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A CLASS NUMBER FORMULA OF IWASAWA'S MODULES

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§ 0. Introduction.

Let p be an odd prime number which will be fixed throughout the following. Let k be a finite extension of $\boldsymbol{Q}$ and k_∞ be the cyclotomic $\boldsymbol{Z}_p$-extension $k\boldsymbol{Q}_\infty$ of k, where $\boldsymbol{Q}_\infty$ is the unique $\boldsymbol{Z}_p$-extension of $\boldsymbol{Q}$ (c.f. [6]). For any $n\geqq 0$, let k_n be the unique extension of k in k_∞ of degree p^n over k: $k=k_0\subset k_1\subset\cdots\subset k_\infty$, and let $\Gamma_n=\mathrm{Gal}(k_\infty/k_n)$. Let A_n be the p-Sylow subgroup of the ideal class group of k_n and D_n be the subgroup of A_n consisting of ideal classes containing ideals $\prod\mathfrak{P}^{m(\mathfrak{P})}$, where $\mathfrak{P}$ runs over all primes of k_n lying over p and $m(\mathfrak{P})\in\boldsymbol{Z}$. Let A_n' be the factor group A_n/D_n (c.f. [6]).

We assume that k is a CM field. Then k_∞ is also a CM field. Let j denote the complex conjugation of k_∞. For any $\boldsymbol{Z}[\{1, j\}]$-module M, let

$$M^-=\{a\in M\,|\,(1+j)a=0\}\,.$$

(0.1) DEFINITION. Let $A_\infty^-=\varinjlim A_n^-$ and $A_\infty'^-=\varinjlim A_n'^-$, with respect to the natural maps induced from inclusion maps $k_n\to k_m$ for $m\geqq n\geqq 0$.

In [3] Greenberg, and in [2] Ferrero and Greenberg have proved that, if k is abelian over $\boldsymbol{Q}$, then the order of $(A_\infty'^-)^{\Gamma_n}$ is finite for any $n\geqq 0$. We shall compute its order by using p-adic L-functions associated to k when the degree of k over $\boldsymbol{Q}$ is prime to p.

In the following, we assume that k is a finite imaginary abelian extension of $\boldsymbol{Q}$ whose degree is prime to p. Let G denote the Galois group $\mathrm{Gal}(k/\boldsymbol{Q})$ and $\hat{G}$ be its character group $\mathrm{Hom}(G, \bar{\boldsymbol{Q}}_p^\times)$, where $\bar{\boldsymbol{Q}}_p$ is a fixed algebraic closure of $\boldsymbol{Q}_p$, We also consider $\hat{G}$ as the set of primitive Dirichlet characters with values in $\bar{\boldsymbol{Q}}_p$ which are associated to the extension $k/\boldsymbol{Q}$ by class field theory. Let ω be the Teichmüller character module p. Take $\phi\in\hat{G}$ with $\phi\neq\omega$ and $\phi(j)=-1$. Let $L_p(s;\omega\phi^{-1})$ be the p-adic L-function attached to $\omega\phi^{-1}$. For $\kappa\in 1+p\boldsymbol{Z}_p$ with $\kappa\notin 1+p^2\boldsymbol{Z}_p$, using Iwasawa's construction of p-adic L-functions, we have the unique power series $f(T;\omega\phi^{-1})\in\Lambda_\phi$ such that

$$f(\kappa^{s-1};\omega\phi^{-1})=L_p(s;\omega\phi^{-1})\,,$$

where $\boldsymbol{Z}_p[\phi]=\boldsymbol{Z}_p[\{\text{all values of }\phi\}]$ and $\Lambda_\phi=\boldsymbol{Z}_p[\phi][[T]]$. We note that

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$$f(0\,;\omega\phi^{-1})=L_p(0\,;\omega\phi^{-1})=(1-\phi^{-1}(p))L(0\,;\phi^{-1})\,.$$

(0.2) DEFINITION. We define $\hat{f}(T\,;\omega\phi^{-1})\in\Lambda_\phi$ by

$$\hat{f}(T\,;\omega\phi^{-1})=\begin{cases} f(T\,;\omega\phi^{-1})/T & \text{if} \quad \phi(p)=1\,, \\ f(T\,;\omega\phi^{-1}) & \text{otherwise}. \end{cases}$$

Ferrero and Greenberg [2] have proved that $\hat{f}(0\,;\omega\phi^{-1})\neq 0$. Then we see that

$$\hat{f}(\zeta-1\,;\omega\phi^{-1})\neq 0 \quad \text{for all } \zeta \text{ with } \zeta^{p^n}=1 \text{ and } n\geqq 0\,.$$

Hence the order of

$$\Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_n)$$

is finite, where $\omega_n=(1+T)^{p^n}-1$.

