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ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN 1/2

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0. Introduction. Let $f(z)$ be meromorphic in the plane. Throughout this note, we shall assume familiarity with elementary aspects and notations of Nevanlinna's theory and in particular with the meaning of the symbols

$$m(r, f),\quad n(r, a, f),\quad N(r, a, f),\quad T(r, f),\quad \delta(a, f).$$

Further we define

$$M(r, f)=\max_{|z|=r}|f(z)|,\quad m^*(r, f)=\min_{|z|=r}|f(z)|,$$

and denote by ρ and μ, respectively, the order and lower order of $f(z)$.

A real-valued function $L(r)$ defined for all $r\geqq r_0$ $(\geqq 0)$ is said to be a slowly varying function (at ∞) if

(i) $L(r)$ is positive and continuous in $r_0\leqq r<\infty$

and (ii) $\lim_{r\to\infty}\dfrac{L(\lambda r)}{L(r)}=1$, for every fixed $\lambda>0$.

This concept was introduced by Karamata [4]. He proved that for each $\varepsilon>0$,

$$r^{\varepsilon}L(r)\to\infty,\quad r^{-\varepsilon}L(r)\to 0\quad (r\to\infty). \tag{1}$$

Let $f(z)$ be an entire function of order ρ, $0<\rho<1/2$. If $f(z)$ is of minimal type, then a well-known theorem of Kjellberg [5] implies that

$$\log m^*(r, f)>\cos\pi\rho\log M(r, f)$$

on an unbounded sequence of r. If $f(z)$ is of mean type, then the following result is valid. (See [2].)

THEOREM A. *Let $h(r)$ $(r\geqq r_0)$ be a slowly varying function such that $h(r)\to 0$ $(r\to\infty)$ and*

$$\int_{r_0}^{\infty}\frac{h(t)}{t}dt=\infty. \tag{2}$$

If $f(z)$ is an entire function of order ρ $(0<\rho<1/2)$ and mean type, then

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$$\log m^*(r, f) > \cos \pi\rho(1-h(r)) \log M(r, f)$$

on a sequence of $r \to \infty$.

If $h(r)$ does not satisfy condition (2), then there is an entire function of order ρ $(0<\rho<1/2)$ and mean type for which

$$\log m^*(r, f) < \cos \pi\rho(1-h(r)) \log M(r, f) \quad (r \geqq r_0).$$

In Theorem A, Barry [2, p. 45] assumes also that $h'(r) > -O(r^{-1})$ $(r \to \infty)$. However, this condition is unnecessary (cf. [7]).

It is natural to consider the analogous problems to the above results for meromorphic functions. That is, for a meromorphic function $f(z)$, what can we say about the relation between $\log m^*(r, f)$ and $T(r, f)$?

In this note we shall prove the following two results.

THEOREM 1. *Let* $f(z)$ *be meromorphic of order* ρ $(0<\rho<1/2)$ *and minimal type. Assume that there is a* $\delta \in (0, 1]$ *such that*

$$\cos \pi\rho - 1 + \delta > 0 \tag{3}$$

and

$$N(r, \infty, f) \leqq (1-\delta) T(r, f) + O(\log r) \quad (r \to \infty). \tag{4}$$

Then

$$\log m^*(r, f) > \frac{\pi\rho}{\sin \pi\rho} (\cos \pi\rho - 1 + \delta) T(r, f) - O(\log r)$$

on an unbounded sequence of r.

THEOREM 2. *Let* $h(r)$ $(r \geqq r_0)$ *be a slowly varying function satisfying* $h(r) \to 0$ $(r \to \infty)$ *and* (2). *Let* $f(z)$ *be meromorphic of order* ρ $(0<\rho<1/2)$ *and mean type. Assume that there is a* $\delta \in (0, 1]$ *satisfying* (3) *and* (4). *Then*

$$\log m^*(r, f) > \frac{\pi\rho}{\sin \pi\rho} (\cos \pi\rho - 1 + \delta)(1-h(r)) T(r, f)$$

on a sequence of $r \to \infty$.

