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# SPECTRAL GEOMETRY OF CR-MINIMAL SUBMANIFOLDS IN THE COMPLEX PROJECTIVE SPACE

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**Introduction.** In the first part of this paper we will study an isometric imbedding of the complex projective space in the Euclidean space, see [7].

In the second part we use this imbedding and the total mean curvature theory, see [4], in order to obtain certain boundaries of the volume and the first eigenvalue of the spectrum of $CR$-minimal closed submanifolds of the complex projective space, such as certain characterizations of some of these submanifolds, in function of these geometric invariants. We give a $\lambda_1$-characterization of totally geodesic complex submanifolds, a spectral reduction of codimension theorem for totally real submanifolds and some other results.

Manifolds are assumed to be connected and dimension $n\geqq 2$ unless mentioned otherwise. For the necessary knowledge and notations of the geometry of submanifolds, see [2], and for spectral geometry, see [1].

## 1. An imbedding of the complex projective space in the Euclidean space.

Let $HM(n)=\{A\in gl(n, \boldsymbol{C})/\bar{A}=A^t\}$ be the set of $nxn$-Hermitian matrices. $HM(n)$ is a $n^2$-dimensional linear subspace of $gl(n, \boldsymbol{C})$. We define in $HM(n)$ the metric

$$g(A, B)=2\operatorname{trace}(AB) \qquad \text{for all} \quad A, B \quad \text{in} \quad HM(n).$$

Let $\boldsymbol{CP}^n=\{A\in HM(n+1)/AA=A, \operatorname{trace} A=1\}$ and $U(n)$ be the unitary group.

LEMMA 1.1. *$\boldsymbol{CP}^n$ is a submanifold of $HM(n+1)$ diffeomorphic to $U(n+1)/U(1)\times U(n)$.*

*Proof.* Let $A$ be in $\boldsymbol{CP}^n$. Since $A$ is a Hermitian matrix, there exists $P$ in $U(n+1)$ such that

$$PAP^{-1}=\begin{pmatrix} h_0 & & \\ & \ddots & \\ & & h_n \end{pmatrix}.$$

As $PAP^{-1}=(PAP^{-1})^2$, $h_i=h_i^2$, so that $h_i=0$ or $h_i=1$, but $\operatorname{trace}(PAP^{-1})=1$, therefore there exists an index $i_0$ such that $h_{i_0}=1$ and $h_i=0$ for all $i\neq i_0$.

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Hence, we see that there exists $P$ in $U(n+1)$ such that

$$PAP^{-1}=\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}=A_0 .$$

We will say that $A_0$ is the origin of $\boldsymbol{CP}^n$. Moreover $\boldsymbol{CP}^n$ is the orbit of $A_0$ for the action of $U(n+1)$ over $HM(n+1)$ given by $(P, A)\mapsto PAP^{-1}$, where $P$ is in $U(n+1)$ and $A$ is in $HM(n+1)$. The isotropy subgroup of $A_0$ is $U(1)\times U(n)$. Therefore $\boldsymbol{CP}^n\cong U(n+1)/U(1)\times U(n)$. (Q.E.D.)

For any $A$ in $\boldsymbol{CP}^n$, we denote by $T_A(\boldsymbol{CP}^n)$ the tanget space of $\boldsymbol{CP}^n$ at $A$ identified by means of the immersion with a subspace of $HM(n+1)$. In the same way we denote by $T_A^{\perp}(\boldsymbol{CP}^n)$ the normal space of $\boldsymbol{CP}^n$ in $HM(n+1)$ at the point $A$.

LEMMA 1.2. *For any point $A$ in $\boldsymbol{CP}^n$, we have*

$$T_A(\boldsymbol{CP}^n)=\{X\in HM(n+1)/XA+AX=X\} , \tag{1.1}$$

$$T_A^{\perp}(\boldsymbol{CP}^n)=\{Z\in HM(n+1)/AZ=ZA\} . \tag{1.2}$$

*Proof*. Let $\alpha: \Gamma\to\boldsymbol{CP}^n$ be a curve such that $\alpha(0)=A$ and $\alpha'(0)=X$, where $\Gamma$ will denote an open interval of real numbers which contains 0. Then from $\alpha(t)\alpha(t)=\alpha(t)$ we obtain $XA+AX=X$. Therefore we have one inclusion. Since the applications $L_p: HM(n+1)\to HM(n+1)$ given by $A\mapsto PAP^{-1}$, where $P$ is in $U(n+1)$, are isometries, it is enough to establish this equalities at the origin. Now we will compute the dimension of the subspace $\{X\in HM(n+1)/XA_0+A_0X=X\}$.