For a finite set A, let ${}^\#A$ denote the cardinality of A. A representation of a group G will be called ***Q***$_p$*-irreducible* if it is defined over $\boldsymbol{Q}_p$ and irreducible over $\boldsymbol{Q}_p$. A character of G will be called ***Q***$_p$*-irreducible* if it is the character of some $\boldsymbol{Q}_p$-irreducible representation of G.

(0.3) THEOREM. *Assume that*

(1) *$k/\boldsymbol{Q}$ is a finite abelian extension,*

(2) *k is imaginary, and*

(3) *the degree $[k:\boldsymbol{Q}]$ is prime to p.*

Then we have

$${}^\#(A'^{-}_\infty)^{\Gamma_n}={}^\#\bigoplus_{\Phi}\Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_n) \qquad \textit{for all} \quad n\geqq 0\,,$$

where Φ runs over all $\boldsymbol{Q}_p$-irreducible characters of $G=\mathrm{Gal}(k/\boldsymbol{Q})$ such that $\Phi\neq\omega$, $\Phi(j)\neq\boldsymbol{Q}(1)$ and ϕ is an absolutely irreducible component of Φ.

For $a,\,b\in\boldsymbol{Q}_p^\times$, we write $a\underset{p}{\sim}b$ if $\mathrm{ord}_p(a)=\mathrm{ord}_p(b)$. Note that

$${}^\#\Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_n)\underset{p}{\sim}\prod_{\psi}\prod_{\zeta^{p^n}=1}\hat{f}(\zeta-1\,;\omega\psi^{-1})\,, \tag{0.4}$$

where ψ runs over all "conjugates" of ϕ over $\boldsymbol{Q}_p$.

(0.5) *Remark.* When no prime of the maximal real subfield k^+ of k lying over p splits in k, our formula in Theorem (0.3) is a direct consequence of the analytic class number formula for k (c.f. [1]). But, if there exist some primes of k^+ lying over p which split in k, then $(A^-_\infty)^{\Gamma_n}$ is an infinite group and $f(0\,;\omega\phi^{-1})$ vanishes for some ϕ.

(0.6) *Remark.* To prove Theorem (0.3), we use essentially Gauss sums, Gross-Koblitz formula concerning a relation between Gauss sums and special values of Morita's p-adic Γ-function in [4], and Ferrero-Greenberg formula concerning

$L'_p(0;\chi)$ in [2].

(0.7) *Remark.* The assumption (3) is not essential. In fact, to prove Theorem (2.1), we need not assume that the degree of k over $\boldsymbol{Q}$ is prime to p.

(0.8) *Remark.* In [2], Ferrero and Greenberg proved Theorem (0.3) when k is imaginary quadratic and $n=0$.

We define fundamental Iwasawa's modules in § 1. In § 2, we reduce Theorem (0.3) to Theorem (2.1). In § 3, we introduce an essential exact sequences of Iwasawa's modules following [3]. And in § 4, p-adic reguraters are defined. In § 5, following [4], we define Gauss sums, which we use to combine orders of two modules in Theorem (2.1). And the group of Gauss sums is introduced. In § 6, we prove Theorem (2.1). In § 7, some examples are given.

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Notations.

As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. For a prime number p, $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. Let $\bar{\boldsymbol{Q}}$ (resp. $\bar{\boldsymbol{Q}}_p$) be an fixed algebraic closures of $\boldsymbol{Q}$ (resp. $\boldsymbol{Q}_p$). We also fix embeddings $\bar{\boldsymbol{Q}}\hookrightarrow\boldsymbol{C}$ and $\rho:\bar{\boldsymbol{Q}}\hookrightarrow\bar{\boldsymbol{Q}}_p$.

§ 1. Iwasawa's modules.

Let $k/\boldsymbol{Q}$ be as in Theorem (0.3). Since the degree of k over $\boldsymbol{Q}$ is prime to p, all primes of k lying over p are totally ramified in k. Since $(A'^{-}_{\infty})^{\Gamma_n}$ is finite for $n\geqq 0$, and the natural maps $A'^{-}_n\to A'^{-}_m$ are injective for $m\geqq n\geqq 0$, we have

(1.1) LEMMA. *For any integer $n\geqq 0$, there exists an integer m_0 such that $(A'^{-}_{\infty})^{\Gamma_n}=(A'^{-}_m)^{\Gamma_n}$ for all $m\geqq m_0$.*