For the case that $f(z)$ is entire—in this case we can choose $\delta = 1$—Theorems 1 and 2 have been proved in [7].

1. Lemmas

LEMMA A. ([1, p. 189]) *Let* $f(z)$ *be meromorphic in the plane and such that for some* ρ, $0<\rho<1$, *either*

$$\pi\rho N(r, 0, f) \leqq \sin \pi\rho \log M(r, f) + \pi\rho \cos \pi\rho N(r, \infty, f)$$

or

$$\sin \pi\rho \log m^*(r, f) \leqq \pi\rho \cos \pi\rho N(r, 0, f) - \pi\rho N(r, \infty, f)$$

for all large r. Then $\liminf_{r\to\infty} T(r, f)/r^{\rho} > 0$.

LEMMA B. ([5, p. 280]) *Let $f(z)$ be a nonconstant meromorphic function. Then for a given $\varepsilon>0$ $(0<\varepsilon<1)$ and every sufficiently large value r_0,*

$$N(r, a, f) \geqq T(r, f) - 2T(r, f)^{(1+\varepsilon)/2}$$

for $r \geqq r_0$, with the possible exception of a set E_0 of values a whose capacity is at most

$$\exp(-\varepsilon T(r_0, f)^{\varepsilon}/4).$$

LEMMA C. (cf. [7] and [2, p. 54]) *Let $h(r)$ $(r \geqq r_0)$ be a slowly varying function satisfying $h(r) \to 0$ $(r\to\infty)$ and (2). Set $h(t)=(t/r_0)h(r_0)$ $(r_0 \geqq t \geqq 0)$, and let*

$$L(r) = \exp\left\{\tilde{\delta}\int_1^r \frac{h(t)}{t}\, dt\right\} \quad (\tilde{\delta}: \text{ a positive constant}).$$

Consider the positive harmonic function $H(z)$ in $\boldsymbol{C}-(-\infty, 0]$ defined by

$$H(re^{i\theta}) = \frac{1}{\pi}\int_0^{\infty} \frac{r^{1/2}(r+s)s^{\rho}L(s)\cos\dfrac{\theta}{2}}{s^{1/2}(r^2+s^2+2sr\cos\theta)}\, ds \quad (0<\rho<1/2).$$

Then we have

$$\frac{H(r)}{H(-r)} < \frac{1+[1+o(1)]\tilde{\delta}h(r)C(\rho)\cos\pi\rho}{\cos\pi\rho}$$

$$(r\to\infty;\ C(\rho): \text{ a positive constant depending only on } \rho).$$

If $f(z)$ is an entire function of order ρ, mean type and all of whose zeros are negative, then there is an unbounded sequence $r=r_n$ such that

$$0 < \frac{\log|f(r)|}{\log|f(-r)|} \leqq \frac{H(r)}{H(-r)}$$

and

$$\frac{N(r, 0, f)}{\log|f(-r)|} < \frac{1}{\pi\rho\cot\pi\rho\left\{1-\dfrac{\sqrt{2}\,\pi\rho}{\tan\pi\rho}\tilde{\delta}C_1(\rho)h(r)\right\}\left\{1-\dfrac{4\tilde{\delta}^2h^2(r)}{\rho^2\tan^2\pi\rho}\right\}}$$

$$(C_1(\rho): \text{ a positive constant depending only on } \rho).$$

2. Proof of Theorem 1. Since $f(z)$ is of order ρ $(0<\rho<1/2)$ and minimal type, we deduce from Lemma A that

$$\sin\pi\rho\log m^*(r, f) > \pi\rho\cos\pi\rho N(r, 0, f) - \pi\rho N(r, \infty, f) \tag{2.1}$$

on an unbounded sequence of r. We denote this sequence by $\{r_n\}_1^\infty \uparrow \infty$.