For any $X\in HM(n+1)$ we put

$$X=\begin{pmatrix} a & b \\ \bar{b}^t & c \end{pmatrix} \quad \text{where} \quad a\in\boldsymbol{R},\ b\in\boldsymbol{C}^n \quad \text{and} \quad c\in HM(n) .$$

Then $XA_0+A_0X=X$ if and only if $a=0$ and $c=0$, so that

$$X=\begin{pmatrix} 0 & b \\ \bar{b}^t & 0 \end{pmatrix}, \qquad \text{with} \quad b\in\boldsymbol{C}^n .$$

The real dimension of this subspace is $2n=\dim T_A(\boldsymbol{CP}^n)=\dim U(n+1)/U(1)\times U(n)$ and so we have (1.1).

A vector $Z$ is in $T_{A_0}^{\perp}(\boldsymbol{CP}^n)$ if and only if $2\operatorname{trace}(XZ)=0$ for all $X\in T_{A_0}(\boldsymbol{CP}^n)$. Let

$$Z=\begin{pmatrix} x & y \\ \bar{y}^t & z \end{pmatrix}.$$

Then, $2\,\mathrm{trace}\,(XZ)=4\,\mathrm{Real\,trace}\,(b\bar{y}^t)$. Therefore $g(X, Z)=0$ for all $X$ in $T_{A_0}(\boldsymbol{C}P^n)$, if and only if $y=0$.

On the other hand, $ZA_0=A_0Z$ if and only if $y=0$. (Q. E. D.)

*Remark* 1.3. The vector fields given by $A\mapsto A$ and $A\mapsto I$ (where $I$ denotes the identity matrix) are normal to $\boldsymbol{C}P^n$. The vector fields given by $A\mapsto AQ+QA-2AQA$ are tangent to $\boldsymbol{C}P^n$ for all $Q$ in $HM(n+1)$.

Hence forth, we will use the following relations which can be obtained by direct calculus. Let $A$ be in $\boldsymbol{C}P^n$ and $X, Y$ in $T_A(\boldsymbol{C}P^n)$. Then $AXY=XYA$, $AXA=0$, $X(I-2A)=-(I-2A)X$, $(I-2A)^2=I$, $(I-2A)XY=XY(I-2A)$.

PROPOSITION 1.4. *Let $D$ be the Riemannian connection of $HM(n+1)$, $\nabla$ the induced connection in $\boldsymbol{C}P^n$, $\tilde{\sigma}$ the second fundamental form of the immersion, $\nabla^\perp$ and $\boldsymbol{A}$ the normal connection and the Weingarten endomorphism and $\widetilde{H}$ the mean curvature vector of $\boldsymbol{C}P^n$. Then*

$$\nabla_X Y=A(D_XY)+(D_XY)A-2A(D_XY)A\,, \tag{1.3}$$

$$\tilde{\sigma}(X, Y)=(XY+YX)(I-2A)\,, \tag{1.4}$$

$$\nabla^\perp_X Z=D_XZ+2A(D_XZ)A-(D_XZ)A-A(D_XZ)\,, \tag{1.5}$$

$$\boldsymbol{A}_Z X=(XZ-ZX)(I-2A)\,, \tag{1.6}$$

$$\widetilde{H}=\frac{1}{2n}[I-(n+1)A]\,, \tag{1.7}$$

*where $X$ and $Y$ are tangent vector fields to $\boldsymbol{C}P^n$, and $Z$ is a normal vector field to $\boldsymbol{C}P^n$.*

*Proof*. Let $\nabla$ and $\tilde{\sigma}$ be as in (1.3) and (1.4). Let $X$ be any vector in $T_A(\boldsymbol{C}P^n)$ and $Y$ any tangent vector field to $\boldsymbol{C}P^n$. If $\alpha: \Gamma\to\boldsymbol{C}P^n$ is a curve which satisfies $\alpha(0)=A$ and $\alpha'(0)=X$, we have $\alpha(t)Y(t)+Y(t)\alpha(t)=Y(t)$. Therefore

$$XY+YX+A(D_XY)+(D_XY)A=D_XY\,. \tag{1.8}$$

On the other hand, we have $\alpha(t)Y(t)\alpha(t)=0$. Therefore

$$XYA+A(D_XY)A+AYX=0\,. \tag{1.9}$$

From (1.8) and (1.9), we get $D_XY=\nabla_XY+\tilde{\sigma}(X, Y)$.

A simple calculations proves that $\nabla_XY$ (resp. $\tilde{\sigma}(X, Y)$) is tangent (resp. normal) to $\boldsymbol{C}P^n$. Then we have (1.3) and (1.4).

