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UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS, II

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0. Introduction. Let f and g be meromorphic functions. If f and g have the same a-points with the same multiplicities, we denote this by $f=a\rightleftarrows g=a$ for simplicity's sake. And we denote the order of f by ρ_f.

In [5] Ozawa proved the following result.

Theorem A. *Let f and g be entire functions. Assume that $\rho_f, \rho_g<\infty$, $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $\delta(0, f)>1/2$. Then $fg\equiv 1$ unless $f\equiv g$.*

It is natural to ask whether the order restriction of f and g in Theorem A can be removed or not. In our previous paper [6] we showed the following fact.

Theorem B. *Let f and g be entire functions. Assume that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $\delta(0, f)>5/6$. Then $fg\equiv 1$ unless $f\equiv g$.*

In this paper we shall show first that in Theorem A the order restriction of f and g can be removed perfectly.

Theorem 1. *Let f and g be entire functions. Assume that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $\delta(0, f)>1/2$. Then $fg\equiv 1$ unless $f\equiv g$.*

In Theorem 1, the estimate "$\delta(0, f)>1/2$" is best possible. In fact, consider $f=e^{\alpha}(1-e^{\alpha})$, $g=e^{-\alpha}(1-e^{-\alpha})$ with a nonconstant entire function α. Then $f=-ge^{3\alpha}$, $f-1=(g-1)e^{2\alpha}$ and $\delta(0, f)=1/2$. $f\not\equiv g$ and $fg\not\equiv 1$ are evident.

In place of Theorem 1, we prove more generally the following

Theorem 2. *Let f and g be meromorphic functions satisfying $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $f=\infty\rightleftarrows g=\infty$. If*

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)+N(r, \infty, f)}{T(r, f)}<1/2,$$

then $f\equiv g$ or $fg\equiv 1$.

In order to state our second result, we introduce a notation: If k is a positive integer or ∞, let

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$$E(a,\ k,\ f)=\{z\in\mathbf{C},\ z \text{ is a zero of } f-a \text{ of order} \leqq k.\},$$

where **C** is the complex plane.

In [7], we showed the following

THEOREM C. *Let f and g be nonconstant entire functions such that $f=0\rightleftarrows g=0$ and $f=1\rightleftarrows g=1$. Further assume that there exists a complex number a $(\neq 0,\ 1)$ satisfying $E(a,\ k,\ f)=E(a,\ k,\ g)$, where k is a positive integer $(\geqq 2)$ or ∞. Then f and g must satisfy one of the following four relations:*

(i) $f\equiv g$, (ii) $(f-1/2)(g-1/2)\equiv 1/4$ (This occurs only for $a=1/2$.),
(iii) $fg\equiv 1$ $(a=-1)$, (iv) $(f-1)(g-1)\equiv 1$ $(a=2)$,

We shall extend this result for meromorphic functions.

THEOREM 3. *Suppose that f and g are nonconstant meromorphic in* **C** *such that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $f=\infty\rightleftarrows g=\infty$. Further assume that there exists a complex number a $(\neq 0,\ 1)$ satisfying $E(a,\ k,\ f)=E(a,\ k,\ g)$, where k is a positive integer $(\geqq 2)$ or ∞. Then f and g must satisfy one of the following seven relations:*

(i) $f\equiv g$, (ii) $f+g\equiv 1$ (This occurs only for $a=1/2$.),
(iii) $fg\equiv 1$ $(a=-1)$, (iv) $f+g\equiv 2$ $(a=2)$, (v) $(f-1)(g-1)\equiv 1$ $(a=2)$,
(vi) $f+g\equiv 0$ $(a=-1)$, (vii) $(f-1/2)(g-1/2)\equiv 1/4$ $(a=1/2)$.

We remark that Theorem 3 has been proved by Gundersen [1] for the case $k=\infty$. Theorem 3 is an improvement of a well known theorem of Nevanlinna [3, p 122].

