cross product for the vectors in \( C_+ = \mathbb{R}^7 \) is defined by

\[ x \times y = (x, y)e_0 + xy, \quad x, y \in C_+. \]

Then the multiplication table is given by the following:

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j )</td>
<td>0</td>
<td>( e_3 )</td>
<td>( -e_2 )</td>
<td>( e_5 )</td>
<td>( -e_4 )</td>
<td>( e_7 )</td>
<td>( -e_6 )</td>
</tr>
<tr>
<td>2</td>
<td>( -e_3 )</td>
<td>0</td>
<td>( e_1 )</td>
<td>( e_6 )</td>
<td>( -e_7 )</td>
<td>( -e_4 )</td>
<td>( e_5 )</td>
</tr>
<tr>
<td>3</td>
<td>( e_2 )</td>
<td>( -e_1 )</td>
<td>0</td>
<td>( -e_7 )</td>
<td>( -e_6 )</td>
<td>( e_5 )</td>
<td>( e_4 )</td>
</tr>
<tr>
<td>4</td>
<td>( -e_5 )</td>
<td>( -e_6 )</td>
<td>( e_7 )</td>
<td>0</td>
<td>( e_1 )</td>
<td>( e_2 )</td>
<td>( -e_3 )</td>
</tr>
<tr>
<td>5</td>
<td>( e_4 )</td>
<td>( e_7 )</td>
<td>( e_6 )</td>
<td>( -e_1 )</td>
<td>0</td>
<td>( -e_3 )</td>
<td>( -e_2 )</td>
</tr>
<tr>
<td>6</td>
<td>( -e_7 )</td>
<td>( e_4 )</td>
<td>( -e_5 )</td>
<td>( -e_2 )</td>
<td>( e_3 )</td>
<td>0</td>
<td>( e_1 )</td>
</tr>
<tr>
<td>7</td>
<td>( e_6 )</td>
<td>( -e_5 )</td>
<td>( -e_4 )</td>
<td>( e_3 )</td>
<td>( e_2 )</td>
<td>( -e_1 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Considering \( S^6 \) as \( \{ x \in C_+ ; (x, x) = 1 \} \), the canonical almost complex structure \( J \) on \( S^6 \) is defined by

\[ J \circ U = x \times U, \]

where \( x \in S^6 \) and \( U \in T_x(S^6) \) (the tangent space of \( S^6 \) at \( x \)).

The above almost complex structure \( J \) together with the induced Riemannian metric \( \langle , \rangle \) on \( S^6 \) from the inner product \( (, ) \) on \( C_+ = \mathbb{R}^7 \) gives rise to a nearly Kaehlerian structure on \( S^6 \). The group \( G_2 \) acts on \( S^6 \) transitively as the group of automorphisms of the nearly Kaehlerian structure \( (J, \langle , \rangle) \) (cf. [3]). It is well known that \( S^6 \) does not admit any Kaehlerian structures.
4. Proofs of Theorems A, B and C

Let $M$ be an almost complex submanifold of a 6-dimensional unit sphere $\tilde{M}=S^6$ with the canonical nearly Kaehlerian structure $(J, \langle, \rangle)$. Then it follows that $\dim M=2$ and hence $M$ is a Kaehlerian manifold of complex dimension 1 with respect to the induced structure from $S^6$. We denote by $K$ the Gaussian curvature of $M$. Then, from (2.4) and (3.4), we get

$$K=1-\frac{\|\sigma\|^2}{2},$$

where $\|\sigma\|$ denotes the length of the second fundamental form $\sigma$.

Codazzi equation (2.5) implies in particular

$$\nabla_X^2 \sigma(Y, Z)=\nabla_Y^2 \sigma(X, Z).$$

From (2.1), (2.2) and (3.2), we get

$$\nabla_\sigma(J\sigma(X, Y))=\frac{1}{4}JN(Z, \sigma(X, Y))+J(-A_{\sigma(X,Y)}Z+\nabla_\sigma(X, Y)),$nabla_\sigma(X, JY)=-A_{\sigma(X,Y)}Z+\nabla_\sigma(X, JY).$$

From (4.3), taking account of (3.1), (3.3) and (3.4), we get

$$\frac{1}{4}JN(Z, \sigma(X, Y))=(\nabla_\sigma(X, JY)-J(\nabla_\sigma(X, Y)).$$

Since $\dim M=2$, from (3.1) and (3.4), we get easily

$$N(Z, \sigma(X, Y))=N(Y, \sigma(X, Z)).$$

Let $M'=\{p\in M; \sigma\neq 0 \text{ at } p\}$. Then $M'$ is an open set of $M$.