Let Z be the decomposition group of p for $k/\boldsymbol{Q}$. Recall that $G=\mathrm{Gal}(k/\boldsymbol{Q})$ and $\hat{G}=\mathrm{Hom}(G,\bar{\boldsymbol{Q}}_p^{\times})$. For any $\phi\in\hat{G}$, $\mathrm{Tr}\,\phi$ denotes the $\boldsymbol{Q}_p$-irreducible character of G which contains ϕ as an absolutely irreducible component, and $e(\mathrm{Tr}\,\phi)$ denotes the orthogonal idempotent in $\boldsymbol{Z}_p[G]$ associated to $\mathrm{Tr}\,\phi$. We consider A_n, D_n, and A'_n as $\boldsymbol{Z}_p[G]$-modules in the natural way. Since all primes of k lying over p are totally ramified in k_n, we have

(1.2.) LEMMA. *If the restriction $\phi|Z$ is not trivial, then $e(\mathrm{Tr}\,\phi)D_n=0$ for all $n\geqq 0$.*

Following [3], we have

(1.3) LEMMA. *For* $m \geqq n \geqq 0$, *we have*

$$D_m^-/D_n^- \cong ((\boldsymbol{Z}_p/p^{m-n}\boldsymbol{Z}_p)[G/Z])^- \qquad \text{as } \boldsymbol{Z}_p[G]\text{-modules.}$$

(1.4) LEMMA. *For* $m \geqq n \geqq 0$,

$$0 \longrightarrow D_n^- \longrightarrow A_n^- \xrightarrow{\alpha} (A_m^-)^{\Gamma_n}/D_m^- \longrightarrow 0$$

is an exact sequence of $\boldsymbol{Z}_p[G]$*-modules, where* α *is induced by the canonical inclusion* $A_n^- \to A_m^-$.

For $m \geqq n \geqq 0$, put

$$M_n^{(m)} = \{a \in A_m^- \mid (s-1)a \in D_m^-\},$$

where s is a generator of $\mathrm{Gal}(k_m/k_n)$ (c.f. [3]). Define a homomorphism $\beta : M_n^{(m)} \to D_m^-$ by $\beta(a) = (s-1)a$. Then $D_m^- \subset \mathrm{Ker}\,\beta = (A_m^-)^{\Gamma_n}$. We have an exact sequence of $\boldsymbol{Z}_p[G]$-modules:

$$0 \longrightarrow (A_m^-)^{\Gamma_n}/D_m^- \longrightarrow M_n^{(m)}/D_m^- \longrightarrow D_m^- . \tag{1.5}$$

From Lemma (1.4) and since $M_n^{(m)}/D_m^- = (A_m'^-)^{\Gamma_n}$, we have an exact sequence of $\boldsymbol{Z}_p[G]$-modules:

$$0 \longrightarrow A_n'^- \longrightarrow (A_m'^-)^{\Gamma_n} \longrightarrow D_m^- . \tag{1.6}$$

§ 2. Reduction.

In this section, we reduce Theorem (0.3) to the following theorem.

(2.1) THEOREM. *Suppose that*

(1) $k/\boldsymbol{Q}$ *is a finite abelian extension,*

(2) k *is imaginary, and*

(3) p *is totally decomposed in* $k/\boldsymbol{Q}$.

Then we have

$$^{\#}(A_\infty'^-)^{\Gamma_n} = {}^{\#}\bigoplus_{\Phi} \Lambda_\phi/(\hat{f}(T;\omega\phi^{-1}), \omega_n) \qquad \text{for} \quad n \geqq 0,$$

where Φ *and* ϕ *are as in Theorem* (0.3).

In Theorem (2.1), we need not assume that the degree $[k:\boldsymbol{Q}]$ is prime to p, and it is essential that p is totally decomposed in $k/\boldsymbol{Q}$.

Let $k/\boldsymbol{Q}$ satisfy the conditions (1), (2), and (3) in Theorem (0.3). For $n \geqq 0$, let $\boldsymbol{Q}_n$ be the n-th layer of the unique $\boldsymbol{Z}_p$-extension of $\boldsymbol{Q}$ (c.f. [6]). Since $[k:\boldsymbol{Q}]$ is prime to p, we see that $k_n = k\boldsymbol{Q}_n$.

(2.2) LEMMA. $\quad {}^{\#}A_n^- \underset{p}{\sim} \prod_{\phi}\prod_{\eta} L(0;\phi^{-1}\eta^{-1}) \qquad \text{for} \quad n \geqq 0,$

where ϕ rnns over all characters of Gal$(k/\boldsymbol{Q})$ *such that* $\phi(j)=-1$ *and* $\phi\neq\omega$, *and* η *runs over all characters of* Gal$(\boldsymbol{Q}_n/\boldsymbol{Q})$.