First, we show that

$$\log m^*(r_n, f) \geqq 0 \tag{2.2}$$

for sufficiently large n. By (2.1), (4) and (3)

$$\begin{aligned} &\sin \pi\rho \log m^*(r, f) + \pi\rho \cos \pi\rho m(r, 1/f) \\ &> \pi\rho \cos \pi\rho T(r, f) - \pi\rho N(r, \infty, f) - O(1) \\ &> \pi\rho[\cos \pi\rho - 1 + \delta] T(r, f) - O(\log r) > 0 \quad (r = r_n,\ n \geqq n_0). \end{aligned} \tag{2.3}$$

Assume that $\log m^*(r_k, f) > 0$ for some $k \geqq n_0$. Then

$$m(r_k, 1/f) \leqq \max_{0 \leqq \theta \leqq 2\pi} \log |1/f(r_k e^{i\theta})| = -\log m^*(r_k, f).$$

From this and (2.3) it follows that

$$\sin \pi\rho \log m^*(r_k, f) - \pi\rho \cos \pi\rho \log m^*(r_k, f) > 0.$$

Hence $\tan \pi\rho < \pi\rho$, a contradiction. This proves (2.2), which in particular implies that

$$m(r_n, 1/f) = 0. \quad (n \geqq n_0). \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain

$$\log m^*(r_n, f) > \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta) T(r_n, f) - O(\log r_n) \quad (n \to \infty).$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2; Preliminaries.

3.1. Assume first that there is a positive number $\lambda < \rho$ such that

$$\liminf_{r \to \infty} T(r, f)/r^\lambda = 0.$$

Then as in the proof of Theorem 1 we deduce that

$$\log m^*(r_n, f) > \frac{\pi\lambda}{\sin \pi\lambda}(\cos \pi\lambda - 1 + \delta) T(r_n, f) - O(\log r_n), \tag{3.1}$$

where $\{r_n\}_1^\infty$ is a suitable increasing sequence tending to ∞. Since the function

$$\frac{x}{\sin x}(\cos x - \alpha) \quad (\alpha\text{: a nonnegative constant})$$

decreases strictly as $x \in [0, \pi/2)$ increases, we deduce from (3.1) that

$$\log m^*(r_n, f) > \frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho - 1 + \delta)T(r_n, f)$$

for sufficiently large n. Hence, in what follows, we may consider only the case that

(3.2) $$\liminf_{r\to\infty} T(r, f)/r^{\lambda} > 0$$

for all $\lambda \in (0, \rho)$.

3.2. By Lemma B, there is a complex number a such that

(3.3) $$N(r, a, f) \geqq T(r, f) - 2T(r, f)^{3/4} \quad (r \geqq R_0 > 0).$$

Put

(3.4) $$F(z) = f(z) - a\,.$$

It is clear that $F(z)$ satisfies all the assumptions of Theorem 2. Assume now that the conclusion of Theorem 2 holds for $F(z)$, i.e. there is a sequence $\{r_n\}_1^{\infty} \uparrow \infty$ such that

(3.5) $$\log m^*(r_n, F) > \frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho - 1 + \delta)(1 - h(r_n))T(r_n, F)\,.$$

By (3.2), (3.4), (3.5) and (3)

(3.6) $$T(r_n, F),\ \log m^*(r_n, F) > O(r_n^{\lambda}) \quad (n \to \infty)$$

for any fixed $\lambda \in (0, \rho)$. Further by (1) for each $\varepsilon > 0$,

(3.7) $$r^{\varepsilon} h(r) \to \infty \quad (r \to \infty).$$

From (3.5), (3.6) and (3.7) follows that

$$\begin{aligned}\frac{\log m^*(r_n, f)}{T(r_n, f)} &\geqq \frac{\log m^*(r_n, F) - O(1)}{T(r_n, F) + O(1)} \\ &> \frac{\log m^*(r_n, F)[1 - O(r_n^{-\lambda})]}{T(r_n, F)[1 + O(r_n^{-\lambda})]} \\ &> \frac{\log m^*(r_n, F)[1 - o(h(r_n))]}{T(r_n, F)[1 + o(h(r_n))]} \\ &> \frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho - 1 + \delta)[1 - (1 + o(1))h(r_n)] \quad (n \to \infty).\end{aligned}$$

This implies that we may prove Theorem 2 only for $F(z)$.