We now assume that $M'\neq \emptyset$ (i.e., $M$ is not totally geodesic in $S^6$). Let $\{X_1, X_2=JX_1\}$ be a local field of orthonormal frame on a neighborhood of a point $p\in M'$ in $M$. If we put

$$\nabla_{X_i} X_j=\sum_{k=1}^{3} B_{1, j, k} X_k, \quad 1\leq i, j \leq 2,$nabla\sigma(X_1, X_i)=-B_{i, k} X_k, \quad 1\leq i, j, k \leq 2.$$

Taking account of (3.1), (3.3), (3.4) and (3.9), we may put

$$(\nabla_{X_1} \sigma)(X_1, X_1)=a\sigma(X_1, X_1)+b\sigma(X_1, X_2)$$

$$+c\frac{1}{4}N(X_1, \sigma(X_1, X_1))+\frac{d}{4}N(X_2, \sigma(X_1, X_1)).$$
\((\nabla_{X_1} \sigma)(X_1, X_2) = a' \sigma(X_1, X_1) + b' \sigma(X_1, X_2) + \frac{c'}{4} N(X_1, \sigma(X_1, X_1)) + \frac{d'}{4} N(X_2, \sigma(X_1, X_1)).\)

Then, from (4.8), taking account of (2.5), (3.1), (3.4) and (4.4), we get

\[(4.9)\quad a' = -b, \quad b' = a, \quad c' = d, \quad d' = -c - 1.\]

Thus, from (4.8), taking account of (3.3), (3.4) and (4.9), we get

\[(4.10)\quad a = \frac{1}{\|\sigma\|} X_1 \|\sigma\|, \quad b = -\frac{1}{\|\sigma\|} X_2 \|\sigma\|.

From (4.6), (4.7) and (4.10), we get

\[\llbracket X_1, X_2 \rrbracket \|\sigma\| = X_1(X_2 \|\sigma\|) - X_2(X_1 \|\sigma\|),\]

and hence

\[(4.11)\quad X_2 a + X_1 b + a B_{121} + b B_{212} = 0.\]

Taking account of (3.4), (4.6) and (4.7), we get easily

\[(4.12)\quad \sum_{i=1}^{2}(\nabla_{X_i} \sigma)(X_i, X_i) = 0.\]

From (4.8) with (4.9), taking account of (2.5), (3.1), (3.3)\textasciitilde(3.6) and (4.12), we get

\[(4.13)\quad \|\nabla' \sigma\|^2 = \sum_{i=1, j \neq i}^{2} \langle(\nabla_{X_i} \sigma)(X_j, X_k), (\nabla_{X_j} \sigma)(X_i, X_k)\rangle = 4 \langle(\nabla_{X_1} \sigma)(X_1, X_1), (\nabla_{X_1} \sigma)(X_1, X_1)\rangle + \langle(\nabla_{X_2} \sigma)(X_2, X_1), (\nabla_{X_2} \sigma)(X_1, X_1)\rangle = (2a^2 + b^2) + (2c^2 + c + d^2) + 1\|\sigma\|^2.\]

From (4.10) and (4.13), we get

\[(4.14)\quad a^2 + b^2 = \|\grad (\log \|\sigma\|)\|^2,\]

\[(4.15)\quad c^2 + c + d^2 = \frac{1}{2\|\sigma\|^2} (\|\nabla' \sigma\|^2 - 2\|\grad \|\sigma\|^2 - \|\sigma\|^2).\]

We put

\[F = \|\grad (\log \|\sigma\|)\|^2\]

and

\[G = \frac{1}{2\|\sigma\|^2} (\|\nabla' \sigma\|^2 - 2\|\grad \|\sigma\|^2 - \|\sigma\|^2).\]
Then, from (4.15), we have easily