Let $\nabla^\perp$ and $\boldsymbol{A}$ be as in (1.5) and (1.6). Let $Z$ be any normal vector field to $\boldsymbol{C}P^n$. We have $\alpha(t)Z(t)=Z(t)\alpha(t)$, then

$$XZ+A(D_XZ)-(D_XZ)A-ZX=0\,,$$

$$\begin{aligned}\boldsymbol{A}_Z X&=(XZ-ZX)(I-2A)=[(D_XZ)A-A(D_XZ)](I-2A)\\&=2A(D_XZ)A-(D_XZ)A-A(D_XZ)=\nabla^\perp_X Z-D_XZ\,.\end{aligned}$$

From (1.1) (resp. (1.2)) we see that $A_Z X$ (resp. $\nabla^{\perp}_X Z$) is tangent (resp. normal), hence we have (1.5) and (1.6).

It is enough, to verify (1.7) at the origin.

Let $\{E_1, \cdots E_n, E_1^*, \cdots E_n^*\}$ be an orthonormal base in $T_A(CP^n)$ defined by

$$E_k = \frac{1}{2}\begin{array}{c} \\ \\ \\ (k) \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & (k) & & & \end{array} \\ \left(\begin{array}{c|cccccc} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \; 0 \\ \hline 0 & & & & & & \\ \vdots & & & & & & \\ 1 & & & & 0 & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{array}\right) \end{array},$$

$$E_k^* = \frac{\sqrt{-1}}{2}\begin{array}{c} \\ \\ \\ (k) \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} & & & (k) & \end{array} \\ \left(\begin{array}{c|ccccc} 0 & 0 & \cdots & 0 & 1 & \cdots \; 0 \\ \hline 0 & & & & & \\ \vdots & & & & & \\ -1 & & & 0 & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right) \end{array}.$$

A direct calculation proves that

$$\tilde{H}_{A_0} = \frac{1}{2n}\left(\begin{array}{c|ccc} -n & 0 & \cdots & 0 \\ \hline 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array}\right) = \frac{1}{2n}[I-(n+1)A_0]. \qquad \text{(Q.E.D.)}$$

LEMMA 1.5. a) *Let $f$ be the diffeomorphism obtained in lemma 1.1. Then $f$ is an isometry when we consider on $U(n+1)/U(1)\times U(n)$ the Fubini-Study metric with holomorphic sectional curvature $c=1$, and on $CP^n$ the metric induced by that on $HM(n+1)$.*

b) *The complex structure induced by the isometry $f$ in $CP^n$ is given by $JX=\sqrt{-1}(I-2A)X$, for all $X$ in $T_A(CP^n)$.*

*Proof.* a) Since both metrics are $U(n+1)$-invariant, it is enough to see that the differential of $f$ at the origin is an isometry between the corresponding tangent spaces.

Let $[P]$ be the coset of $P\in U(n+1)$ in $U(n+1)/U(1)\times U(n)$. Then $f([P]) = PA_0P^{-1}$ and so

$$T_0(U(n+1)/U(1)\times U(n)) = \left\{\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix} / a\in C^n\right\}, \qquad 0=[I].$$

The Fubini-Study metric of the constant holomorphic sectional curvature $c=1$ at the origin is given by

$$g_0\left(\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\bar{b}^t & 0 \end{pmatrix}\right)=2\,\mathrm{trace}\begin{pmatrix} a\bar{b}^t & 0 \\ 0 & \bar{a}^t b \end{pmatrix}.$$

Let $\alpha: \Gamma \to U(n+1)$ be a curve such that $\alpha(0)=I$ and $\alpha'(0)=\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}$. We consider the curve $\beta: \Gamma \to U(n+1)/U(1)\times U(n)$ given by $\beta(t)=[\alpha(t)]$. $df_0\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}=(f\beta)'(0)=\alpha'(0)A_0\overline{\alpha(0)}^t+\alpha(0)A_0\overline{\alpha'(0)}^t=\begin{pmatrix} 0 & -a \\ -\bar{a}^t & 0 \end{pmatrix}$, and we have $g_{A_0}\left((df)_0\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}, (df)_0\begin{pmatrix} 0 & b \\ -\bar{b}^t & 0 \end{pmatrix}\right)=2\,\mathrm{trace}\begin{pmatrix} a\bar{b}^t & 0 \\ 0 & \bar{a}^t b \end{pmatrix}$. This show a).