3.3. Let $\{a_n\}$, $\{b_n\}$ $(a_n, b_n \neq 0)$ be the sequences of the zeros and poles of $F(z)$. Then we can write

(3.8) $$F(z) = cz^p \frac{\Pi(1 - z/a_n)}{\Pi(1 - z/b_n)} \equiv cz^p \frac{P(z)}{Q(z)} \equiv cz^p F_1(z) \qquad (c \neq 0,\ p: \text{an integer}).$$

From (4) and (3.8) we deduce that

$$N(r,\infty,F_1)\leqq(1-\delta)T(r,F_1)+O(\log r)\quad(r\to\infty). \tag{3.9}$$

Assume that there is a sequence $\{r_n\}_1^\infty\uparrow\infty$ such that

$$\log m^*(r_n,F_1)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r_n))T(r_n,F_1). \tag{3.10}$$

Then as in § 3.2, we have from (3.10), (3.8), (3.6) and (3.7)

$$\begin{aligned}\frac{\log m^*(r_n,F)}{T(r_n,F)}&\geqq\frac{\log m^*(r_n,F_1)-O(\log r_n)}{T(r_n,F_1)+O(\log r_n)}\\&>\frac{\log m^*(r_n,F_1)[1-O(r_n^{-\lambda}\log r_n)]}{T(r_n,F_1)[1+O(r_n^{-\lambda}\log r_n)]}\\&>\frac{\log m^*(r_n,F_1)[1-o(h(r_n))]}{T(r_n,F)[1+o(h(r_n))]}\\&>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)[1-(1+o(1))h(r_n)]\quad(n\to\infty).\end{aligned}$$

Hence it suffices to show that the conclusion of Theorem 2 holds for $F_1(z)$.

4. Proof of Theorem 2; Conclusion.

Define $\hat{P}(z)$ and $\hat{Q}(z)$ as follows:

$$\hat{P}(z)=\Pi(1+z/|a_n|),\quad \hat{Q}(z)=\Pi(1+z/|b_n|).$$

First, we show that $\hat{P}(z)$ is of order ρ and mean type. By (3.8), (3.4) and (3.3)

$$\begin{aligned}\log M(r,\hat{P})&\geqq m(r,\hat{P})\geqq N(r,0,\hat{P})=N(r,0,F_1)\\&\geqq(1-o(1))T(r,F_1)\quad(r\to\infty).\end{aligned} \tag{4.1}$$

On the other hand,

$$\begin{aligned}\log M(r,\hat{P})&=r\int_0^\infty\frac{N(t,0,\hat{P})}{(t+r)^2}dt\leqq N(r,0,\hat{P})+r\int_r^\infty\frac{N(t,0,\hat{P})}{t^2}dt\\&\leqq T(r,F_1)+r\int_r^\infty\frac{T(t,F_1)}{t^2}dt=O(r^\rho)\quad(r\to\infty).\end{aligned} \tag{4.2}$$

(4.1) and (4.2) implies that $\hat{P}(z)$ is of order ρ and mean type.

Now, we apply Lemma C to $\hat{P}(z)$. Then we see that there is a sequence $\{r_n\}_1^\infty\uparrow\infty$ such that

$$O<\frac{\log\hat{P}(r_n)}{\log|\hat{P}(-r_n)|}\leqq\frac{H(r_n)}{H(-r_n)}<\frac{1+[1+o(1)]\tilde{\delta}h(r_n)C(\rho)\cos\pi\rho}{\cos\pi\rho}\quad(n\to\infty) \tag{4.3}$$

and

$$(4.4)\qquad \frac{N(r_n,\,0,\,F_1)}{\log|\hat{P}(-r_n)|}<\frac{1}{\pi\rho\cot\pi\rho\left\{1-\dfrac{\sqrt{2}\,\pi\rho}{\tan\pi\rho}\tilde{\delta}C_1(\rho)h(r_n)\right\}\left\{1-\dfrac{4\tilde{\delta}^2h^2(r_n)}{\rho^2\tan^2\pi\rho}\right\}}\quad (n\to\infty).$$