1. Lemmas. In this section we state three lemmas. The first lemma is due to Niino and Ozawa [4].

LEMMA 1. *Let $\{\alpha_j\}_1^p$ be a set of non-zero constants and $\{g_j\}_1^p$ a set of entire functions satisfying*

$$\sum_{j=1}^{p}\alpha_j g_j\equiv 1.$$

Then

$$\sum_{j=1}^{p}\delta(0,\ g_j)\leqq p-1.$$

The second lemma is very straightforward, but important for the proof of Theorem 2.

LEMMA 2. *Let f be a nonconstant meromorphic function. Put*

$$F=f''/f-2(f'/f)^2.$$

Then

$$N(r, \infty, F) \leqq 2N(r, 0, f) + N(r, \infty, f).$$

Proof. Let a be a pole of F. Then it is clear that a is a zero or a pole of f.

Case 1. Assume that a is a zero of f with multiplicity $n \geqq 1$. In this case we have

$$f(z) = g(z)(z-a)^n$$

with a meromorphic function $g(z)$ satisfying $g(a) \neq 0, \infty$. Hence

$$F(z) = -\frac{n(n+1)}{(z-a)^2} - 2\frac{n}{z-a}\frac{g'(z)}{g(z)} + \frac{g''(z)}{g(z)} - 2\left(\frac{g'(z)}{g(z)}\right)^2.$$

Case 2. Assume that a is a pole of f with multiplicity $n \geqq 1$. Then we have

$$f(z) = g(z)(z-a)^{-n}$$

with a meromorphic function $g(z)$ satisfying $g(a) \neq 0, \infty$. Hence

$$F(z) = -\frac{n(n-1)}{(z-a)^2} + 2\frac{n}{z-a}\frac{g'(z)}{g(z)} + \frac{g''(z)}{g(z)} - 2\left(\frac{g'(z)}{g(z)}\right)^2.$$

The above two simple computations combine to show that

$$N(r, \infty, F) \leqq 2N(r, 0, f) + N(r, \infty, f).$$

The third lemma, which is due to Hiromi and Ozawa [2], plays an important role for the proof of Theorem 3.

LEMMA 3. *Let $h_0, h_1, \cdots, h_m$ be meromorphic functions and $k_1, k_2, \cdots, k_m$ be entire functions. Suppose that*

$$T(r, h_j) = o\left(\sum_{n=1}^{m} T(r, e^{k_n})\right) \qquad j = 0, 1, \cdots, m$$

holds outside a set of finite linear measure. If an identity

$$\sum_{n=1}^{m} h_n(z) e^{k_n(z)} \equiv h_0(z)$$

holds, then for suitable constants $\{C_n\}_1^m$, not all zero,

$$\sum_{n=1}^{m} C_n h_n(z) e^{k_n(z)} \equiv 0.$$

2. Proof of Theorem 2. By assumption, we have

(2.1) $$f = e^{\alpha} g, \qquad f - 1 = e^{\beta}(g-1)$$

with two entire functions α and β.

(A) Suppose that $e^{\beta}\equiv c(\neq 0)$. If f has at least one zero, (2.1) implies $c=1$, i.e. $f\equiv g$. If f has no zeros and $c\neq 1$, we have

$$f-cg=1-c\neq 0\,.$$

Put $f_1=f^{-1}$, $g_1=g^{-1}$. Then f_1, g_1 are entire functions satisfying

$$g_1=\frac{cf_1}{1-(1-c)f_1}\,.$$

Since g_1 is an entire function, $1-(1-c)f_1=e^{\gamma}$, where γ is entire. Hence

$$f=f_1^{-1}=\frac{1-c}{1-e^{\gamma}}\,.$$

Thus

$$N(r,\infty,f)=N(r,1,e^{\gamma})=(1+o(1))T(r,e^{\gamma})=(1+o(1))(T(r,f) \qquad (r\notin E,\ r\to\infty)\,.$$

(Here and throughout this paper, the letter E will denote sets of finite linear measure which will not necessarily be the same at each occurrence.)

This is impossible.

