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ROTATIONALLY INVARIANT CYLINDRICAL MEASURES I

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§0. Introduction.

Since Gross [4, 1962] introduced the concept of measurable norm, it has been extensively studied by many researchers (see for example [3, 4, 5, 6, 7]).

In a real separable Hilbert space H there is a finitely additive cylindrical measure γ, say the canonical Gaussian cylindrical measure, which is analogous to the normal distribution in the finite dimensional case. Gross [5] showed that if H is completed with respect to any of his measurable semi-norms, as defined in [4], then γ gives rise to a countably additive Borel measure on the Banach space obtained from H by means of the semi-norm. Dudley [2] showed that if the polar of the closed unit semi-ball is a compact GC-set, then the semi-norm is measurable in Gross' sense. Furthermore, using Dudley's result just above mentioned, Dudley-Feldman-Le Cam [3] proved the converse of Gross' result.

Here we shall generalize the above result for the rotationally invariant cylindrical measures. There are some inequalities for Gaussian cylindrical measures, known from Gross [4], which play an important role in the present circle of ideas. In this paper we shall begin to prove the similar inequalities concerning rotationally invariant cylindrical measures instead of Gaussian cylindrical measures.

On the other hand, Dudley-Feldman-Le Cam [3] introduced another measurability for semi-norms. They denoted Gross' definition by "*measurable by projections*" and the latter by "*measurable*". We shall use their expression. Badrikian-Chevet [1] have offered the problem whether these two concepts of measurability coincide exactly with each other. Our result will answer partially this problem.

Finally we add that this report contains [9] and improves the main result.

§1. Cylindrical measures.

Let E be a real separable Banach space, E^* its topological dual and $\mathscr{B}(E)$ the Borel σ-algebra of E. We use $(\cdot, \cdot)$ to denote the natural pairing between E^* and E.

DEFINITION 1. Let $\{\xi_1, \cdots, \xi_n\}$ be a finite system of elements of E^*. Then by Ξ we denote the operator $: x\in E\mapsto((\xi_1, x), \cdots, (\xi_n, x))\in R^n$. A set $Z\subset E$ is

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said to be a *cylindrical set* if there are $\xi_1, \cdots, \xi_n \in E^*$ and $B \in \mathcal{B}(R^n)$ such that $Z = \Xi^{-1}(B)$. Let C_E denote the collection of all cylindrical sets of E.

DEFINITION 2. A map μ from the algebra of all cylindrical sets into $[0, 1]$ is called a *cylindrical measure* if it satisfies the two following conditions:

(1) $\mu(E)=1$;

(2) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed finite system in E^*. Then each such restriction is countably additive.

Let H be a real separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$.

DEFINITION 3. The *Gaussian cylindrical measure* γ^t on H with parameter $t \in R$ is the cylindrical measure defined as follows:

$$\gamma^t(C) = (\sqrt{2\pi}\, t)^n \int_D \exp(-|x|^2/2t^2)\,dx$$

for $C = \{x \in H : Px \in D\}$, where P is a finite dimensional orthogonal projection of H, $n = \dim PH$, $D \in \mathcal{B}(PH)$ and dx is the Lebesgue measure on PH. Especially, γ^1 is called the canonical Gaussian cylindrical measure and simply denoted by γ.

DEFINITION 4. Let μ be a cylindrical measure on H. If $\mu(C) = \mu(u(C))$ for $C \in C_H$ and unitary operator u of H, μ is called a *rotationally invariant cylindrical measure*.

Obviously γ^t is rotationally invariant, but it is not necessarily countably additive on (H, C_H).

Let (Ω, m) be a probability measure space, and $L^0(\Omega, m; R)$ be the linear space of all real valued random variables. Given any cylindrical measure μ on E, there exist a probability measure space (Ω, m) and a linear map $\Lambda: E^* \to L^0(\Omega, m; R)$ such that $\mu \circ \xi^{-1} = m \circ (\Lambda(\xi))^{-1}$ for every $\xi \in E^*$. We call Λ the random function associated with μ. Conversely, for any linear random function $\Lambda: E^* \to L^0(\Omega, m; R)$, there exists uniquely a cylindrical measure μ on E satisfying that $\mu \circ \xi^{-1} = m \circ (\Lambda(\xi))^{-1}$ for every $\xi \in E^*$.