As in § 1, let Z be the decomposition group of p for $k/\boldsymbol{Q}$. Let X_1 (resp. X_2) be the set of all $\boldsymbol{Q}_p$-irreducible characters $\mathrm{Tr}\,\phi$ of $G=\mathrm{Gal}(k/\boldsymbol{Q})$ such that $\phi(j)=-1$ and $\phi(p)=1$ (resp. $\phi(p)\neq1$). Let $m\geqq n\geqq0$. Put $A_{m,i}'^{-}=\bigoplus_{\Phi\in X_i}e(\Phi)A_m'^{-}$, and $A_{m,i}^{-}=\bigoplus_{\Phi\in X_i}e(\Phi)A_m^{-}$ for $i=1$ and 2. Then $A_m'^{-}=A_{m,1}'^{-}\oplus A_{m,2}'^{-}$, and $A_m^{-}=A_{m,1}^{-}\oplus A_{m,2}^{-}$. From Lemma (1.2) and (1.6), we see that

$$(A_{m,2}'^{-})^{\Gamma^n}=A_{n,2}'^{-}=A_{n,2}^{-}\,. \tag{2.3}$$

Let $A(k^Z)_n$ denote the p-Sylow subgroup of the ideal class group of $(k^Z)_n$, where $(k^Z)_n$ is the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension of the fixed field k^Z of Z. Then

$$A(k^Z)_n^{-}\cong A_{n,1}^{-}\quad\text{and}\quad A(k^Z)_m'^{-}\cong A_{m,1}'^{-}\,. \tag{2.4}$$

By Lemma (2.2) for k^Z, we have

$$^{\#}A(k^Z)_n^{-}\underset{p}{\sim}\prod_{\phi}\prod_{\eta}L(0\,;\phi^{-1}\eta^{-1})\,, \tag{2.5}$$

where ϕ runs over all characters of Gal$(k/\boldsymbol{Q})$ such that $\phi(j)=-1$, $\phi\neq\omega$, and $\phi|Z=1$, and η runs over all characters of Gal$(\boldsymbol{Q}_n/\boldsymbol{Q})$.

In the rest of this section, we shall prove the following lemma.

(2.6) LEMMA. *Theorem* (0.3) *follows from Theorem* (2.1).

Proof. Assume that k satisfies the conditions (1), (2), and (3) in Theorem (0.3). For any $n\geqq0$, there exists an integer $m\geqq n$ such that

$$(A(k^Z)_\infty'^{-})^{\Gamma^n}=(A(k^Z)_m'^{-})^{\Gamma^n}\quad\text{and}\quad(A_\infty'^{-})^{\Gamma^n}=(A_m'^{-})^{\Gamma^n}\,.$$

Hence, by (2.3) and (2.4), we have

$$(A_\infty'^{-})^{\Gamma^n}=(A_{m,1}'^{-})^{\Gamma^n}\oplus(A_{m,2}'^{-})^{\Gamma^n}\cong(A(k^Z)_\infty'^{-})^{\Gamma^n}\oplus A_{n,2}^{-}\,. \tag{2.7}$$

By Theorem (2.1) for k^Z, we have

$$\begin{aligned}{}^{\#}(A(k^Z)_\infty'^{-})^{\Gamma^n}&={}^{\#}\bigoplus_{\Psi}\Lambda_\psi/(\hat{f}(T\,;\omega\psi^{-1}),\,\omega_n)\\&={}^{\#}\bigoplus_{\Phi\in X_1}\Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_n)\,,\end{aligned} \tag{2.8}$$

where Ψ and ψ are as in Theorem (2.1) with respect to k^Z. On the other hand, since $A_n^{-}=A_{n,1}^{-}\oplus A_{n,2}^{-}\cong A(k^Z)_n^{-}\oplus A_{n,2}^{-}$, from (2.2) and (2.5), we see that

$$^{\#}A_{n,2}^{-}\underset{p}{\sim}\prod_{\phi|Z\neq1}\prod_{\eta}L(0\,;\phi^{-1}\eta^{-1})\underset{p}{\sim}\prod_{\phi|Z\neq1}\prod_{\zeta^{p^n}=1}\hat{f}(\zeta-1\,;\omega\phi^{-1})\,. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9), we obtain Lemma (2.6).

§ 3 The group of imaginary p-units.

In the following, we assume that k satisfies the conditions (1)-(3) in Theorem (2.1). For $n\geqq 0$, let H_n be the group of p-units of k_n:

$$H_n=\{\alpha\in k_n^\times \mid (\alpha)=\text{product of primes of } k_n \text{ lying over } p\}.$$

Let $m\geqq n\geqq 0$. Let $N_{m,n}: k_m\rightarrow k_n$ be the norm map. Recall that

$$M_n^{(m)}=\{a\in A_m^- \mid (s-1)a\in D_m^-\},$$

where s is a generator of $\mathrm{Gal}(k_m/k_n)$.