(B) Suppose that $e^{\alpha-\beta}\equiv c(\neq 0)$. If $c=1$, we have $f\equiv g$. If $c\neq 1$, (2.1) gives

$$f=\frac{-c(e^{\beta}-1)}{c-1}\,.$$

Thus

$$N(r,0,f)=N(r,1,e^{\beta})=(1+o(1))T(r,e^{\beta})=(1+o(1))T(r,f) \qquad (r\notin E,\ r\to\infty)\,.$$

This is untenable.

(C) Suppose neither e^{β} nor $e^{\alpha-\beta}$ are constants. In this case, we have from (2.1)

$$f=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}\,,\qquad g=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}e^{-\alpha}\,. \tag{2.2}$$

Now, we use the argument of impossibility of Borel's identity. (cf. [3]) Put $\varphi_1=f$, $\varphi_2=-fe^{\beta-\alpha}$ and $\varphi_3=e^{\beta}$. Then by (2.2)

$$\varphi_1+\varphi_2+\varphi_3\equiv 1\,,\qquad \varphi_1^{(n)}+\varphi_2^{(n)}+\varphi_3^{(n)}\equiv 0 \qquad (n=1,2)\,. \tag{2.3}$$

Further put

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ \varphi_1'/\varphi_1 & \varphi_2'/\varphi_2 & \varphi_3'/\varphi_3 \\ \varphi_1''/\varphi_1 & \varphi_2''/\varphi_2 & \varphi_3''/\varphi_3 \end{vmatrix}\,,\qquad \Delta'=\begin{vmatrix} \varphi_2'/\varphi_2 & \varphi_3'/\varphi_3 \\ \varphi_2''/\varphi_2 & \varphi_3''/\varphi_3 \end{vmatrix}\,. \tag{2.4}$$

Assume first that $\Delta\equiv 0$. Then by (2.3)

$$0=\begin{vmatrix}\varphi_1 & \varphi_2 & \varphi_3\\ \varphi_1' & \varphi_2' & \varphi_3'\\ \varphi_1'' & \varphi_2'' & \varphi_3''\end{vmatrix}=\begin{vmatrix}\varphi_1 & \varphi_2 & 1\\ \varphi_1' & \varphi_2' & 0\\ \varphi_1'' & \varphi_2'' & 0\end{vmatrix}=\begin{vmatrix}\varphi_1' & \varphi_2'\\ \varphi_1'' & \varphi_2''\end{vmatrix}.$$

This implies $\varphi_2=C\varphi_1+D$ (C, D: constants), i.e. $-fe^{\beta-\alpha}=Cf+D$. If $C\neq 0$, we have

$$f=\frac{-D}{C+e^{\beta-\alpha}}.$$

so that $N(r,\infty,f)=(1+o(1))T(r,f)$ $(r\notin E,\ r\to\infty)$, a contradiction. Hence C must vanish, i.e. $f=-De^{\alpha-\beta}$. Substituting this into (2.3), we have

$$-De^{\alpha-\beta}+e^{\beta}=1-D.$$

Using Lemma 1, we have $D=1$ and $e^{\beta}=e^{\alpha-\beta}$. It follows from these and (2.2) that $fg\equiv 1$.

Assume next that $\Delta\not\equiv 0$. Then by (2.4) $\varphi_1=f=\Delta'/\Delta$. Thus

$$\begin{aligned}m(r,f)&\leqq m(r,\Delta')+m(r,\Delta^{-1})\\&\leqq m(r,\Delta')+m(r,\Delta)+N(r,\infty,\Delta)+O(1).\end{aligned}\tag{2.5}$$

Here we estimate $m(r,\Delta')$ and $m(r,\Delta)$. By (2.1)

$$T(r,e^{\beta})\leqq T(r,f)+T(r,g)+O(1)$$

$$\begin{aligned}T(r,e^{\beta-\alpha})&\leqq T(r,e^{\beta})+T(r,e^{-\alpha})\\&\leqq 2T(r,f)+2T(r,g)+O(1).\end{aligned}$$