**Lemma 4.1.** $G \geq -\frac{1}{4}$ on $M'$.

From (2.6), taking account of (2.1), (2.2), (3.1)~(3.4), (3.7), (3.8), (4.1), (4.5)~(4.9), we get

\[ \frac{1}{8} \| \sigma \|^2 = \langle R^e(X_1, X_2) \sigma(X_1, X_1), \sigma(X_2, X_2) \rangle \]
\[ = \frac{\| \sigma \|^2}{4} (X_1 a - X_2 b - b B_{121} + a B_{212} - 2G - 1) \]
\[ + \frac{1}{2} (X_1 B_{212} - X_2 B_{121} - B_{121} B_{212} + B_{212} B_{121}) \]
\[ = \frac{\| \sigma \|^2}{4} (X_1 a - X_2 b - b B_{121} + a B_{212} - 1 - 2G - 2K), \]

and hence

\[ \langle 4.16 \rangle \]
\[ X_1 a - X_2 b - b B_{121} + a B_{212} = 2G + 3K. \]

Similarly, we get

\[ \langle 4.17 \rangle \]
\[ X_1 d - X_2 c = 3(2c + 1) B_{121} - 6d B_{212} - 2a d - (2c + 1)b, \]
\[ \langle 4.18 \rangle \]
\[ X_1 c + X_2 d = -6d B_{121} - 3(2c + 1) B_{212} + 2bd - (2c + 1)a. \]

**Lemma 4.2.** $\Delta(\log \| \sigma \|) = 2G + 3K$ on $M'$.

**Proof.** From (4.6), (4.7), (4.10) and (4.16), we get

\[ \Delta \| \sigma \| = X_1(X_1 \| \sigma \|) + X_2(X_2 \| \sigma \|) + B_{121} X_1 \| \sigma \| + B_{212} X_1 \| \sigma \|
\]
\[ = \| \sigma \| (X_1 a - X_2 b - b B_{121} + a B_{212} + a^2 + b^2)
\]
\[ = \| \sigma \| (F + 2G + 3K), \]

and hence

\[ \Delta(\log \| \sigma \|) = (1/\| \sigma \|) \Delta \| \sigma \| - \| \text{grad}(\log \| \sigma \|) \|^2 \]
\[ = 2G + 3K. \quad \text{Q.E.D.} \]

Let $\{ E_1, E_2 = J E_2 \}$ be an orthonormal basis of $T_p(M)$, $p \in M'$ and $\gamma_i = \gamma_i(t_i)$ ($1 \leq i \leq 2$) be the geodesics in $M'$ such that

\[ \gamma_i(0) = p \quad \text{and} \quad \frac{d\gamma_i}{dt_i}(0) = E_i, \quad 1 \leq i \leq 2. \]

Then, we may easily see that there exists an orthonormal frame field $\{ X_i, X_i = JX_i \}$ near $p$ in $M'$ such that
(4.19) \[ X_i = E_i \quad (1 \leq i \leq 2) \] at \( p \),

and

\[ X_1 = \frac{d\gamma_1}{dt_1} \text{ along } \gamma_1, \quad X_2 = \frac{d\gamma_2}{dt_2} \text{ along } \gamma_2. \]

From (4.19), we get

(4.20) \[ B_{121} = 0 \text{ along } \gamma_1 \text{ and } B_{212} = 0 \text{ along } \gamma_2. \]

From (4.17) and (4.18), taking account of (4.19) and (4.20), we get

(4.21)

\[
E_1(X_1^d) - E_1(X_2^c) = -(2c + 1)E_1^d - 2dE_1^a - 6dE_2B_{212} - 2bE_2^c - 2aE_2^d,
\]

\[
E_2(X_1^c) + E_2(X_2^d) = -(2c + 1)E_2^a + adE_2^b - 6dE_2B_{212} - 2aE_2^c + 2bE_2^d.
\]

From (4.21), taking account of (4.11), (4.16) and (4.20), we get

(4.22) \[ d = -4dG - 2aE_1^d + 2bE_2^d - 2bE_1^c - 2aE_2^c. \]