(3.1) DEFINITION. We define a homomorphism

$$\varphi_n^{(m)}: M_n^{(m)} \longrightarrow H_n^{1-j}/N_{m,n}(H_m^{1-j})$$

in the following way (c.f. [1, 3]). Let $c\in M_n^{(m)}$ and let $\mathfrak{A}\in c$. Then $\mathfrak{A}^{1-s}=(\alpha)\mathfrak{B}$ for some $\alpha\in k_m^\times$ and some ideal $\mathfrak{B}$ which is a product of primes of k_m lying over p. Define

$$\varphi_n^{(m)}(c)=N_{m,n}(\alpha^{1-j}) \bmod N_{m,n}(H_m^{1-j}).$$

This is well-defined (c.f. [3]), and we have

(3.2) LEMMA ([3]). (1) $\mathrm{Ker}\,\varphi_n^{(m)}=(A_m^-)^{\Gamma_n}$, *and*
(2) $\mathrm{Im}\,\varphi_n^{(m)}=(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j})/N_{m,n}(H_m^{1-j})$.

Proof. (1) See [3],

(2) By definition of $\varphi_n^{(m)}$, $\mathrm{Im}\,\varphi_n^{(m)}\subset(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j})/N_{m,n}(H_m^{1-j})$. Take any $\alpha\in k_m^\times$ such that $N_{m,n}(\alpha)\in H_n^{1-j}$. Then $(N_{m,n}(\alpha))$ is an ideal of k_n which is a product of primes of k_n lying over p. Since each prime of k_n lying over p is totally ramified in k_m/k_n, there exists an ideal $\mathfrak{B}$ of k_m which is a product of primes of k_m lying over p such that $(N_{m,n}(\alpha))=\mathrm{N}_{m,n}(\mathfrak{B})$. Then $N_{m,n}(\alpha\mathfrak{B}^{-1})=(1)$. Thus there exists an ideal $\mathfrak{A}$ of k_m such that $(\alpha)\mathfrak{B}^{-1}=\mathfrak{A}^{1-s}$. Let r be an integer prime to p such that the class of $\mathfrak{A}^{r(1-j)}$ is contained in A_m^-. Put $a=$class of $\mathfrak{A}^{r(1-j)}$. Then $a\in M_n^{(m)}$ and $\varphi_n^{(m)}(a)=N_{m,n}(\alpha^{2r(1-j)}) \bmod N_{m,n}(H_m^{1-j})$. Since $(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j})/N_{m,n}(H_m^{1-j})$ is a finite abelian p-group, $\mathrm{Im}\,\varphi_n^{(m)}=(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j})/N_{m,n}(H_m^{1-j})$.

(3.3) COROLLARY.

$$0\longrightarrow A_n'^- \longrightarrow (A_m'^-)^{\Gamma_n} \xrightarrow{\tilde{\varphi}} \mathrm{Im}\,\varphi_n^{(m)} \longrightarrow 0$$

is an exact sequence of $\boldsymbol{Z}_p[G]$*-modules, where* $\tilde{\varphi}$ *is induced from* $\varphi_n^{(m)}$ *since* $\varphi_n^{(m)}(D_m^-)=0$ (c.f. (1.5) *and* (1.6)).

§ 4. p-adic regurators.

In this section, we shall define p-adic regurators for certain subgroups of H_0^{1-j}. Assume that k satisfies the conditions (1)-(3) in Theorem (2.1). For $n\geqq 0$, let E_n be the unit group of k_n and let $\mathbf{P}_n$ be the subgroup $\{(\alpha)\mid\alpha\in H_n\}$ of the ideal group of k_n. From a natural exact sequence $0\to E_n\to H_n\to\mathbf{P}_n\to 0$, we have an exact sequence $0\to E_n\cap H_n^{1-j}\to H_n^{1-j}\to\mathbf{P}_n^{1-j}\to 0$. Let $\mu(k_n)$ denote the group of all roots of unity in k_n, then we have $E_n\cap H_n^{1-j}=\mu(k_n)\cap H_n^{1-j}$. Hence, we have (letting $n=0$)