By the second fundamental theorem,

$$\begin{aligned}(1-o(1))T(r,g)&\leqq N(r,0,g)+N(r,1,g)+N(r,\infty,g)\\&\leqq N(r,0,f)+N(r,1,f)+N(r,\infty,f)\\&\leqq(3+o(1))T(r,f)\qquad(r\notin E,\ r\to\infty).\end{aligned}$$

Hence

$$T(r,\varphi_3)=T(r,e^{\beta})\leqq(4+o(1))T(r,f)\qquad(r\notin E,\ r\to\infty),$$

$$T(r,\varphi_2)\leqq T(r,f)+T(r,e^{\beta-\alpha})\leqq(9+o(1))T(r,f)\qquad(r\notin E,\ r\to\infty),$$

Therefore

$$m(r,\Delta'),\ m(r,\Delta)=O(\log rT(r,f))\qquad(r\notin E,\ r\to\infty).$$

Substituting these into (2.5), we have

$$m(r,f)\leqq N(r,\infty,\Delta)+O(\log rT(r,f))\qquad(r\notin E,\ r\to\infty).\tag{2.6}$$

Also, a direct computation shows that

$$\Delta=[f''/f-2(f'/f)^2](\beta'-\alpha')+(f'/f)[(\beta')^2-(\alpha')^2-2(\beta'-\alpha')$$
$$-(\beta''-\alpha'')]+\beta'(\beta''-\alpha'')+\beta'(\beta'-\alpha')-(\beta'-\alpha')[\beta''+(\beta')^2].$$

It follows from this and Lemma 2 that

$$N(r, \infty, \Delta)\leqq 2N(r, 0, f)+N(r, \infty, f). \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$T(r, f)\leqq 2[N(r, 0, f)+N(r, \infty, f)]+O(\log rT(r, f)) \qquad (r\notin E, r\to\infty).$$

Hence,

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)+N(r, \infty, f)}{T(r, f)}\geqq 1/2.$$

This is a contradiction.

This completes the proof of Theorem 2

3. Proof of Theorem 3. By assumption we have with two entire functions α and β

$$f=e^{\alpha}g, \qquad f-1=e^{\beta}(g-1). \tag{3.1}$$

We divide our argument into the following five cases.

(A) $\beta(z)$ is a constant. (B) $\alpha(z)-\beta(z)$ is a constant.
(C) $\alpha(z)$ is a constant. (D) $\beta(z)-a(\beta(z)-\alpha(z))$ is a constant.
(E) None of $\beta(z)$, $\alpha(z)-\beta(z)$, $\alpha(z)$ and $\beta(z)-a(\beta(z)-\alpha(z))$ are constants.

(A) Suppose that $e^{\beta}\equiv c(\neq 0)$. If f has a zero, $c=1$. Hence $f\equiv g$. If f has no zeros and $c\neq 1$, (3.1) implies

$$f=\frac{1-c}{1-e^{\gamma}}, \qquad g=\frac{f-(1-c)}{c}, \tag{3.2}$$

where γ is a nonconstant entire function. Assume first that $a=1-c$. In this case, $f=a$ has no roots, so that $E(a, k, g)=\emptyset$ $(k\geqq 2)$. By (3.2)

$$g=\frac{a}{1-a}\cdot\frac{1}{e^{-\gamma}-1}.$$

Hence, if $a\neq 2$, $g=a$ has infinitely many simple roots, a contradiction. On the other hand, if $a=2$, $g=a$ has no roots, and we have from (3.2)

$$g\equiv 2-f, \qquad f=\frac{2}{1-e^{\gamma}}.$$

Next, assume that $a\neq 1-c$. In this case, $f=a$ has infinitely many simple roots. Hence by (3.2)

$$a=\frac{a-(1-c)}{c},$$

which implies $a=1$, a contradiction.