Let $\mathcal{L}^0(\Omega, m; R)$ be the family of all m-measurable real valued functions, and ϕ be the canonical map of $\mathcal{L}^0(\Omega, m; R)$ into $L^0(\Omega, m; R)$. We call a subset $D \subset E^*$ a continuity set of Λ if there exists a map $\lambda: E^* \to \mathcal{L}^0(\Omega, m; R)$ such that $\Lambda = \phi \circ \lambda$ and a map $x \in D \mapsto \lambda(x, \omega) \in R$ is continuous for all $\omega \in \Omega \backslash N$, where N is an m-null set. A subset $A \subset L^0(\Omega, m; R)$ is said to be bounded in L^0 if there exists $g \in L^0(\Omega, m; R)$ such that $|f| \leqq g$ for all $f \in A$. A subset $D \subset E^*$ is called a bounded set of Λ if the set $\{\Lambda(x) : x \in D\}$ is bounded in L^0 (cf. [1]).

Given a cylindrical measure μ on H, there exists a linear random function Λ associated with μ. A subset $D \subset H$ is said to be a *μ-continuity set* (resp. a *μ-bounded set*) if D is a continuity set (resp. a bounded set) of Λ. Usually a γ-continuity set is called a *GC-set* and a γ-bounded set is called a *GB-set*.

§ 2. Two measurabilities of semi-norms and main results.

First of all, we shall present two definitions of measurability which are interpreted in the introduction.

Let H be a real separable Hilbert space, $FD(H)$ the family of all finite dimensional subspaces of H, μ a cylindrical measure on H and $p(\cdot)$ be a continuous semi-norm on H.

DEFINITION 5([4]). We say that $p(\cdot)$ is *μ-measurable by projections* if for every $\varepsilon>0$, there exists $G\in FD(H)$ such that $\mu(N_\varepsilon\cap F+F^\perp)\geqq 1-\varepsilon$ whenever $F\in FD(H)$ and $F\perp G$, where $N_\varepsilon=\{x\in H: p(x)<\varepsilon\}$ and $F^\perp$ is the orthogonal complement of F.

DEFINITION 6([3]). A continuous semi-norm $p(\cdot)$ is said to be *μ-measurable* if for every $\varepsilon>0$, there exists $G\in FD(H)$ such that $\mu(P_F(N_\varepsilon)+F^\perp)\geqq 1-\varepsilon$ whenever $F\in FD(H)$ and $F\perp G$, where P_F is the orthogonal projection of H onto F.

If $p(\cdot)$ is μ-measurable by projections, then it is μ-measurable. However, the converse is still open. Dudley-Feldman-Le Cam showed in [3] that the above two measurabilities are equivalent with respect to γ.

THEOREM A. *Let $p(\cdot)$ be a continuous semi-norm on H, E be the Banach space obtained from H by means of $p(\cdot)$ and i be the canonical map of H into E. The following statements are equivalent*

(1) *$p(\cdot)$ is γ-measurable;*
(2) *$\gamma\circ i^{-1}$ is countably additive on (E, C_E);*
(3) *The polar of the closed unit semi-ball of p is a compact GC-set;*
(4) *$p(\cdot)$ is γ-measurable by projections.*

The conditions (1) and (2) in Theorem A are equivalent for every cylindrical measure as well as γ.

The purpose of this paper is a generalization of Theorem A. Here we shall present the main theorem.

THEOREM 1. *Let H be a real separable Hilbert space, μ be a rotationally invariant cylindrical measure on H not δ_0, and $p(\cdot)$ be a continuous semi-norm on H. E is the induced Banach space from $p(\cdot)$ and H, and i is the canonical map from H into E. The following statements are equivalent.*

(1) *$p(\cdot)$ is μ-measurable;*
(2) *$\mu\circ i^{-1}$ is countably additive on (E, C_E);*
(3) *The polar of the set $\{x\in H: p(x)\leqq 1\}$ is a compact μ-continuity set;*
(4) *$p(\cdot)$ is μ-measurable by projections.*

The essential part of this theorem is (3)$\Rightarrow$(4). Therefore, we had better prove the following theorem before Theorem 1.

Let C be a subset of H. We denote C° the polar of C and $p_{C^\circ}(x)=\inf\{\rho>0: x\in\rho C^\circ\}$ for $x\in H$.