b) The complex structure $\tilde{J}$ at the origin of $U(n+1)/U(1)\times U(n)$ is given by $\tilde{J}\begin{pmatrix} 0 & a \\ -\bar{a}^t & 0 \end{pmatrix}=\sqrt{-1}\begin{pmatrix} 0 & -a \\ -\bar{a}^t & 0 \end{pmatrix}$, see [6]. Let $\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}$ be a vector in $T_{A_0}(\boldsymbol{CP}^n)$. Therefore the complex structure induced in $\boldsymbol{CP}^n$ is given by

$$J\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}=df_0\tilde{J}(df_0)^{-1}\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}=\sqrt{-1}\begin{pmatrix} 0 & -a \\ \bar{a}^t & 0 \end{pmatrix}.$$

On the other hand

$$\sqrt{-1}\,(I-2A_0)\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}=\sqrt{-1}\begin{pmatrix} 0 & -a \\ \bar{a}^t & 0 \end{pmatrix}. \qquad \text{(Q.E.D.)}$$

The following proposition resumes some properties of the immersion. For other properties, see [5], [7].

PROPOSITION 1.6. *The immersion of $\boldsymbol{CP}^n$ in $HM(n+1)$ verifies the following properties:*

a) *It is an isometric $U(n+1)$-equivariant imbedding.*

b) *$\tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y)$ and $\nabla\tilde{\sigma}=0$, that is, the second fundamental form is parallel.*

c) *It is minimal in the sphere $S$, whose center is $[1/(n+1)]I$ and whose radius is $\sqrt{2n/(n+1)}$.*

*Proof.* a) It is a consequence of lemma 1.1 and 1.5.

b) It is easy to see that $\tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y)$ for all $X, Y\in T_A(\boldsymbol{CP}^n)$. Let $X, Y_1, Y_2$ be any three vector fields tangent to $\boldsymbol{CP}^n$. Then we have

$$\begin{aligned}(\nabla\tilde{\sigma})_X(JY_1, JY_2)&=\nabla_X\tilde{\sigma}(JY_1, JY_2)-\tilde{\sigma}(\nabla_X JY_1, JY_2)-\tilde{\sigma}(JY_1, \nabla_X JY_2)\\ &=\nabla_X\tilde{\sigma}(Y_1, Y_2)-\tilde{\sigma}(\nabla_X Y_1, Y_2)-\tilde{\sigma}(Y_1, \nabla_X Y_2)\end{aligned}$$

$$=(\nabla\tilde{\sigma})_X(Y_1, Y_2).$$

Therefore we have $(\nabla\tilde{\sigma})_X(Y, JY)=0$, for all $Y$ in $T_A(\boldsymbol{C}P^n)$, and so from Codazzi's equation $(\nabla\tilde{\sigma})_Y(X, JY)=0$. If we choose $X=JY$, we have $0=(\nabla\tilde{\sigma})_Y(JY, JY)=(\nabla\tilde{\sigma})_Y(Y, Y)$. Hence $\nabla\tilde{\sigma}=0$.

c) If $A$ is in $\boldsymbol{C}P^n$ then $g\left(A-\frac{1}{n+1}I,\ A-\frac{1}{n+1}I\right)=\frac{2n}{n+1}$. Therefore $\boldsymbol{C}P^n$ is included in $S$. Let $\tilde{H}$ be the mean curvature vector of $\boldsymbol{C}P^n$ in $HM(n+1)$. $\tilde{H}=\frac{1}{2n}[I-(n+1)A]=-\frac{n+1}{2n}\left(A-\frac{1}{n+1}I\right)$. Therefore $\boldsymbol{C}P^n$ is minimal in $S$, see [2]. (Q. E. D.)

LEMMA 1.7. *Let $E_1$, $E_2$ be any two vectors in $T_A(\boldsymbol{C}P^n)$ such that $g(E_1, E_2)=0$ and $g(E_1, E_1)=g(E_2, E_2)=1$. Then*

a) $g(\tilde{\sigma}(E_1, E_1), \tilde{\sigma}(E_1, E_1))=1$,

b) $1/2\leqq g(\tilde{\sigma}(E_1, E_1), \tilde{\sigma}(E_2, E_2))\leqq 1$.

*Moreover if we have $g(E_1, JE_2)=0$, then*

c) $g(\tilde{\sigma}(E_1, E_1), \tilde{\sigma}(E_2, E_2))=1/2$,

d) $g(\tilde{\sigma}(E_1, E_2), \tilde{\sigma}(E_1, E_2))=1/4$.