(B) Suppose that $e^{\alpha-\beta}\equiv c(\neq 0)$. If $c=1$, we have $f\equiv g$. If $c\neq 1$, (3.1) gives

(3.3) $$g=\frac{f}{(1-c)f+c},\qquad f=\frac{c(1-e^{\beta})}{c-1},\qquad g=\frac{e^{-\beta}-1}{c-1}.$$

By the same reasoning as in (A), we deduce from (3.3) that $c=-1$, $a=1/2$, and

$$g\equiv\frac{f}{2f-1},\qquad f=\frac{1-e^{\beta}}{2}.$$

(C) Suppose that $e^{\alpha}\equiv c(\neq 0)$. If $c=1$, we have $f\equiv g$. If $c\neq 1$, (3.1) gives

(3.4) $$g=\frac{f}{c},\qquad f=\frac{c(1-e^{\beta})}{c-e^{\beta}},\qquad g=\frac{1-e^{\beta}}{c-e^{\beta}}.$$

By the same reasoning as in (A), we deduce from (3.4) that $c=-1$, $a=-1$, and

$$g\equiv -f,\qquad f=\frac{1-e^{\beta}}{1+e^{\beta}}.$$

(D) Suppose that $\beta(z)=a(\beta(z)-\alpha(z))+C$, where C is a constant. By (3.1)

(3.5) $$f=\frac{1-e^{\beta}}{1-e^{\gamma}},\qquad g=\frac{1-e^{\beta}}{1-e^{\gamma}}e^{\gamma-\beta}=\frac{1-e^{-\beta}}{1-e^{-\gamma}},$$

where $\gamma\equiv\beta-\alpha$.

Assume first that there exists a sequence $\{w_n\}$ satisfying

(3.6) $$f(w_n)=a,\qquad e^{\gamma(w_n)}\neq 1.$$

Let w be an element of $\{w_n\}$. Clearly

(3.7) $$e^{\beta(w)}\neq 1,\qquad e^{\beta(w)}\neq e^{\gamma(w)}.$$

By (3.5), (3.6) and (3.7), $g(w)\neq a$. Hence, by assumption, w is a zero of $f-a$ with multiplicity $\geqq k+1$ ($\geqq 3$). Then an elementary calculation shows that

$$\gamma'(w)=\gamma''(w)=\cdots=\gamma^{(k)}(w)=0.$$

Here, we show that

(3.8) $$\#\{\gamma(w_n)\}=1.$$

If the set $\{\gamma(w_n)\}$ contains γ_1 and γ_2 ($\gamma_1\neq\gamma_2$), all the roots of $\gamma(z)=\gamma_j$ ($j=1, 2$) satisfy $f(z)=a$, $e^{\gamma(z)}\neq 1$. Then the above reasoning shows that $\gamma^{(i)}(z)=0$, $i=1, 2, \cdots, k$. Hence

$$\Theta(\gamma_j, \gamma)=1-\overline{\lim_{r\to\infty}}\frac{\bar{N}(r, \gamma_j, \gamma)}{T(r, \gamma)}\geqq\frac{k}{k+1}\qquad (j=1, 2),$$

and so

$$\sum_c \Theta(c, \gamma) \geqq \Theta(\gamma_1, \gamma) + \Theta(\gamma_2, \gamma) + \Theta(\infty, \gamma) > 2 .$$

This is a contradiction. Thus (3.8) holds.

Let $\{z_n\}$ be the sequence satisfying

$$e^{\gamma(z_n)} = e^{\beta(z_n)} = 1 . \tag{3.9}$$

We claim here that

$$\# \{\gamma(z_n)\} \leqq 1 . \tag{3.10}$$

If γ_1, γ_2 $(\gamma_1 \neq \gamma_2) \in \{\gamma(z_n)\}$, then by (3.9)

$$\gamma_j = 2 l_j \pi i , \qquad a\gamma_j + C = 2 s_j \pi i \qquad (j = 1, 2) ,$$

where l_1, l_2, s_1, s_2 are integers such that $l_1 \neq l_2$, $s_1 \neq s_2$. Hence

$$a = \frac{s_1 - s_2}{l_1 - l_2}$$

is a rational number. By (3.8) $\{\gamma(w_n)\} = \{\delta_1\}$, where δ_1 is a complex number. Since $\gamma(z)$ is a nonconstant entire function, $\gamma(z)$ omits at most one finite value. Hence $\gamma(z) = \delta_1 + 2(l_1 - l_2)\pi i$ or $\gamma(z) = \delta_1 - 2(l_1 - l_2)\pi i$ has roots. This implies that $\delta_1 + 2(l_1 - l_2)\pi i \in \{\gamma(w_n)\}$ or $\delta_1 - 2(l_1 - l_2)\pi i \in \{\gamma(w_n)\}$. This is a contradiction.