THEOREM 2. *Let μ be a rotationally invariant cylindrical measure on H and C be a compact convex balanced subset of H. If C is a μ-continuity set, then the semi-norm p_{C° is μ-measurable by projections.*

Remark. It is easy to see that p_{C° is continuous.

We need several lemmas and propositions for the proof of the above two theorems. They will be shown in successive sections.

§3. Gross' inequalities and rotationally invariant cylindrical measures.

In [4], Gross showed several inequalities for the canonical Gaussian cylindrical measure. These facts are basic to some results of Dudley [2] as well as [4]. In this section we shall show the generalization of these facts for rotationally invariant cylindrical measures. It will be an important tool in the proof of Theorems 1 and 2.

We present the theorem concerning the characterization of rotationally invariant cylindrical measures (cf. [1] and [10]).

Let γ_n be the canonical Gaussian measure on R^n and m_n be the Lebesgue measure on R^n. We denote by $|\cdot|_n$ the usual norm on R^n. Let $\{\xi_1, \cdots, \xi_n\}$ be a finite system of H and Ξ be the operator $x\in H\mapsto(\langle\xi_1, x\rangle, \cdots, \langle\xi_n, x\rangle)\in R^n$. Put $\mu_{\xi_1\cdots\xi_n}(B)=\mu(\Xi^{-1}(B))$ for $B\in\mathcal{B}(R^n)$.

THEOREM B([10]). *Suppose that H is an infinite dimensional real separable Hilbert space. Let μ be a rotationally invariant cylindrical measure on H. Then there exists a Borel probability measure σ_μ on $[0, \infty)$ such that*

$$(*)\qquad \mu_{e_1\cdots e_n}(A)$$

$$=\int_{t>0}\gamma_n(A/t)d\sigma_\mu(t)+\sigma_\mu(\{0\})\delta_0(A)$$

$$=\int_A\left(\int_{t>0}(\sqrt{2\pi}\ t)^{-n}\exp(-|x|_n^2/2t^2)d\sigma_\mu(t)\right)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)$$

for every $A\in\mathcal{B}(R^n)$ and every finite orthonormal system $\{e_1, \cdots, e_n\}$ of H.

Now we start with the following lemma.

LEMMA 1. *Under the hypothesis of Theorem B, there exists a non-negative function $\Phi(r)$ defined on $[0, \infty)$ such that*

$$\mu_{e_1\cdots e_n}(A)=\int_0^\infty\Phi(r)m_n(A\cap S_r)dr+\sigma_\mu(\{0\})\delta_0(A),$$

where $S_r=\{x\in R^n : |x|_n\leqq r\}$.

Proof. Define $\phi(t, r)$ and $\psi(r)$ by

$$\phi(t, r)=(\sqrt{2\pi}\, t)^{-n}\exp(-r^2/2t^2) \qquad \text{and} \quad \psi(r)=\int_{t>0}\phi(t, r)d\sigma_\mu(t)\,.$$

For every $a\in[0, \infty)$,

$$\begin{aligned}\psi(a)&=\int_{t>0}\phi(t, a)d\sigma_\mu(t)\\&=\int_{t>0}\left(\int_a^\infty -\frac{\partial\phi(t, r)}{\partial r}dr\right)d\sigma_\mu(t)\\&=\int_a^\infty\left(\int_{t>0}-\frac{\partial\phi(t, r)}{\partial r}d\sigma_\mu(t)\right)dr\,.\end{aligned}$$

Let $\Phi(r)=\int_{t>0}-\frac{\partial\phi(t, r)}{\partial r}d\sigma_\mu(t)$. Then by Theorem B

$$\begin{aligned}\mu_{e_1\cdots e_n}(A)&=\int_A\psi(|x|_n)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)\\&=\int_A\left(\int_0^\infty\Phi(r)1_{[|x|_n,\infty)}(r)dr\right)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)\\&=\int_0^\infty\left(\int_A\Phi(r)1_{[|x|_n,\infty)}(r)dm_n(x)\right)dr+\sigma_\mu(\{0\})\delta_0(A)\\&=\int_0^\infty\Phi(r)m_n(A\cap S_r)dr+\sigma_\mu(\{0\})\delta_0(A)\,.\end{aligned}$$

Thus we have the desirable result. □

Let us denote by $\|\cdot\|$ the operator norm.