*Proof.* Let $E_1=\begin{pmatrix} 0 & a \\ \bar{a}^t & 0 \end{pmatrix}$ and $E_2=\begin{pmatrix} 0 & b \\ \bar{b}^t & 0 \end{pmatrix}$. Then $g(E_1, E_1)=1$ if and only if $a\bar{a}^t=1/4$, $g(E_1, E_2)=0$ if and only if $a\bar{b}^t=\sqrt{-1}\,h$, where $h\in\boldsymbol{R}$. Moreover $g(E_1, JE_2)=0$ if and only if $a\bar{b}^t=0$. Now a), c) and d) are obvious.

b) $g(\tilde{\sigma}(E_1, E_1), \tilde{\sigma}(E_2, E_2))=8\,\mathrm{trace}\,(E_1^2E_2^2)=8\,\mathrm{trace}\begin{pmatrix} 1/16 & 0 \\ 0 & \sqrt{-1}\,h\bar{a}^tb \end{pmatrix}=1/2$ $+8h^2$. But $h^2=|a\bar{b}^t|^2\leqq|a|^2|b|^2=1/16$. (Q. E. D.)

## 2. $CR$-minimal submanifolds in the complex projective space.

For $CR$-submanifolds see for example [4]. In the following we write $M^{2n+p}$ for a $CR$-submanifold of $\boldsymbol{C}P^n$, where $2n=\dim\mathcal{D}$ and $p=\dim\mathcal{D}^\perp$, $\mathcal{D}$ being the holomorphic distribution and $\mathcal{D}^\perp$ the totally real distribution of $M$.

LEMMA 2.1. a) *Let $M^n$ be a submanifold of $\boldsymbol{C}P^m$. Let $H^\perp$ be the normal component of the mean curvature vector of $M^n$ in $HM(m+1)$ to $\boldsymbol{C}P^m$. Then*

(2.1) $$(n+1)/2n\leqq g(H^\perp, H^\perp)\leqq 1.$$

b) *Let $M^{2n+p}$ be a $CR$-submanifold of $\boldsymbol{C}P^m$. Let $H^\perp$ be as in* a). *Then*

$$g(H^\perp, H^\perp)=[(2n+p)^2+4n+p]/2(2n+p)^2\,. \tag{2.2}$$

*Proof*. a) Let $\{E_1, \cdots E_n\}$ be an orthonormal base of $T_A(M^n)$ where $A$ is any point in $M^n$. Let $\tilde{\sigma}$ be the second fundamental form of $\boldsymbol{CP}^m$ in $HM(m+1)$. Then $H^\perp=\frac{1}{n}\sum_i \tilde{\sigma}(E_i, E_i)$. By using lemma 1.7 we have (2.1).

b) We can choose an orthonormal base of $T_A(M)$ of the type $\{E_1, \cdots E_n, JE_1, \cdots JE_n, F_1, \cdots F_p\}$, where $E_i, JE_i$ are in $\mathcal{D}$ and $F_j$ is in $\mathcal{D}^\perp$. From lemma 1.7, we have (2.2). (Q. E. D.)

LEMMA 2.2. *Let $M^{2n+p}$ be a CR-submanifold of $\boldsymbol{CP}^m$, $\tilde{\sigma}$ the second fundamental form of $\boldsymbol{CP}^m$ in $HM(m+1)$ and $\tilde{\sigma}_M$ its restriction to $M$. Then*

$$g(\tilde{\sigma}_M, \tilde{\sigma}_M)=(1/4)[(2n+p)^2+4n+3p]\,. \tag{2.3}$$

The proof can be obtained by using lemma 1.7. From the expresion of the scalar curvature for submanifolds in the Euclidean space, we obtain the following

COROLLARY 2.3. *Let $M^{2n+p}$ be a CR-submanifold of $\boldsymbol{CP}^m$. Let $H$ be the mean curvature vector of $M^{2n+p}$ in $\boldsymbol{CP}^m$, $r$ the scalar curvature of $M^{2n+p}$, and $\sigma$ the second fundamental form of $M^{2n+p}$ in $\boldsymbol{CP}^m$. Then*

$$r=[(2n+p)^2+4n-p]/4+(2n+p)^2g(H, H)-g(\sigma, \sigma)\,. \tag{2.4}$$

B. Y. Chen has proved the following theorems:

THEOREM A. [2]. *Let $M$ be an $n$-dimensional closed submanifold of $E^m$. Then we have*

$$\int_M \alpha^n dv \geqq c_n\,, \tag{2.5}$$

*where $\alpha=\sqrt{g(H, H)}$ is the mean curvature of $M$ and $c_n$ is the volume of unit $n$-sphere. The equality holds if and only if $M$ is imbedded as an ordinary $n$-sphere in an affine $(n+1)$-space.*

For an isometric immersion of a closed manifold $M$ in the Euclidean space $x: M\to E^m$, we put $x=(x_1, \cdots x_m)$, where $x_i$ is the $i$-th coordinate function of $M$ in $E^m$. We call an isometric immersion $x$ is of order $k$ if each coordinate function $x_i$ of $x$ is an eigenfunction of the Laplace Beltrami operator of $M$ corresponding to eigenvalue $\lambda_k$.