$$\mu(k)H_0^{1-j}/\mu(k)\cong\mathbf{P}_0^{1-j}. \tag{4.1}$$

We note that $\mathbf{P}_0^{1-j}$ is a free $\boldsymbol{Z}$-module of rank $g=[k:\boldsymbol{Q}]/2$. Assume that M is a submodule of $\mu(k)H_0^{1-j}$ such that $\mu(k)M/\mu(k)$ has rank g. Let $m_1, m_2, \cdots, m_g$ be a system of elements of $\mu(k)M$ such that $m_1 \bmod \mu(k), \cdots, m_g \bmod \mu(k)$ are $\boldsymbol{Z}$-basis of $\mu(k)M/\mu(k)$. Let $s_1, s_2, \cdots, s_g$ be a system of representatives of $G/\{1, j\}$. Let $\log_p$ denote the p-adic logarithm from $\boldsymbol{Q}_p^\times$ into $\boldsymbol{Q}_p$ normalized by $\log_p p=0$ and $\log_p\zeta=0$ for $\zeta^{p-1}=1$ (c.f. [5]). Recall that $\rho:\bar{\boldsymbol{Q}}\hookrightarrow\bar{\boldsymbol{Q}}_p$ is the fixed embedding. Then $\rho(k)\subset\boldsymbol{Q}_p$ by the assumption (3) in Theorem (2.1).

(4.2) DEFINITION. We define the p-adic regurator of M by

$$R_p(M)=\det\begin{pmatrix}\log_p\rho(s_1m_1), & \cdots, & \log_p\rho(s_1m_g)\\ \vdots & & \vdots\\ \log_p\rho(s_gm_1), & \cdots, & \log_p\rho(s_gm_g)\end{pmatrix}\quad \text{up to } \pm 1.$$

This definition is independent of the choices of $(s_1, \cdots, s_g)$ and $(m_1, \cdots, m_g)$.

(4.3) LEMMA. *Let $M_1\subset M_2$ be submodules of $\mu(k)H_0^{1-j}$ such that $R_p(M_1)\neq 0$. Then, $R_p(M_2)\neq 0$, and*

$$\frac{R_p(M_1)}{R_p(M_2)}=(\mu(k)M_2:\mu(k)M_1)\qquad \textit{up to } \pm 1.$$

§ 5. Gauss sums.

In this section, we recall Gauss sums in [4]. Assume that k satisfies the conditions (1)-(3) in Theorem (2.1). Let N be the conductor of $k/\boldsymbol{Q}$. Since p is totally decomposed in k, N is prime to p. Let $K=\boldsymbol{Q}(\mu_N)$, $G_N=\mathrm{Gal}(K/\boldsymbol{Q})$, $H=\mathrm{Gal}(K/k)$, and let D be the decomposition group of p for $K/\boldsymbol{Q}$, where μ_N denote the group of all N-th roots of unity in $\bar{\boldsymbol{Q}}$. For any $t\in\boldsymbol{Z}$ such that $(t, N)=1$, define $s_t\in G_N$ by $s_t(\zeta)=\zeta^t$ for all $\zeta\in\mu_N$. Then $D=\langle s_p\rangle$. Let v be the place of $\bar{\boldsymbol{Q}}$ corresponding to the fixed enbedding $\rho:\bar{\boldsymbol{Q}}\hookrightarrow\bar{\boldsymbol{Q}}_p$. Let $\mathfrak{p}$ (resp. $\mathfrak{p}_N$) be the prime of k (resp. K) which is the restriction of v to k (resp. K).

(5.1) DEFINITION. Put

$$\mathcal{A}_N=\frac{1}{N}\boldsymbol{Z}/\boldsymbol{Z}-\{0 \bmod \boldsymbol{Z}\} \quad \text{and} \quad \mathbf{A}_N=\mathrm{Map}(\mathcal{A}_N, \boldsymbol{Z}).$$

For $a=(t/N \bmod \boldsymbol{Z})\in\mathcal{A}_N$, let $\delta_{t/N}=\delta_a\in\mathbf{A}_N$ be the map defined by $\delta_a(a)=1$ and $\delta_a(b)=0$ for $b\in\mathcal{A}_N$ with $b\neq a$. The group G_N acts on $\mathcal{A}_N$ and $\mathbf{A}_N$ by

$$s_t(t'/N \bmod \boldsymbol{Z})=tt'/N \bmod \boldsymbol{Z} \qquad \text{for} \quad s_t\in G_N\,, \quad t'/N \bmod \boldsymbol{Z}\in\mathcal{A}_N$$

and

$$(s\alpha)(a)=\alpha(s^{-1}a) \qquad \text{for} \quad s\in G_N\,, \quad \alpha\in\mathbf{A}_N\,, \quad a\in\mathcal{A}_N\,.$$

We define the Gauss sum $g(\alpha, \mathfrak{p}_N, \Psi\circ\mathrm{Tr})=g(\alpha, \mathfrak{p}_N)$ as in (1.3) and (1.4) of [4].