LEMMA 2. *Suppose the hypothesis of Theorem B as ever. Let u be a linear symmetric invertible operator of* R^n *onto* R^n *and C be a closed convex balanced set in* R^n. *If* $\|u^{-1}\|\leqq 1$ *then* $\mu_{e_1\cdots e_n}(u(C))\geqq\mu_{e_1\cdots e_n}(C)$.

Proof. Clearly, when we want to show the above inequality we can neglect the second part of the representation of $\mu_{e_1\cdots e_n}$ which has been obtained in the previous lemma. In [4], Gross proved that $m_n(u(C)\cap S_r)\geqq m_n(C\cap S_r)$ for all $r>0$. Therefore, by Lemma 1 we have $\mu_{e_1\cdots e_n}(u(C))\geqq\mu_{e_1\cdots e_n}(C)$. □

Successive two lemmas will be deduced from Lemma 2, then their proofs will have the same processes as the case of the canonical Gaussian cylindrical measure.

LEMMA 3. *Let E be an n-dimensional Hilbert space, where n is a natural* *number, and* σ *be a Borel probability measure on* $[0, \infty)$. *Let* μ *be a probability measure on* E *defined by*

$$\mu(A)=\int_A\left(\int_{t>0}(\sqrt{2\pi}\, t)^{-n} \exp(-|x|_E^2/2t^2)d\sigma(t)\right)dm(x)+\sigma(\{0\})\delta_0(A)$$

for every $A\in\mathcal{B}(E)$, *where* $|\cdot|_E$ *denotes the norm of* E *and* m *the Lebesgue measure on* E. *Let* E_1 *be a linear subspace of* E *and* C *be a closed convex balanced subset of* E. *Then* $\mu(C)\leqq\mu(C\cap E_1+E_1^\perp)$.

Remark. It is clear that μ is rotationally invariant.

Proof. We can assume that $E_1\subsetneqq E$. Let P be the orthogonal projection of E onto E_1 and I be the identity operator of E. Define $P^\perp=I-P$ and $T_k=kP^\perp+P$ for every integer $k>1$. Then T_k is the linear symmetric invertible operator of E. Clearly we have $\|T_k^{-1}\|<1$. Let $\{e_i\}_{i=1,2,\cdots,n}$ be an orthonormal basis of E. Then the mapping

$$x=\sum_{i=1}^{n} x_i e_i \longmapsto (x_1, \cdots, x_n)$$

defines an isomorphism from E onto R^n. Thus Lemma 2 says that $\mu(T_k(C))\geqq\mu(C)$ for every integer $k>1$. By Fatou's lemma, we have $\mu(\limsup_{k\to\infty} T_k(C))\geqq\mu(C)$. On the other hand, $P^{-1}(C)=P^{-1}(C\cap E_1)=\{C\cap E_1+E_1^\perp\}$. Then in order to complete the proof we only have to show that $P^{-1}(C)\supset\limsup_{k\to\infty} T_k(C)$, i.e., $E\backslash P^{-1}(C)\subset\liminf_{k\to\infty} T_k(E\backslash C)$. To see this, take any $x\in E\backslash P^{-1}(C)$. Then there exists a number $\varepsilon>0$ such that $S_\varepsilon+Px\subset E\backslash C$ where $S_\varepsilon=\{x\in E: |x|_E<\varepsilon\}$. Choose an integer k such that $k>|P^\perp x|_E/\varepsilon$. So $T_k^{-1}x=Px+(1/k)P^\perp x\in Px+S_\varepsilon\subset E\backslash C$. Hence we have $x\in\liminf_{k\to\infty} T_k(E\backslash C)$ and $E\backslash P^{-1}(C)\subset\liminf_{k\to\infty} T_k(E\backslash C)$. □

Remark. Given any Borel probability measure σ on $[0, \infty)$, there exists a rotationally invariant cylindrical measure μ on a Hilbert space H satisfying the relation (*) in Theorem B. At this time we call μ the rotationally invariant cylindrical measure induced by σ. Especially if H is infinite dimensional, the above correspondence between σ and μ is a bijection.

LEMMA 4. *Let* H *be a real separable Hilbert space,* σ *be a Borel probability measure on* $[0, \infty)$ *and* μ *be the rotationally invariant cylindrical measure induced by* σ *on* H. *Let* u *be a continuous linear operator of* H, C_0 *be a closed convex balanced set of some finite dimensional subspace of* H *and* C *be the cylindrical set with the base* C_0.