THEOREM B. [3]. *Let $x: M\to E^m$ be an isometric immersion of a closed $n$-dimensional Riemannian manifold $M$ into $E^m$. The total mean curvature of $x$ satisfies*

$$\int_M \alpha^n dv \geqq \left(\frac{\lambda_1}{n}\right)^{n/2} \operatorname{vol}(M)\,, \tag{2.6}$$

*where* vol $(M)$ *denotes the volume of* $(M, g)$ *and* $\lambda_1$ *denotes the first eigenvalue of the Laplace-Beltrami operator of* $(M, g)$ *acting on differentiable functions in* $C^\infty(M)$. *The equality holds if and only if there is a vector* $c$ *in* $E^m$ *such that* $x-c$ *is an imbedding of order* 1.

COROLLARY 2.4. *Let* $M^n$ *be a closed minimal submanifold of* $CP^m$. *Then we have*

$$\mathrm{vol}\,(M) \geqq c_n\,. \tag{2.7}$$

*Proof*. Let $H$ be the mean curvature vector of $M^n$ in $HM(m+1)$. Let $H^\perp$ be the same as in lemma 2.1. Since $M^n$ is minimal in $\boldsymbol{CP}^m$, $H=H^\perp$. Now we use theorem A and lemma 2.1. (Q. E. D.)

COROLLARY 2.5. *Let* $M^{2n+p}$ *be a closed CR-minimal submanifold of* $CP^m$. *Then we have*

$$\left[\frac{(2n+p)^2+4n+p}{2(2n+p)^2}\right]^{n+p/2} \mathrm{vol}\,(M) \geqq c_{2n+p}\,. \tag{2.8}$$

*The equality holds if and only if* $M=\boldsymbol{CP}^1$ *is imbedded as a totally geodesic complex submanifold in* $\boldsymbol{CP}^m$.

*Proof*. By using theorem A and lemma 2.1 we obtain (2.8).

We suppose that the equality holds. Then $M$ is isometric to a sphere of radius $R$. We have $\mathrm{vol}(M)=R^{2n+p}c_{2n+p}$, and then $R^2=2(2n+p)^2/[(2n+p)^2+4n+p]$. Let $c$ and $r$ be the sectional curvature and the scalar curvature of $M$ respectively. Then $c=1/R^2$ and

$$r=c(2n+p-1)(2n+p)\,. \tag{2.9}$$

From corollary 2.3

$$r \leqq (1/4)[(2n+p)^2+4n-p]\,. \tag{2.10}$$

From (2.9) and (2.10) we have

$$[(2n+p)^2+4n](2n+p-2)+p(6n+3p-2) \leqq 0\,.$$

But this occurs if and only if $n=1$ and $p=0$. Therefore $M$ is a unit 2-sphere imbedded as complex submanifold in $\boldsymbol{CP}^m$. Since $M$ and $\boldsymbol{CP}^m$ have the same holomorphic sectional curvature $c=1$, we get that $M$ is totally geodesic in $\boldsymbol{CP}^m$.

The converse is trivial because $\boldsymbol{CP}^1$ is imbedded in $HM(2)$ as a standard sphere. (Q. E. D.)

The following corollaries can be obtained from theorem B and lemma 2.1.

COROLLARY 2.6. *Let* $M^n$ *be a closed minimal submanifold of* $\boldsymbol{CP}^m$. *Then we have*

$$\lambda_1 \leqq n\,. \tag{2.12}$$

COROLLARY 2.7. *Let* $M^{2n+p}$ *be a closed CR-minimal submanifold of* $\boldsymbol{CP}^m$.

*Then we have*

$$\lambda_1 \leqq [(2n+p)^2+4n+p]/2(2n+p)\,. \tag{2.13}$$

*In particular, if $M^{2n}$ is a compact complex (resp. $M^p$ is a closed minimal totally real) submanifold of $\boldsymbol{CP}^m$ then*

$$\lambda_1 \leqq n+1 \tag{2.14}$$

$$(resp.\ \lambda_1 \leqq (p+1)/2)\,. \tag{2.15}$$

The following result gives a complete classification of the $CR$-minimal submanifolds of $\boldsymbol{CP}^m$ which are minimal in some sphere of $HM(m+1)$.