Similarly, we get

(4.23) \[ c = -(2c + 1)G + 2bE_1^d + 2aE_2^d - 2aE_1^c + 2bE_2^c. \]

On one hand, from (4.17), (4.18) and (4.20), we get

(4.24)

\[
(E_1c)^a = -(E_2c)(E_2^d) - (2c + 1)aE_1c + 2bdE_2c,
\]

\[
(E_2c)^a = (E_2c)(E_1^d) + 2adE_2c + (2c + 1)bE_1c,
\]

\[
(E_2d)^a = (E_2c)(E_1^d) - 2adE_1d - (2c + 1)bE_2d,
\]

\[
(E_2d)^a = -(E_1c)(E_2^d) - (2c + 1)aE_2d + 2bdE_2d.
\]

From (4.17), (4.18) and (4.24), we get

(4.25) \[ 2((E_2c)(E_1^d) - (E_1c)(E_2^d)) \]

\[ = -F(4G + 1) + (E_1c)^a + (E_2c)^a + (E_2d)^a + (E_2d)^a. \]

Thus, from (4.21)~(4.25), we get

(4.26) \[ \Delta G = 2(-(F + G)(4G + 1) - 2aE_1G + 2bE_2G + (E_1c)^a + (E_2c)^a + (E_2d)^a + (E_2d)^a). \]

**Lemma 3.** The following holds on \( M' \).

(4.27) \[ J(4G + 1)^a = 24(4G + 1)(-4G + 1)^2G + 6\| \text{grad } G \| ^2. \]
Proof. By the definition of the function $G$, we get

\begin{equation}
E_i G = (2c+1)E_i c + 2d E_i d, \quad 1 \leq i \leq 2.
\end{equation}

From (4.17), (4.18) and (4.28), we get

\begin{align*}
(4.29) \quad (4G+1)E_1 c &= (2c+1)E_1 G - 2d E_2 G + 2b d (4G+1), \\
(4G+1)E_2 c &= 2d E_1 G + (2c+1)E_2 G + 2a d (4G+1), \\
(4G+1)E_1 d &= 2d E_1 G + (2c+1)E_2 G - (2c+1)b (4G+1), \\
(4G+1)E_2 d &= -(2c+1)E_1 G + 2d E_2 G - (2c+1)a (4G+1).
\end{align*}

From (4.29), taking account of the definitions of the functions $F$ and $G$, we get

\begin{align*}
(4.30) \quad (4G+1)^4 (E_1 c)^2 + (E_2 c)^2 + (E_1 d)^2 + (E_2 d)^2 \\
&= -2(4G+1)^3 F + \|\text{grad } G\|^2 \\
&\quad + a (4G+1) E_1 G - b (4G+1) E_2 G.
\end{align*}

Thus, from (4.26) and (4.30), we have finally (4.27). Q.E.D.

We are now in a position to prove Theorems A, B and C. First, we shall prove Theorem A. We denote by $\nu^k_p$ the $k$-th normal space and by $\sigma^k_p$ the $k$-th fundamental form of the isometric immersion $\psi$ at $p \in M'$. Then from (4.8) with (4.9), we see that $\nu^k_p$ and $\nu^2_p$ are generated respectively by \{ $\sigma^k_p(E_1, E_1) = \sigma(E_1, E_1)$, $\sigma^k_p(E_1, E_2) = \sigma(E_1, E_2)$ \} and \{ $\sigma^2_p(E_1, E_1, E_2) = (c/4) N(E_1, \sigma(E_1, E_1))$, $(d/4) N(E_2, \sigma(E_1, E_1))$, $\sigma^2_p(E_2, E_1, E_1) = (d/4) N(E_2, \sigma(E_1, E_1)) - (c+1)/4 N(E_2, \sigma(E_1, E_1))$ \}, where $E_2 = JE_1$.