By (3.3), (3.4) and (3.8)

$$N(r,\,0,\,F_1)\geqq T(r,\,F_1)-T(r,\,F_1)^{3/4}-O(\log r)\quad (r\to\infty),$$

and so we have

$$(4.5)\qquad \begin{aligned} T(r,\,F_1)&\leqq N(r,\,0,\,F_1)+2T(r,\,F_1)^{3/4}\\ &\leqq N(r,\,0,\,F_1)+O(r^{(3/4)\rho})\quad (r\to\infty). \end{aligned}$$

From (3.9) and (4.5) it follows that

$$N(r,\,\infty,\,F_1)\leqq(1-\delta)N(r,\,0,\,F_1)+O(r^{(3/4)\rho})\quad (r\to\infty),$$

which gives

$$(4.6)\qquad \log\hat{Q}(r)=r\int_0^\infty\frac{N(t,\,\infty,\,F_1)}{(t+r)^2}dt\leqq(1-\delta)\log\hat{P}(r)+O(r^{(3/4)\rho})\quad (r\to\infty).$$

By (4.3), (4.1), (3.2) and (3.7)

$$(4.7)\qquad \begin{aligned} \frac{r_n^{(3/4)\rho}}{\log|\hat{P}(-r_n)|}&\leqq O\left(\frac{r_n^{(3/4)\rho}}{\log\hat{P}(r_n)}\right)<O\left(\frac{r_n^{(3/4)\rho}}{T(r_n,\,F_1)}\right)\\ &<O(r_n^{-\varepsilon})\quad (0<\varepsilon<(1/4)\rho)\\ &=o(h(r_n))\,. \end{aligned}$$

Hence by (4.6), (4.3) and (4.7)

$$(4.8)\qquad \begin{aligned} \log m^*(r_n,\,F_1)&\geqq\log|\hat{P}(-r_n)|-\log\hat{Q}(r)\\ &\geqq\log|\hat{P}(-r_n)|-(1-\delta)\log\hat{P}(r_n)-O(r_n^{(3/4)\rho})\\ &\geqq\log|\hat{P}(-r_n)|\left\{1-(1-\delta)\frac{H(r_n)}{H(-r_n)}-o(h(r_n))\right\}\\ &\geqq\log|\hat{P}(-r_n)|\left\{1-(1-\delta)\frac{1}{\cos\pi\rho}-(\tilde{\delta}C(\rho)+o(1))h(r_n)\right\}\\ &\qquad (n\to\infty). \end{aligned}$$

On the other hand, we deduce from (4.5), (4.4) and (4.7) that

$$\begin{aligned} T(r_n,\,F_1)&\leqq N(r_n,\,0,\,F_1)+O(r_n^{(3/4)\rho})\\ &\leqq\log|\hat{P}(-r_n)|\left[\frac{1}{\pi\rho\cos\pi\rho\left\{1-\dfrac{\sqrt{2}\,\pi\rho}{\tan\pi\rho}\tilde{\delta}C_1(\rho)h(r_n)\right\}\left\{1-\dfrac{4\tilde{\delta}^2h^2(r_n)}{\rho^2\tan^2\pi\rho}\right\}}\right.\\ &\qquad\left.+o(h(r_n))\right]\quad (n\to\infty). \end{aligned}$$

Combining (4.8) and (4.9), we obtain

$$\frac{\log m^*(r_n, F_1)}{T(r_n, F_1)} \geqq \left\{1-(1-\delta)\frac{1}{\cos \pi\rho}-(\tilde{\delta}C(\rho)+o(1))h(r_n)\right\}$$

$$\times \pi\rho \cot \pi\rho \left\{1-\frac{\sqrt{2}\,\pi\rho}{\tan \pi\rho}\tilde{\delta}C_1(\rho)h(r_n)\right\}\left\{1-\frac{4\tilde{\delta}^2 h^2(r_n)}{\rho^2 \tan^2 \pi\rho}\right\}\{1-o(h(r_n))\}$$

$$> \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho-1+\delta)(1-h(r_n)) \quad (n\to\infty),$$

if $\tilde{\delta}$ (>0) is sufficiently small.