THEOREM 2.8. *Let $M^{2n+p}$ be a $CR$-minimal submanifold of $\boldsymbol{CP}^m$. Then $M^{2n+p}$ is minimal in some sphere of $HM(m+1)$ if and only if one of the following cases holds:*

a) *$p=0$ and $M^{2n}$ is a totally geodesic complex submanifold of $\boldsymbol{CP}^m$.*

b) *$n=0$ and $M^p$ is a totally real submanifold of $\boldsymbol{CP}^m$ for which there exists a totally geodesic complex submanifold $\overline{M}^{2p}$ of $\boldsymbol{CP}^m$, such that $M^p$ is a totally real submanifold of $\overline{M}^{2p}$.*

*Proof*. We suppose that $M^{2n+p}$ is minimal in a certain sphere $S$ of $HM(m+1)$. If $Q$ denotes the center of $S$, we can suppose that $Q$ is a diagonal matrix (otherwise we can use an isometry of $HM(m+1)$ of the type $A \mapsto PAP^{-1}$, where $P$ is in $U(m+1)$). Let $H$ be the mean curvature vector of $M$ in $HM(m+1)$. From the minimality of $M$ in $S$ we have $H=h\cdot(A-Q)$, for any $A$ in $M$ where $h$ is a real number with $h\neq 0$. It is clear that $Q\in T_A^{\perp}(\boldsymbol{CP}^m)$. Therefore $AQ=QA$ for any $A$ in $M$. That is, $M$ is contained in the linear subspace, $L$ of $HM(m+1)$, which is defined by the equation $AQ=QA$. We put

$$Q=\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_1 & & & \\ & & & \ddots & & \\ & & & & a_r & \\ & & & & & \ddots \\ & & & & & & a_r \end{pmatrix}.$$

Then

$$\boldsymbol{CP}^m \cap L = \left\{ \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} \Big/ \begin{matrix} A_i A_i = A_i \\ \sum_i \operatorname{trace} A_i = 1 \end{matrix} \right\}.$$

Since $A_iA_i=A_i$, we see that trace $A_i$ is a natural number. Hence for any $A$ in $\boldsymbol{CP}^m \cap L$ there exists an index $j$ such that trace $A_j=1$ and trace $A_i=0$ for all $i\neq j$, which implies $A_i=0$ and

$$\boldsymbol{CP}^m \cap L = \left\{ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & A_j & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \middle/ \begin{matrix} A_j A_j = A_j \\ \text{trace } A_j = 1 \end{matrix} \right\}.$$

Therefore $M$ is contained in a connected component of $\boldsymbol{CP}^m \cap L$. Each of these component is evidently a totally geodesic complex submanifold of $\boldsymbol{CP}^m$ (it is a $\boldsymbol{CP}^q$, $q \leqq m$), and $M$ is a minimal submanifold of the sphere $S \cap L$. Consequently the problem is reduced to the study of $CR$-minimal submanifolds of $\boldsymbol{CP}^q$ which are minimal in some sphere of $HM(q+1)$ whose center is $aI$ where $a$ is a real number and $I$ is the $(q+1)\times(q+1)$-identity matrix.

We have $H=h\cdot(A-aI)$. As $M$ is contained in the sphere we know that

$$g(H, A-aI)=-1\,,$$

and since $M$ is $CR$-minimal in $\boldsymbol{CP}^q$,

$$g(H, H)=\frac{(2n+p)^2+4n+p}{2(2n+p)^2}\,.$$

Therefore

$$h=-\frac{(2n+p)^2+4n+p}{2(2n+p)^2}\,,$$

(2.16) $$g(A-aI, A-aI)=\frac{2(2n+p)^2}{(2n+p)^2+4n+p}$$

for all $A$ in $M$. On the other hand,

(2.17) $$\begin{aligned} g(A-aI, A-aI)&=g(A, A)-2ag(A, I)+a^2 g(I, I)\\ &=2(q+1)a^2-4a+2\,. \end{aligned}$$

From (2.16) and (2.17) we obtain

$$(q+1)[(2n+p)^2+4n+p]a^2-2[(2n+p)^2+4n+p]a+4n+p=0\,.$$

Since the discriminate of this equation must $\geqq 0$, we get

$$(2n+p)^2+4n+p-(q+1)(4n+p)\geqq 0\,,$$

that is $(2n+p)^2 \geqq q(4n+p)$. But $q \geqq n+p$, and so

$$(2n+p)^2 \geqq (4n+p)(n+p)\,.$$

Therefore $4np \geqq 5np$, which implies $n=0$ or $p=0$.