Now, consider the function

$$F(z) \equiv 1 - a - e^{\beta} + a e^{\gamma} = (f - a)(1 - e^{\gamma}) . \tag{3.11}$$

By the second fundamental theorem

$$\begin{aligned} N(r, 1-a, F) \leqq T(r, F) &\leqq N(r, 0, F) + N(r, \infty, F) + N(r, 1-a, F) - N(r, 0, F') \\ &\quad + o(T(r, F)) = N(r, 0, F) + N(r, a, e^{\beta-\gamma}) - N(r, 0, e^{\beta-\gamma} - 1) \\ &\quad + o(T(r, F)) = N(r, 0, F) + o(T(r, e^{\beta-\gamma})) + o(T(r, F)) \\ &\qquad\qquad (r \notin E,\ r \to \infty) . \end{aligned}$$

Hence

$$N(r, 0, F) \geqq (1 - o(1)) T(r, F) \geqq (1 - o(1)) T(r, e^{\beta-\gamma}) \qquad (r \notin E,\ r \to \infty) . \tag{3.12}$$

Let $\{x_n\}$ be the roots of $F(z) = 0$ with multiplicity $\geqq 3$. Then x_n is a root of $F'(z) = e^{\gamma}\{a\gamma' - \beta' e^{\beta-\gamma}\} = \beta' e^{\gamma}\{1 - e^{\beta-\gamma}\} = 0$ with multiplicity $\geqq 2$. Applying the second fundamental theorem to $G = \beta'(1 - e^{\beta-\gamma})$, we have

$$\begin{aligned} (1 + o(1)) T(r, G) &\leqq \bar{N}(r, 0, G) + \bar{N}(r, \infty, G) + \bar{N}(r, 0, \beta' e^{\beta-\gamma}) \\ &= \bar{N}(r, 0, G) + o(T(r, e^{\beta-\gamma})) \\ &= \bar{N}(r, 0, G) + o(T(r, G)) \qquad (r \notin E,\ r \to \infty) , \end{aligned}$$

which implies

$$T(r, G)=(1+o(1))N(r, 0, G)=(1+o(1))\bar{N}(r, 0, G) \qquad (r \notin E, r\to\infty).$$

Hence

$$\lim_{\substack{r\to\infty\\ r\notin E}} \frac{N_1(r, 0, G)}{N(r, 0, F)}=\lim_{\substack{r\to\infty\\ r\notin E}} \frac{N_1(r, 0, G)}{T(r, e^{\beta-\gamma})}=\lim_{\substack{r\to\infty\\ r\notin E}} \frac{N_1(r, 0, G)}{T(r, G)}=0. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\bar{N}(r, 0, F)\geqq\frac{1}{2}\{N(r, 0, F)-N_1(r, 0, G)\}=(1/2-o(1))T(r, e^{\beta-\gamma}) \qquad (r\notin E, r\to\infty). \tag{3.14}$$

Further, we claim that

$$\{z : F(z)=0\}=\{w_n\}\cup\{z_n\}. \tag{3.15}$$

By (3.6) and (3.11) $F(w_n)=0$. By (3.9) and (3.11) $F(z_n)=0$. Hence $\{w_n\}\cup\{z_n\}\subset\{z : F(z)=0\}$. Conversely, assume that $F(z)=0$. If $e^{\gamma(z)}\neq 1$, then $f(z)=a$, i.e. $z\in\{w_n\}$. If $e^{\gamma(z)}=1$, then $e^{\beta(z)}=1$, i.e. $z\in\{z_n\}$. Hence $\{z : F(z)=0\}\subset\{w_n\}\cup\{z_n\}$.