This completes the proof of Theorem 2.

5. A counterexample. It is natural to ask in Theorem 2 whether the condition (2) is necessary or not. As the answer to this question, we give the following example.

Let $h(r)$ $(r\geqq r_0)$ be a slowly varying function such that $h(r)\to 0$ $(r\to\infty)$ and

$$\int_{r_0}^{\infty}\frac{h(t)}{t}dt<\infty . \tag{5.1}$$

Let ρ and δ be numbers with $0<\rho<1/2$, $1-\cos \pi\rho<\delta\leqq 1$. Then there is a meromorphic function $f(z)$ satisfying the following conditions (i)-(iv).

(i) *$f(z)$ is of order ρ and mean type.*

(ii) $\delta(\infty, f)=\delta$.

(iii) $N(r, \infty, f)\leqq(1-\delta)T(r, f)+O(\log r)$ $(r\to\infty)$.

(iv) $\log m^*(r, f)<\dfrac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho-1+\delta)(1-h(r))T(r, f)$

for all sufficiently large r.

For convenience, we state two lemmas which will be used to construct the above example.

LEMMA D ([3]). *Let $g(z)$ be meromorphic in the plane. For a measurable set $I\subset[0, 2\pi)$, define*

$$m(r, g, I)=\frac{1}{2\pi}\int_I \log^+|g(re^{i\theta})|\,d\theta \quad (r>0).$$

Then

$$m(r, g, I)\leqq 22T(2r, g)|I|\left[1+\log^+\frac{1}{|I|}\right],$$

where $|I|$ is the Lebesgue measure of I.

LEMMA E ([2], [7]). *Let g be an entire function of genus zero, all of whose zeros are negative and such that $g(0)=1$ and $n(r, 0, g)=[r^\rho L(r)]$, where $L(r)$ is* *defined as in Lemma C. Then we have*

$$N(r,\ 0,\ g)=\frac{r^{\rho}L(r)}{\rho}\Big[1-\frac{\tilde{\delta}(1+o(1))}{\rho}h(r)\Big]+O(\log r)\quad(r\rightarrow\infty),$$

$$\log M(r,\ g)=r^{\rho}L(r)\Big[\frac{\pi}{\sin\pi\rho}+\tilde{\delta}C_{1}(\rho)(1+o(1))h(r)\Big]+O(\log r)$$

$$(r\rightarrow\infty,\ C_{1}(\rho)\equiv\sum_{n=0}^{\infty}(-1)^{n}\{(n+1-\rho)^{-2}-(n+\rho)^{-2}\}),$$

and

$$\log|g(re^{i\theta(r)})|=r^{\rho}L(r)\Big[\frac{\pi\cos\pi\rho}{\sin\pi\rho}-\tilde{\delta}\frac{\pi^{2}}{\sin^{2}\pi\rho}(1+o(1))h(r)\Big]+O(\log r)$$

$$(r\rightarrow\infty,\ \theta(r)\equiv\pi-r^{-K}\ with\ K>1).$$

Construction of a counterexample. Let α and β be numbers such that $\alpha\cos\pi\rho>\beta\geqq 0$. Let

$$P(z)=\prod_{n=1}^{\infty}\Big(1+\frac{z}{a_{n}}\Big),\quad Q(z)=\sum_{n=1}^{\infty}\Big(1-\frac{z}{b_{n}}\Big)\quad(a_{n},\ b_{n}>0)$$

be canonical products satisfying $n(r,\ 0,\ P)=[\alpha r^{\rho}L(r)]$, $n(r,\ 0,\ Q)=[\beta r^{\rho}L(r)]$, respectively. Set $\delta=(\alpha-\beta)/\alpha$ $(>1-\cos\pi\rho)$. Then we shall show that $f(z)\equiv P(z)/Q(z)$ satisfies all the conditions (i)-(iv).