(5.2) *Note*. Let $\alpha\in N\mathbf{A}_N$. Then $g(\alpha, \mathfrak{p}_N)$ is contained in K^D. The action of G_N on $g(\alpha, \mathfrak{p}_N)$ is given by

$$g(\alpha, \mathfrak{p}_N)^s=g(s\alpha, \mathfrak{p}_N) \qquad \text{for} \quad s\in G_N\,.$$

For $x\in\boldsymbol{R}$, let $\langle x\rangle$ be the unique real number such that $0\leqq\langle x\rangle<1$ and $x-\langle x\rangle\in\boldsymbol{Z}$. For $a=(t/N \bmod \boldsymbol{Z})\in\mathcal{A}_N$, let $\langle a\rangle=\langle t/N\rangle$, and for $\alpha\in\mathbf{A}_N$, let $n(\alpha)=\sum_{a\in\mathcal{A}_N}\alpha(a)\langle a\rangle$.

Let Γ_p be the p-adic Γ-function defined by Morita. As in [4], we define $\Gamma_p:\mathbf{A}_N\to\boldsymbol{Z}_p$ by

$$\Gamma_p(\alpha)=\prod_{a\in\mathcal{A}_N}\Gamma_p(\langle a\rangle)^{\alpha(a)} \qquad \text{for} \quad \alpha\in\mathbf{A}_N\,.$$

The following theorem was proved by Gross and Koblitz [4].

(5.3) THEOREM. *If $n(\alpha)\in\boldsymbol{Z}$, then*

$$\rho(g(\alpha, \mathfrak{p}_N))=(-p)^{n(\sum_{s\in D}s\alpha)}\Gamma_p(\sum_{s\in D}s\alpha) \qquad \text{in} \quad \boldsymbol{Q}_p\,.$$

(5.4) COROLLARY. *Let $\alpha\in N\mathbf{A}_N$. Then*

$$\log_p\rho(g(\alpha, \mathfrak{p}_N))=\sum_{s\in D}\log_p\Gamma_p(s\alpha)\,.$$

Let $X^-=\{\phi\in\hat{G}\,|\,\phi(j)=-1\}$. Let M be a divisor of N. We put $X_M=\{\phi\in X^-\,|$ conductor of $\phi=M\}$ and $H_M=\mathrm{Gal}(\boldsymbol{Q}(\mu_N)/\boldsymbol{Q}(\mu_M))\subset G_N$. Recall that $2g=[k:\boldsymbol{Q}]={}^{\#}G={}^{\#}(G_N/H)$. Fix a system of representatives $\{s_1, \cdots, s_{2g}\}$ of G_N/H $(s_i\in G_N$, $1\leqq i\leqq 2g)$. For $\phi\in X^-$, let

$$e(\phi)=\frac{1}{{}^{\#}G}\sum_{s\in G}\phi(s^{-1})s\in\bar{\boldsymbol{Q}}_p[G]$$

and

$$\tilde{e}(\phi)=\frac{1}{{}^{\#}G}\sum_{i=1}^{2g}\phi((s_iH)^{-1})s_i\in\bar{\boldsymbol{Q}}_p[G_N]\,.$$

Then we see that ${}^{\#}G\sum_{\phi\in X_M^-}e(\phi)\in \boldsymbol{Z}[G]$ and ${}^{\#}G\sum_{\phi\in X_M^-}\tilde{e}(\phi)\in \boldsymbol{Z}[G_N]$.

(5.5) DEFINITION. Let $\mathbf{A}_k$ be the submodule of $N\mathbf{A}_N$ generated by

$$\{{}^{\#}G\sum_{\phi\in X_M^-}e(\phi)N\delta_{1/M}\,|\,M|N\}\,.$$

(5.6) DEFINITION. We define the Gauss sum $g_k(\alpha)$ of k associated to $\alpha\in N\mathbf{A}_N$ by

$$g_k(\alpha)=N_{K^D/k}(g(\alpha,\,\mathfrak{p}_N))$$

and the group $\mathcal{G}_k$ of Gauss sums of k by

$$\mathcal{G}_k=\{g_k(\alpha)\,|\,\alpha\in\mathbf{A}_k\}\,.$$

We define a $\boldsymbol{Z}[G_N]$-homomorphism $S_k: N\mathbf{A}_N\to\boldsymbol{Z}[G]$ by

$$S_k(\alpha)=\sum_{s\in G_N}n(s\alpha)(sH)^{-1}\qquad\text{for}\quad \alpha\in N\mathbf{A}_N\,.$$

We have Stickelberger relations for k.