If $G(p) \neq 0$, then it follows that $\dim \nu^k_p = 2$, $\dim \nu^2_p = 2$, and hence the degree of the immersion $\phi$ is 3. So, we assume that $G = 0$ on $M'$. Let $p$ be any point of $M'$ and define $E$ by

\begin{equation}
\| (\nabla_p \sigma)(E, E) \| = \max_{X \in N^1(F_p')} \| (\nabla_p \sigma)(X, X) \|.
\end{equation}

Let \{ $X_1, X_2 = JX_1$ \} be an orthonormal frame field near $p$ satisfying the condition (4.19) for the basis \{ $E_1 = E, E_2 = JE$ \} at $p$. Then, we may easily see that $d = 0$ (and hence $c^2 + c = 0$) at $p$. We may assume that $c = -1$ at $p$. We put

\begin{equation}
\zeta = - \frac{d}{4} N(X_1, \sigma(X_1, X_1)) + \frac{c}{4} N(X_2, \sigma(X_1, X_1)) \quad \text{near } p.
\end{equation}

Then, taking account of (3.1), (3.7), (3.8), (4.2), (4.8), (4.9), (4.20) and (4.29), we get

\begin{align*}
(4.31) \quad \sigma^4_p(E_1, E_1, E_1, E_1) &= -(E_2 d + a) \zeta_p \\
&= -2G(p) \zeta_p = 0.
\end{align*}
Similarly, we get

\[ (4.32) \quad \sigma^\sharp(E_2, E_1, E_4, E_5) = 0. \]

Thus, from (4.31) and (4.32), taking account of (4.12) and the symmetricity of \( \sigma^\sharp \), we have finally \( \sigma^\sharp = 0 \), and hence the degree of \( \phi \) is 3. This completes the proof of Theorem A. Next, we shall prove Theorem B. We assume that the Gaussian curvature \( K \) of \( M \) is constant and \( K \neq 1 \). From (4.1), we get

\[ \| \sigma \|^2 = 2(1-K), \]

and hence from (4.10) and (4.14)

\[ (4.33) \quad F = 0 \quad \text{on} \quad M = M'. \]

Thus, from (4.33) and Lemma 4.2, we get

\[ (4.34) \quad G = -\frac{3}{2} K \quad \text{on} \quad M. \]

From (4.34) and Lemma 4.3, it follows that \( G(4G+1) = 0 \). If \( 4G+1 = 0 \), then, from (4.34), we have \( K = 1/6 \), and otherwise, we have \( K = 0 \). This completes the proof of Theorem B.

Lastly, we shall prove Theorem C. We suppose that \( M \) is compact and \( M' \neq 0 \). Then \( \| \sigma \| \) takes its maximum at some point \( p \in M'. \) Then, from (4.10), we have \( F(p) = 0 \). Thus, from Lemmas 4.1 and 4.2, we have

\[ (4.35) \quad 0 \leq (\Delta \log \| \sigma \|)(p) \leq -\frac{1}{2} + 3K(p), \]

and hence \( K(p) \leq 1/6 \).

Thus, if \( M \) is compact and \( K > 1/6 \) on \( M \), from (4.35), it follows that \( M' = 0 \), and hence the first half of Theorem C is proved. The latter half of Theorem C is immediately followed by using Lemmas 4.1 and 4.2, and Green's theorem. From Lemmas 4.2 and 4.3, taking account of Green's theorem and Gauss-Bonnet theorem, we have the following

**Theorem D.** Assume that \( M \) is compact and \( K \leq 1 \) on \( M \). If the function \( G \) satisfies the inequality \(-1/4 \leq G \leq 0 \) on \( M \), then \( G = 0 \) or \(-1/4 \) on \( M \), and furthermore \( M \) is diffeomorphic to a 2-dimensional torus (resp. a 2-dimensional sphere) in the case where \( G = 0 \) on \( M \) (resp. \( G = -1/4 \) on \( M \)).

We remark that the equality \( G = 0 \) (resp. \( G = -1/4 \) on \( M' \) is equivalent to

\[ (4.36) \quad \Delta \log(1-K) = 6K, \quad \text{on} \quad M' \]

(resp. (4.37) \( \Delta \log(1-K) = -1+6K \) on \( M' \))
5. Some examples

**Example 1.** Let \( M = \{ x \in S^6; x = x_1e_1 + x_2e_2 + x_3e_3 \} \), and \( \iota \) be the inclusion map from \( M \) into \( S^6 \). Then, we may easily see that \((M, \iota)\) is a 2-dimensional almost complex and totally geodesic submanifold of \( S^6 \).