If $\|u\|\leqq1$, *then* $\mu(u^{-1}(C))\geqq\mu(C)$.

Proof. (I) Suppose that H is a finite dimensional Hilbert space and u is a bijection. We can decompose u as $u=I\circ u_1$, where I is a linear isometric operator on H and u_1 is a linear symmetric invertible operator of H such that $\|u\|=\|u_1\|$. Since $\|u_1\|\leqq 1$, we have

$$\mu(u^{-1}(C))=\mu(u_1^{-1}(I^{-1}(C)))\geqq\mu(I^{-1}(C))=(\mu\circ I^{-1})(C)=\mu(C).$$

Hence the proof is complete in this case.

(II) Consider next the case that H is same as (I) and u is a general form. Let $K_1=(u^{-1}(0))^{\perp}$ and $K_2=u(H)$. Let P_{K_1} be the orthogonal projection of H onto K_1 and v be the linear bijection from K_1 onto K_2. It is easy to see that $v\circ P_{K_1}(x)=u(x)$ for all $x\in H$ and that $\|v\|\leqq 1$. Let μ_{K_i} be the rotationally invariant cylindrical measure on the Hilbert space K_i induced by σ, for $i=1, 2$. We have

$$\mu(u^{-1}(C))=\mu(P_{K_1}^{-1}(v^{-1}(C\cap K_2)))=(\mu\circ P_{K_1}^{-1})(v^{-1}(C\cap K_2))$$
$$=\mu_{K_1}(v^{-1}(C\cap K_2)).$$

Since $C\cap K_2$ is a closed convex balanced subset of K_2, we can apply the consequence of (I). Then

$$\mu_{K_1}(v^{-1}(C\cap K_2))\geqq\mu_{K_2}(C\cap K_2).$$

Therefore

$$\mu(u^{-1}(C))\geqq\mu_{K_2}(C\cap K_2)=\mu(C\cap K_2+K_2^{\perp}).$$

Hence Lemma 3 says the desired conclusion.

(III) Now consider the case that H is infinite dimensional. C is a cylindrical set, then there exists a finite dimensional subspace N of H such that $C_0\subset N$ and $P_N^{-1}(C_0)=C$, where P_N is the orthogonal projection of H onto N. It is clear that

$$\|P_N\circ u\|\leqq 1 \quad \text{and} \quad \mu(u^{-1}(C))=\mu(u^{-1}(P_N^{-1}(C_0)))=\mu((P_N\circ u)^{-1}(C_0)).$$

Apply (II), then we have $\mu((P_N\circ u)^{-1}(C_0))\geqq\mu_N(C_0)$, where μ_N is the rotationally invariant cylindrical measure on N induced by σ. Since $\mu_N(C_0)=\mu(C)$, we have $\mu(u^{-1}(C))\geqq\mu(C)$. □

§4. The equivalency of two measurabilities with respect to rotationally invariant cylindrical measures.

In this section, we shall complete the proof of Theorems 1 and 2. Notice that we identify H^* and H. Let us begin with the following lemma.

Let μ be a cylindrical measure on H, $\Lambda: H\to L^0(\Omega, m; R)$ be the linear random function associated with μ and $\Lambda[A]=\sup_{x^*\in A}|\Lambda(x^*)|$ for every subset A of H. Let us denote by $|||S|||$ the cardinal number of S.

LEMMA 5. *For every* $t\geqq 0$,

$$m(\Lambda[A]\leqq t)=\inf_{S}\{\mu(tS^{\circ}): S\subset A,\ |||S|||<\infty\}\ .$$

Proof. Let S be a finite subset of A. Since Λ is the associated random function with μ, we have $\mu(tS^{\circ})=m(\Lambda[S]\leqq t)$ for every $t\geqq 0$. Hence we have the consequence. □

LEMMA 6. (I) *Let u^* be a continuous linear operator of H into H such that $\|u^*\|\leqq 1$. For every $t\geqq 0$ and every subset $A\subset H$, we have $m(\Lambda[A]\leqq t)\leqq m(\Lambda[u^*(A)]\leqq t)$.*

(II) *Let Q_1, Q_2 be two orthogonal projections of H such that $Q_1(H)\subset Q_2(H)$. For every $\varepsilon\geqq 0$ and every subset $B\subset H$, we have $m(\Lambda[Q_1(B)]>\varepsilon)\leqq m(\Lambda[Q_2(B)]>\varepsilon)$.*