Using Lemma E, we have

$$\log|f(re^{i\theta(r)})|\geqq\log|P(re^{i\theta(r)})|-\log Q(-r)$$

$$=r^{\rho}L(r)\Big[\frac{\pi(\alpha\cos\pi\rho-\beta)}{\sin\pi\rho}-\tilde{\delta}\Big(\alpha\frac{\pi^{2}}{\sin^{2}\pi\rho}+\beta C_{1}(\rho)\Big)h(r)-o(h(r))\Big]$$

$$-O(\log r)>0\quad(r>R_{1}).$$

Hence by Lemma D

$$m(r,\ 0,\ f)=\frac{1}{\pi}\int_{\theta(r)}^{\pi}\log^{+}\Big|\frac{1}{f(re^{i\theta})}\Big|d\theta \tag{5.2}$$

$$\leqq 44T(2r,\ 1/f)(\pi-\theta(r))\Big[1+\log^{+}\frac{1}{\pi-\theta(r)}\Big]$$

$$=44T(2r,\ f)r^{-K}[1+K\log r]\quad(r>R_{1}).$$

Since $T(r,\ f)\leqq m(r,\ P)+m(r,\ Q)\leqq\log M(r,\ P)+\log M(r,\ Q)$, we deduce from Lemma E and (5.1) that

$$T(r,\ f)=O(r^{\rho})\quad(r\rightarrow\infty). \tag{5.3}$$

In view of (5.2) and (5.3) we have

$$m(r,\ 0,\ f)=O(r^{\rho+\varepsilon-K})\quad(r\rightarrow\infty) \tag{5.4}$$

for any fixed $\varepsilon>0$.

Thus (5.4) and Lemma E give

$$T(r,\ f)=T(r,\ 1/f)=\frac{\alpha r^{\rho}L(r)}{\rho}\Big[1-\frac{\tilde{\delta}(1+o(1))}{\rho}h(r)\Big]+O(\log r)\quad(r\rightarrow\infty). \tag{5.5}$$

On the other hand,

$$N(r,\ \infty,\ f)=N(r,\ 0,\ Q)=\frac{\beta r^{\rho}L(r)}{\rho}\Big[1-\frac{\tilde{\delta}(1+o(1))}{\rho}h(r)\Big]+O(\log r)\quad(r\rightarrow\infty). \tag{5.6}$$

Combining (5.5) and (5.6) we have

$$\delta(\infty,\ f)=1-\alpha/\beta=\delta.$$

Further,

$$N(r,\ \infty,\ f)=\int_{0}^{r}\frac{[\beta t^{\rho}L(t)]}{t}dt\leqq\frac{\beta}{\alpha}\int_{0}^{r}\frac{\alpha t^{\rho}L(t)}{t}dt$$

$$\leqq\frac{\beta}{\alpha}[N(r,\ 0,\ f)+\log r]\leqq(1-\delta)T(r,\ f)+O(\log r)\quad(r\rightarrow\infty).$$

It remains to show (iv). Using Lemma E and (5.5), we have

$$\begin{aligned}\log m^{*}(r,\ f)&=\log|P(-r)|-\log Q(-r)\\&\leqq r^{\rho}L(r)\Big[\frac{\pi(\alpha\cos\pi\rho-\beta)}{\sin\pi\rho}-\tilde{\delta}\Big(\frac{\alpha\pi^{2}}{\sin^{2}\pi\rho}+\beta C_{1}(\rho)\Big)h(r)+o(h(r))\Big]+O(\log r)\\&\leqq\Big[\frac{\pi(\alpha\cos\pi\rho-\beta)}{\sin\pi\rho}-\tilde{\delta}\Big(\frac{\alpha\pi^{2}}{\sin^{2}\pi\rho}+\beta C_{1}(\rho)\Big)h(r)+o(h(r))\Big]\\&\qquad\times\frac{\rho(T(r,\ f)-O(\log r))}{\alpha\Big[1-\frac{\tilde{\delta}}{\rho}h(r)-o(h(r))\Big]}+O(\log r)\\&=\frac{\rho}{\alpha}\Big[\frac{\pi(\alpha\cos\pi\rho-\beta)}{\sin\pi\rho}-\tilde{\delta}\Big(\frac{\alpha\pi^{2}}{\sin^{2}\pi\rho}+\beta C_{1}(\rho)-\frac{\pi(\alpha\cos\pi\rho-\beta)}{\rho\sin\pi\rho}\Big)h(r)\\&\qquad+o(h(r))\Big]T(r,\ f)+O(\log r).\end{aligned} \tag{5.7}$$