*) Suppose $p=0$. Then we have $q=n$, that is $M^{2n}$ is open in $\boldsymbol{CP}^n$.

*) Suppose $n=0$. Then $p=q$.

Conversely: If $M^{2n}$ is a totally geodesic complex submanifold of $\boldsymbol{CP}^m$, then from proposition 1.6, $M$ is minimal in some sphere. Let $M^p$ be a totally real minimal submanifold of $\boldsymbol{CP}^p$. For any $A \in M$, let $\{E_1, \cdots E_p\}$ be an orthonormal base of $T_A(M)$. Then we have that $\{E_1, \cdots E_p, JE_1, \cdots JE_p\}$ is an orthonormal base of $T_A(CP^p)$. Hence, if $H$ is the mean curvature vector of $M^p$ in $HM(p+1)$ it is easy to see from proposition 1.6, that

$$H = \frac{1}{2p}[I-(p+1)A],$$

and so $M^p$ is minimal in some sphere. (Q.E.D.)

COROLLARY 2.9. *Let $M^{2n+p}$ be a closed CR-minimal submanifold of $\boldsymbol{CP}^m$.* 1) *If $M$ is in the cases* a) *or* b) *of theorem* 2.8, *then* $[(2n+p)^2+4n+p]/2(2n+p)$ *is in* Spec$(M)$.

2) *If $\lambda_1=[(2n+p)^2+4n+p]/2(2n+p)$, then $M$ is imbedded and is in the cases* a) *or* b) *of theorem* 2.8, *where* Spec$(M)$ *is the spectrum of the Laplace-Beltrami operator of $M$ and $\lambda_1$ is the first eigenvalue of this operator.*

*Proof.* 1) From the proof of theorem 2.8 and from a well know theorem of Takahashi [8], if $M$ is minimal in $S$ then $\lambda_k=\dim(M)/R^2$ for some $\lambda_k$ in Spec$(M)$, where $R$ is the radius of $S$. Then $\lambda_k=[(2n+p)^2+4n+p]/2(2n+p)$.

2) From theorem B, we see, by choosing a suitable origen, that the immersion is an imbedding of order 1. In particular it is minimal in some sphere, [8]. Now from theorem 2.8, $M$ is in the cases a) or b). (Q.E.D.)

COROLLARY 2.10. *Let $M^{2n}$ be a complex compact submanifold of $\boldsymbol{CP}^m$. Then we have $\lambda_1 \leqq n+1$. Moreover $M^{2n}$ is totally geodesic in $\boldsymbol{CP}^m$ if and only if $\lambda_1=n+1$.*

*Proof.* We consider corollaries 2.7 and 2.9, and Spec$(\boldsymbol{CP}^n)$, see [1].

COROLLARY 2.11. *Let $M^p$ be a totally real closed minimal submanifold of $CP^m$. Then we have* 1) *If there exists $\overline{M}^{2p}$ such that $\overline{M}^{2p}$ is a totally geodesic complex submanifod of $\boldsymbol{CP}^m$ and $M^p$ is a totally real submanifold of $\overline{M}^{2p}$, then $(p+1)/2$ belongs to* Spec$(M^p)$.

2) *If $\lambda_1=(p+1)/2$, then there exists a totally geodesic complex submanifold $\overline{M}^{2p}$ of $\boldsymbol{CP}^m$ such that $M^p$ is a totally real submanifold of $\overline{M}^{2p}$.*

*Proof.* We consider corollary 2.9.

The author has known that corollaries 2.7 and 2.10 has been recently obtained by N. Ejiri.

## REFERENCES

[1] M. BERGER, P. GAUDUCHON AND E. MAZET, Le spectre d'une variété Riemanniene. Lecture Notes in Math. No. **194**, Springer-Verlag, Berlin 1971.

[2] B.Y. CHEN, Geometry of submanifolds. M. Dekker, New-York 1973.

[3] B.Y. CHEN, On the total curvature of immersed manifolds, IV: Spectrum and total mean curvature. Bull. Math. Acad. Sinica, vol. **7** No. 3, 1979, 301-311.

[4] B.Y. CHEN, Geometry of submanifolds and its applications. Science University of Tokyo, 1981.

[5] J.A. LITTLE, Manifold with planar geodesic. J. Differential Geometry, **11**, 1976, 265-285.

[6] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry. Wiley-Interscience, New-York 1963.

[7] S.S. TAI, Minimum imbedding of compact symmetric spaces of rank one. J. Differential Geometry **2**, 1968, 55-66.

[8] T. TAKAHASHI, Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan, **18**, 1966, 380-385.

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