Proof. (I) By virtue of Lemma 5, we can (and do) suppose that A is finite. Let u be the adjoint of u^*. Lemma 4 says that $\mu(tA^{\circ})\leqq\mu(u^{-1}(tA^{\circ}))$. Therefore,

$$m(\Lambda[A]\leqq t)=\mu(tA^{\circ})\leqq\mu(u^{-1}(tA^{\circ}))=\mu(t(u^*(A))^{\circ})=m(\Lambda[u^*(A)]\leqq t)\ .$$

(II) In order to apply (I) we take $u^*=Q_1$, $A=Q_2(B)$ and $t=\varepsilon$. Then

$$\begin{aligned} &m(\Lambda[Q_2(B)]\leqq\varepsilon)\\ &\qquad\leqq m(\Lambda[Q_1\circ Q_2(B)]\leqq\varepsilon)\\ &\qquad=m(\Lambda[Q_1(B)]\leqq\varepsilon)\ . \end{aligned}$$

Thus $m(\Lambda[Q_1(B)]>\varepsilon)\leqq m(\Lambda[Q_2(B)]>\varepsilon)$. □

LEMMA 7. *Let I be a directed set, $\{\pi_\iota\}_{\iota\in I}$ be a directed family of orthogonal projections of H such that $\{\pi_\iota(x^*)\}_{\iota\in I}$ converges to x^* for each $x^*\in H$, and B be a closed convex balanced subset of H. Then $m(\Lambda[B]\leqq t)=\lim_{\iota} m(\Lambda[\pi_\iota(B)]\leqq t)$ for every $t>0$.*

Proof. It follows from Lemma 6 that

$$m(\Lambda[B]\leqq t)\leqq\liminf_{\iota}\, m(\Lambda[\pi_\iota(B)]\leqq t)\ .$$

Then it is sufficient to show that

$$\limsup_{\iota}\, m(\Lambda[\pi_\iota(B)]\leqq t)\leqq m(\Lambda[B]\leqq t)\ .$$

Now we notice that every rotationally invariant cylindrical measure is of type 0, i.e., the associated linear random function Λ is continuous from H into L^0 equipped with the topology of convergence in probability. Thus $\{\Lambda[\pi_\iota(x^*)]\}_{\iota\in I}$ converges to $\Lambda(x^*)$ in L^0 for each $x^*\in H$. It is also obvious that $\{\Lambda[\pi_\iota(S)]\}_{\iota\in I}$ converges to $\Lambda[S]$ in L^0 for each finite subset S of B. Then

$$\limsup_{\iota}\, m(\Lambda[\pi_\iota(S)]\leqq t)\leqq m(\Lambda[S]\leqq t)\ .$$

Since $m(\Lambda[\pi_\iota(B)]\leqq t)\leqq m(\Lambda[\pi_\iota(S)]\leqq t)$, $\limsup_\iota m(\Lambda[\pi_\iota(B)]\leqq t)\leqq m(\Lambda[S]\leqq t)$ for every S. Using Lemma 5 we have $\limsup_\iota m(\Lambda[\pi_\iota(B)]\leqq t)\leqq m(\Lambda[B]\leqq t)$. $\square$

LEMMA 8. *Let B be a closed convex balanced subset of H, then the following statements are equivalent.*

(1) *Given any $\varepsilon>0$, there exists a subspace $K_\varepsilon\in FD(H)$ such that $m(\Lambda[\pi_L(B)]>\varepsilon)<\varepsilon$ whenever $L\in FD(H)$ and $L\perp K_\varepsilon$, where π_L is the orthogonal projection of H onto L.*

(2) *Given any $\varepsilon>0$, there exists a subspace $K_\varepsilon\in FD(H)$ such that $m(\Lambda[\pi_{K_\varepsilon^\perp}(B)]>\varepsilon)<\varepsilon$.*

Proof. It is clear that $\pi_{K_\varepsilon^\perp}(H)\supset\pi_L(H)$. So it follows from Lemma 6 that

$$m(\Lambda[\pi_{K_\varepsilon^\perp}(B)]>\varepsilon)\geqq m(\Lambda[\pi_L(B)]>\varepsilon)\,.$$