**Example 2.** Let \( M = S^2_{1/6} = \{ (y_1, y_2, y_3) \in \mathbb{R}^3; y_1^2 + y_2^2 + y_3^2 = 6 \} \) and \( \phi_0 \) be a \( C^\infty \) map from \( M \) into \( S^6 \) defined by

\[
\phi_0(y_1, y_2, y_3) = \left( \sqrt{\frac{6}{72}} (2y_1^2 - 3y_1y_2^2 - 3y_1y_3^2) \right)e_1 + \left( \sqrt{\frac{15}{72}} (3y_1^2y_2 - y_1y_2y_3) \right)e_2 \\
+ \left( \sqrt{\frac{15}{72}} (y_2^2 - 2y_2y_3^2) \right)e_3 + \left( \frac{1}{24} (4y_1^2y_2 - y_1^2 - y_2y_3y_3) \right)e_4 \\
+ \left( \frac{1}{24} (4y_1y_2y_2 - y_1y_2y_3) \right)e_5 + \left( \frac{\sqrt{10}}{24} (y_1y_2 - y_1y_3) \right)e_6 \\
+ \left( \frac{\sqrt{10}}{12} y_1y_2y_3 \right)e_7, \quad \text{for} \quad (y_1, y_2, y_3) \in S^2_{1/6}.
\]

Then, we may easily check that \((S^2_{1/6}, \phi_0)\) is a 2-dimensional almost complex submanifold of \( S^6 \) and furthermore, any almost complex submanifold \((S^2_{1/6}, \phi)\) of \( S^6 \) is obtained by \( \phi = \alpha \cdot \phi_0 \) for some \( \alpha \in \mathbb{C}^2 \).

**Example 3.** Let \( M = \mathbb{R}^3 \) be a 2-dimensional Euclidean space with the canonical metric and \( \phi \) be a \( C^\infty \) map from \( \mathbb{R}^3 \) into \( S^6 \) defined by

\[
\phi(u, v) = \sqrt{\frac{2}{3}} \left( \cos \frac{\sqrt{3}}{2} u \right) \left( \sin \frac{\sqrt{3}}{2} v \right) a_1 - \left( \cos \frac{\sqrt{3}}{2} v \right) b_1 \\
+ \left( \sqrt{\frac{2}{3}} \sin \frac{\sqrt{3}}{2} u \right) \left( \sin \frac{\sqrt{3}}{2} v \right) a_2 - \left( \cos \frac{\sqrt{3}}{2} v \right) b_2 \\
+ \left( \sqrt{\frac{1}{3}} \cos \sqrt{2} u \right) a_3 + \left( \sqrt{\frac{1}{3}} \sin \sqrt{2} u \right) b_3,
\]

for \((u, v) \in \mathbb{R}^2\), where \( a_i, b_i \in \mathbb{C}^3 = \mathbb{R}^6 \) such that \( (a_1, a_2) = (a_3, a_4) = (a_5, a_6) = 0 \), \( (b_1, b_2) = (b_3, b_4) = 0 \), \( 1 \leq i, j \leq 3 \), and

\[
a_1 \times b_1 = -b_3, \quad a_1 \times a_4 = b_2, \quad a_3 \times b_1 = -a_2, \\
a_3 \times b_3 = b_3, \quad a_3 \times a_4 = b_1 \times b_2 = -a_2 \times b_3.
\]

For example, \( (a_1, a_2, a_3, b_4, b_5, b_6) = (e_3, -e_5, e_6, -e_7, e_5, e_6) \) satisfies the relations in (5.2). We may easily check that \((\mathbb{R}^3, \phi)\) is a 2-dimensional almost complex submanifold of \( S^6 \).
The above immersion φ induces an immersion \( \psi : T^2 = \mathbb{R}^2/\Gamma \rightarrow S^4 \) in the natural way, where \( \Gamma \) denotes the lattice group in \( \mathbb{R}^2 \) generated by \( \left\{ 2\sqrt{2} \pi (1, 0), 2\sqrt{2} \pi (0, 1) \right\} \).

**References**


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