Now, by (3.8) and (3.10)

$$N(r, \{w_n\})+N(r, \{z_n\})\leqq 2T(r, \gamma)=o(T(r, e^{\beta-\gamma})) \qquad (r\notin E, r\to\infty). \tag{3.16}$$

On the other hand, by (3.15) and (3.14)

$$N(r, \{w_n\})+N(r, \{z_n\})=\bar{N}(r, 0, F)\geqq(1/2-o(1))T(r, e^{\beta-\gamma}) \qquad (r\notin E, r\to\infty),$$

which contradicts (3.16). This implies that if $f(w)=a$, then $e^{\gamma(w)}=1$. Then by (3.11) $e^{\beta(w)}=1$, hence by (3.5) $g(w)=a$.

Now, we show that $f=a$ has at least one root. If not, by (3.11) $F(w)=0$ implies $e^{\gamma(w)}=e^{\beta(w)}=1$, so that $F'(w)=\beta'(w)(e^{\gamma(w)}-e^{\beta(w)})=0$. Hence all the zeros of $F(z)$ has multiplicities$\geqq 2$. Thus by (3.11) and (3.14)

$$N(r, 0, \gamma')\geqq N_1(r, 1, e^{\gamma})\geqq N_1(r, 0, F)\geqq\bar{N}(r, 0, F)\geqq(1/2-o(1))T(r, e^{\beta-\gamma}) \qquad (r\notin E, r\to\infty).$$

This is impossible.

It the same way, we conclude that $g=a$ has at least one root, and if $g=a$, then $e^{-\gamma(w)}=1$, so that by (3.5) $e^{-\beta(w)}=1$, $f(w)=a$. Therefore $E(a, \infty, f)=E(a, \infty, g)\neq\emptyset$. In this case, by a result of Gundersen [1, Theorem 1],

$$g=S(f),$$

where S is a linear transformation which fixes a, a_1 and permutes a_2, a_3, and the cross ratio $(a_2, a_3, a, a_1)=-1$, where $\{a_1, a_2, a_3\}=\{0, 1, \infty\}$. From this we obtain one of the following three relations:

$$g\equiv 1-f \qquad (a=1/2, a_1=\infty),$$

$$g \equiv f^{-1} \qquad (a=-1,\ a_1=1),$$

or

$$g \equiv f/(f-1) \qquad (a=2,\ a_1=0).$$

(E) Suppose that β, $\alpha-\beta$, α, $\beta-a\gamma \not\equiv$ constant, where $\gamma \equiv \beta-\alpha$. Consider the function $F(z)$ (defined by (3.11)) and its logarithmic derivative $H(z)$:

$$H(z)=\frac{F'(z)}{F(z)}. \tag{3.17}$$

Then

$$T(r, H)=o(T(r, F))+\overline{N}(r, 0, F) \qquad (r \notin E,\ r \to \infty). \tag{3.18}$$

By (3.11) $F(w)=0$ implies (i) $f(w)=a$, $e^{\gamma(w)} \neq 1$ or (ii) $e^{\gamma(w)}=e^{\beta(w)}=1$. First, consider the case (i). In this case, $g(w) \neq a$, so that w is a zero of $F(z)$ with multiplicity $\geqq k+1 \geqq 3$. Then w is a zero of $G(z) \equiv a\gamma'-\beta' e^{\beta-\gamma}$ with multiplicity $\geqq k \geqq 2$. Hence, by the second fundamental theorem

$$N(r, \{w\}) \leqq N_1(r, 0, G)=o(T(r, e^{\beta})+T(r, e^{\gamma})) \qquad (r \notin E,\ r \to \infty). \tag{3.19}$$

Next, consider the case (ii). In this case, $f(w)=g(w)$. In particular we note that $e^{\gamma(w)}=e^{\beta(w)}=1$ and $f(w)=g(w)=0, 1, \infty, a$ imply $\beta'(w)=0$, $\alpha'(w)=0$, $\gamma'(w)=0$, $\beta'(w)-a\gamma'(w)=0$, respectively. Hence by (3.18), (3.19) and (3.11)

$$\begin{aligned} T(r, H)=&o(T(r, e^{\beta})+T(r, e^{\gamma}))+\overline{N}(r, 0, \beta'-a\gamma')+\overline{N}(r, 0, \beta') \\ &+\overline{N}(r, 0, \alpha')+\overline{N}(r, 0, \gamma')+N_2(r, 0, f-g), \end{aligned} \tag{3.20}$$

where N_2 counts only those points of N where $f(z)=g(z) \neq 0, 1, \infty, a$.