Thus we have only to show that (1) implies (2). Let $\{M_\varepsilon^n\}_{n=1,2,\dots}$ be a chain of increasing finite dimensional subspaces of H such that $\bigcup_n M_\varepsilon^n$ is dense in H and $\bigcap_n M_\varepsilon^n\supset K_\varepsilon$. Using Lemma 7, we have

$$\begin{aligned} &m(\Lambda[\pi_{K_\varepsilon^\perp}(B)]\leqq\varepsilon)\\ &\quad=\lim_n m(\Lambda[\pi_{M_\varepsilon^n}\circ\pi_{K_\varepsilon^\perp}(B)]\leqq\varepsilon)\\ &\quad=\lim_n m(\Lambda[\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}(B)]\leqq\varepsilon)\,.\end{aligned}$$

By (1), $m(\Lambda[\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}(B)]>\varepsilon)<\varepsilon$ and this implies that $m(\Lambda[\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}(B)]\leqq\varepsilon)>1-\varepsilon$. Therefore we have $m(\Lambda[\pi_{K_\varepsilon^\perp}(B)]\leqq\varepsilon)\geqq 1-\varepsilon$ and so $m(\Lambda[\pi_{K_\varepsilon^\perp}(B)]>\varepsilon)\leqq\varepsilon$. Thus we have (2). $\square$

LEMMA 9. *Let ν be a cylindrical measure (not only rotationally invariant) on H, $\{(\Omega', m'); \Lambda'\}$ be the pair of a probability measure space and a random function associated with ν and B be a closed convex balanced bounded subset of H. Then the following two conditions are equivalent.*

(1) *The semi-norm p_{B° is ν-measurable by projections.*

(2) *Given any $\varepsilon>0$, there exists a subspace $K_\varepsilon\in FD(H)$ such that $m'(\Lambda'[\pi_L(B)]>\varepsilon)<\varepsilon$ whenever $L\in FD(H)$ and $L\perp K_\varepsilon$.*

Proof. It is easy to see that $m'(\Lambda'[\pi_L(B)]\leqq\varepsilon)=\nu(\pi_L^{-1}(\varepsilon B^\circ))=\nu(\varepsilon B^\circ\cap L+L^\perp)$. Also we have $\{x\in H\colon p_{B^\circ}(x)<\varepsilon\}\subset\varepsilon B^\circ=\{x\in H\colon p_{B^\circ}(x)\leqq\varepsilon\}$. Thus the desired conclusion follows immediately. $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let D be a countable dense subset of C. The assumption that C is a μ-continuity set implies the existence of the version $\lambda(x, \omega)$ of Λ (i.e., $\Lambda=\phi\circ\lambda$, where ϕ is the canonical map of $\mathcal{L}^0$ into L^0) which is continuous on C for almost all $\omega\in\Omega$. It follows from the compactness of C that

$$m\Big(\bigcup_{k=1}^{\infty}\bigcap_{\substack{(x,y)\in D\times D\\ |x-y|\leqq 1/k}}\{\omega:|\lambda(x-y,\omega)|\leqq\varepsilon\}\Big)=1$$

for all $\varepsilon>0$. This implies that for any $\varepsilon>0$ there exists a positive number δ such that

$$m\Big(\Big\{\omega:\sup_{\substack{(x,y)\in D\times D\\ |x-y|<\delta}}|\lambda(x-y,\omega)|>\varepsilon\Big\}\Big)<\varepsilon\,.$$

Since D is relatively compact in H, we have a subspace $F\in FD(H)$ such that $\sup_{x\in D}(\inf_{y\in F\cap D}|x-y|)<\delta$. Let π be an orthogonal projection onto F, I be the identity operator of H and $\pi^{\perp}=I-\pi$. It follows from Lemma 6 that

$$m\Big(\Big\{\omega:\sup_{\substack{(x,y)\in D\times D\\ |x-y|<\delta}}|\lambda(\pi^{\perp}(x-y),\omega)|>\varepsilon\Big\}\Big)$$
$$\leqq m\Big(\Big\{\omega:\sup_{\substack{(x,y)\in D\times D\\ |x-y|<\delta}}|\lambda(x-y,\omega)|>\varepsilon\Big\}\Big)\,.$$