Here, we note that

$$A(\rho)\equiv\frac{\alpha\pi^{2}}{\sin^{2}\pi\rho}+\beta C_{1}(\rho)-\frac{\pi(\alpha\cos\pi\rho-\beta)}{\rho\sin\pi\rho}>0. \tag{5.8}$$

In fact,

$$\begin{aligned}A(\rho)&=\frac{\pi^{2}}{\sin^{2}\pi\rho}\Big\{\alpha+\beta\frac{\sin^{2}\pi\rho}{\pi^{2}}C_{1}(\rho)-\frac{\sin\pi\rho(\alpha\cos\pi\rho-\beta)}{\pi\rho}\Big\}\\&=\frac{\pi^{2}}{\sin^{2}\pi\rho}\Big\{\alpha\Big(1-\frac{\sin 2\pi\rho}{2\pi\rho}\Big)+\beta\Big(\frac{\sin\pi\rho}{\pi\rho}+\frac{\sin^{2}\pi\rho}{\pi^{2}}C_{1}(\rho)\Big)\Big\}\end{aligned}$$

$$=\frac{\pi^2}{\sin^2\pi\rho}\left\{\alpha\left(1-\frac{\sin 2\pi\rho}{2\pi\rho}\right)+\beta\frac{\sin^2\pi\rho}{\pi^2\rho^2}\left(\frac{\pi\rho}{\sin\pi\rho}\right.\right.$$

$$\left.\left.+\rho^2\sum_{n=0}^{\infty}(-n)^n[(n+1-\rho)^{-2}-(n+\rho)^{-2}]\right)\right\}$$

$$\geqq\frac{\pi^2\alpha}{\sin^2\pi\rho}\left(1-\frac{\sin 2\pi\rho}{2\pi\rho}\right)+\frac{\beta}{\rho^2}\left[\frac{\pi\rho}{\sin\pi\rho}+\rho^2\left(\frac{1}{(1-\rho)^2}-\frac{1}{\rho^2}\right)\right]$$

$$=\frac{\pi^2\alpha}{\sin^2\pi\rho}\left(1-\frac{\sin 2\pi\rho}{2\pi\rho}\right)+\frac{\beta}{\rho}\left(\frac{\rho^2}{(1-\rho)^2}+\frac{\pi\rho}{\sin\pi\rho}-1\right)>0.$$

Since $T(r, f)\sim(\alpha/\rho)r^{\rho}L(r)$,

$$\log r=o(h(r))T(r, f)\quad(r\to\infty). \tag{5.9}$$

Therefore, choosing $\tilde{\delta}$ large enough, we obtain from (5.7), (5.8) and (5.9)

$$\log m^*(r, f)\leqq\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))T(r, f)\quad(r\geqq R_2).$$

Remark. The method of this section can be used also when we prove the following result.

Let $h(r)$ be given as in Theorem 2. Let ρ and δ be numbers with $0<\rho<1/2$, $1-\cos\pi\rho<\delta\leqq 1$. Then there is a meromorphic function $f(z)$ satisfying the following (i)-(iv).

(i) *$f(z)$ is of order ρ and minimal (maximal) type.*

(ii) $\delta(\infty, f)=\delta$.

(iii) $N(r, \infty, f)\leqq(1-\delta)T(r, f)+O(\log r)\quad(r\to\infty)$.

(iv) $\log m^*(r, f)\leqq\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+h(r)(-h(r)))T(r, f)$

for all sufficiently large r.

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