Here we estimate $N_2(r, 0, f-g)$. By the second fundamental theorem

$$2T(r, f) \leqq \overline{N}(r, 0, f)+\overline{N}(r, 1, f)+\overline{N}(r, \infty, f)+\overline{N}(r, a, f)+o(T(r, f)) \qquad (r \notin E,\ r \to \infty), \tag{3.21}$$

and similarly for g. Let $N(r, a; f, g)$ denote the counting function of the number of common roots of $f=a$ and $g=a$. Then by (3.21) and (3.19)

$$\begin{aligned} &N_2(r, 0, f-g)+\overline{N}(r, 0, f)+\overline{N}(r, 1, f)+\overline{N}(r, \infty, f)+N(r, a; f, g) \\ &\leqq N(r, 0, f-g) \leqq T(r, f-g) \leqq T(r, f)+T(r, g) \leqq \overline{N}(r, 0, f) \\ &\quad +\overline{N}(r, 1, f)+\overline{N}(r, \infty, f)+\overline{N}(r, a; f, g)+o(T(r, e^{\beta})+T(r, e^{\gamma})) \\ &\quad +o(T(r, f)+T(r, g)) \qquad (r \notin E,\ r \to \infty), \quad \text{i.e.} \end{aligned}$$

$$N_2(r, 0, f-g)=o(T(r, e^{\beta})+T(r, e^{\gamma})) \qquad (r \notin E,\ r \to \infty). \tag{3.22}$$

Substituting (3.22) into (3.20), we have

$$T(r, H)=o(T(r, e^{\beta})+T(r, e^{\gamma})) \qquad (r \notin E,\ r \to \infty). \tag{3.23}$$

Now, by (3.11) and (3.17)

$$(\beta'-H)e^{\beta}+a(H-\gamma')e^{\gamma}=(a-1)H. \tag{3.24}$$

Case 1. Assume that $\beta'\equiv H$. In this case $F(z)=De^{\beta}$, where D is a non-zero constant. Hence by (3.11)

$$(D+1)e^{\beta}-ae^{\gamma}=1-a\neq 0.$$

Using Lemma 1, we have $D+1=0$. Then $e^{\gamma}\equiv(a-1)/a$, a contradiction.

Case 2. Assume that $H\equiv\gamma'$. In this case $F(z)=De^{\gamma}$, where D is a non-zero constant. Hence by (3.9)

$$e^{\beta}+(D-a)e^{\gamma}=1-a\neq 0.$$

Using Lemma 1, we have $D-a=0$. Then $e^{\beta}\equiv 1-a$, a contradiction.

Case 3. Assume that $\beta'-H\not\equiv 0$ and $H-\gamma'\not\equiv 0$. In this case, we use Lemma 3. Noting (3.23), we have from (3.24)

$$C_1(\beta'-H)e^{\beta}+C_2(H-\gamma')e^{\gamma}\equiv 0, \tag{3.25}$$

where C_1, C_2 are non-zero constants. Hence

$$e^{\beta}=\frac{C_2}{C_2-aC_1}\frac{(a-1)H}{\beta'-H}, \qquad e^{\gamma}=\frac{C_1}{aC_1-C_2}\frac{(a-1)H}{H-\gamma'}.$$

Therefore by (3.23)

$$T(r, e^{\beta})+T(r, e^{\gamma})\leqq 4T(r, H)+T(r, \beta')+T(r, \gamma')+O(1)=o(T(r, e^{\beta})+T(r, e^{\gamma})) \quad (r\notin E, r\to\infty),$$

a contradiction.

This completes the proof of Theorem 3.

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