For each $x\in D$ we can take $y\in F\cap D$ such that $|x-y|<\delta$ and $\pi^{\perp}y=0$. Therefore,

$$m(\Lambda[\pi^{\perp}(C)]>\varepsilon)=m(\Lambda[\pi^{\perp}(D)]>\varepsilon)$$
$$\leqq m\Big(\Big\{\omega:\sup_{\substack{(x,y)\in D\times D\\ |x-y|<\delta}}|\lambda(\pi^{\perp}(x-y),\omega)|>\varepsilon\Big\}\Big)$$
$$\leqq m\Big(\Big\{\omega:\sup_{\substack{(x,y)\in D\times D\\ |x-y|<\delta}}|\lambda(x-y,\omega)|>\varepsilon\Big\}\Big)\,.$$

Thus we can say that for any $\varepsilon>0$ there exists a finite dimensional orthogonal projection π such that $m(\Lambda[\pi^{\perp}(C)]>\varepsilon)<\varepsilon$. Using Lemmas 8 and 9, we can complete the proof. □

We have the following proposition from the result of Badrikian (cf. [1]).

PROPOSITION 1([1]). *Let H be a real separable Hilbert space, μ be a cylindrical measure on H and $p(\cdot)$ be a continuous semi-norm on H. We denote by E the Banach space induced by H and $p(\cdot)$, and by i the canonical map from H into E. Assume that $\mu\circ i^{-1}$ is countably additive on $(E, \mathcal{C}_E)$. Then the set $\{x\in H: p(x)\leqq 1\}^{\circ}$ is the μ-continuity set and also the μ-bounded set.*

Let ν be a rotationally invariant cylindrical measure on an infinite dimensional space. Then we can write

$$\nu=\alpha\mu+(1-\alpha)\delta_0 \quad (0\leqq\alpha\leqq1),$$

where μ is a rotationally invariant cylindrical measure satisfying that $\sigma_\mu(\{0\})=0$.

PROPOSITION 2. *Let μ be as in the above and $\nu=\alpha\mu+(1-\alpha)\delta_0$ $(0<\alpha\leqq1)$. Let X be a topological linear space separating by its dual and u be a weakly continuous linear operator of H into X. Then the following statements are equivalent.*

(1) *$\mu\circ u^{-1}$ is extensible to a Radon measure on X.*

(2) *$\nu\circ u^{-1}$ is extensible to a Radon measure on X.*

The proof is easy, and so it is omitted.

Remark. (1) A finite Radon measure means a finite Borel measure with inner regularity.

(2) Let μ, X and u be as in Proposition 2. Then (1) of Proposition 2 (in consequence, also (2)) is equivalent to the condition that $\gamma\circ u^{-1}$ is extensible to a Radon measure on X (see [1]).

PROPOSITION 3. *Let μ be a rotationally invariant cylindrical measure on H not δ_0, $p(\cdot)$ be a continuous semi-norm on H, E be the Banach space obtained from H by means of $p(\cdot)$ and $\imath$ be the canonical map of H into E. Put $A=\{x\in H: p(x)\leqq1\}^\circ$. If A is a μ-bounded set, then it is compact.*

Proof. Since A is a μ-bounded set, $\mu\circ\imath^{-1}$ is extensible to a Radon measure for $\sigma(E^{**}, E^*)$ (see [1]). And also the converse is true. Then Proposition 2 and its remark imply that $\gamma\circ\imath^{-1}$ is also extensible to a Radon measure for $\sigma(E^{**}, E^*)$. Therefore, it follows that A is a γ-bounded set, i.e., GB-set. Dudley showed in [2] that every GB-set is relatively compact. Since A is closed, it yields the consequence. □

Our preparations for Theorem 1 have been completed.

Proof of Theorem 1. Put $C=\{x\in H: p(x)\leqq1\}^\circ$. The equivalency of (1) and (2) is well known. Also it is clear that (4) implies (1). (2)$\Rightarrow$(3) is given by Propositions 1 and 3. Therefore it remains only to prove that (3)$\Rightarrow$(4). However, C is convex balanced and $p_{C^\circ}=p$, and so we have it by Theorem 2. □

Using the remark of Proposition 2, we have the following corollary.

COROLLARY. *Let $RI(H)$ be the family of all rotationally invariant cylindrical measures on H and $p(\cdot)$ be a continuous semi-norm on H. If there exists a cylindrical measure μ in $RI(H)\backslash\{\delta_0\}$ such that $p(\cdot)$ is μ-measurable, then for every*

$\nu \in RI(H) \setminus \{\delta_0\}$, $p(\cdot)$ *is* ν-*measurable by projections.*

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