(5.7) If $\alpha\in N\mathbf{A}_N$, then

$$(g_k(\alpha))=\mathfrak{p}^{S_k(\alpha)}\quad\text{in } k\,.$$

For $n\geqq 0$, let $\mathbf{D}_n$ be the subgroup of the ideal group of k_n generated by primes of k_n lying over p, and let $\mathbf{G}_k=\{(g_k(\alpha))\,|\,\alpha\in\mathbf{A}_k\}$. From (5.7), $\mathbf{G}_k$ is a $\boldsymbol{Z}[G]$-submodule of $\mathbf{D}_0$.

(5.8) PROPOSITION.

$$(\mathbf{D}_0^{1-j}:\mathbf{G}_k^{1-j})=(2gN)^g\prod_{M|N}({}^{\#}H_M)^{{}^{\#}X_M^-}\prod_{\phi\in X^-}L(0\,;\phi^{-1})\,.$$

Proof. Since p is totally decomposed in k, we have

$$\mathbf{D}_0^{1-j}=(1-j)\boldsymbol{Z}[G]\cdot\mathfrak{p}\cong(1-j)\boldsymbol{Z}[G]\,.$$

We compute the index $(\mathbf{D}_0^{1-j}:\mathbf{G}_k^{1-j})$ in $(1-j)\boldsymbol{Z}[G]\otimes\bar{\boldsymbol{Q}}_p=(1-j)\bar{\boldsymbol{Q}}_p[G]$ and in $e(\phi)\bar{\boldsymbol{Q}}_p[G]=e(\phi)\bar{\boldsymbol{Q}}_p$ for $\phi\in X^-$, because $(1-j)\bar{\boldsymbol{Q}}_p[G]=\bigoplus_{\phi\in X^-}e(\phi)\bar{\boldsymbol{Q}}_p$. Let $M|N$ and let $\phi\in X_M^-$. For $L|N$, we have

$$\begin{aligned}&e(\phi)S_k({}^{\#}G\sum_{\psi\in X_L^-}\tilde{e}(\psi)N\delta_{1/L})\\&=\begin{cases}e(\phi){}^{\#}GN\sum_{s_t\in G_N}\langle t/M\rangle\phi^{-1}(s_tH) & \text{if}\quad L=M\,,\\ 0 & \text{if}\quad L\neq M\,.\end{cases}\end{aligned}$$

Futhermore we have

$$\sum_{s_t\in G_N}\langle t/M\rangle\phi^{-1}(s_tH)={}^{\#}H_M\sum_{\bar{s}_t\in G_M}\langle t/M\rangle\phi_M^{-1}(\bar{s}_t\bar{H})\,,$$

where $\bar{s}_t = s_t H_M$, $G_M = G_N/H_M$, $\bar{H} = HH_M/H_M$, and ϕ_M is the character of $G_M/\bar{H}$ induced from ϕ. Hence

$$e(\phi)S_k(^{\#}G \sum_{\psi\in X_L^-} \tilde{e}(\psi)N\delta_{1/L}) = \begin{cases} e(\phi)^{\#}GN^{\#}H_M L(0\,;\phi^{-1}) & \text{if } L=M, \\ 0 & \text{if } L\neq M. \end{cases}$$

Since $(1-j)e(\phi)=2e(\phi)$, we have

$$(\mathbf{D}_0^{1-j} : \mathbf{G}_k^{1-j}) = (2gN)^g \prod_{M|N} (^{\#}H_M)^{^{\#}X_M^-} \prod_{\phi\in X^-} L(0\,;\phi^{-1}).$$

We shall compute the p-adic regurator of $\mathcal{G}_k^{1-j}$ by using p-adic L-functions.

(5.9) THEOREM.

$$R_p(\mathcal{G}_k^{1-j}) = (4gN)^g \prod_{M|N} (^{\#}H_M)^{^{\#}X_M^-} \prod_{\phi\in X^-} L_p'(0\,;\omega\phi^{-1}) \quad \textit{up to } \pm 1.$$

We recall a result of Ferrero and Greenberg [2].

(5.10) THEOREM. *Let $M|N$ and let $\phi\in X_M^-$. Then*

$$L_p'(0\,;\omega\phi^{-1}) = \sum_{\bar{s}_t\in G_M} \phi^{-1}(\bar{s}_t)\log_p \Gamma_p(s_t\delta_{1/M}).$$

By (5.10) and (5.4),

$$L_p'(0\,;\omega\phi^{-1}) = \frac{1}{N^{\#}H_M} \sum_{s\in G} \phi^{-1}(s)\log_p \rho(g_k(N\delta_{1/M})^s). \tag{5.11}$$

Put $g_{k,M} = g_k(N\delta_{1/M})$. We have, for $s_t\in G_N$ with $s_tH=s$,

$$\log_p \rho(g_k(N^{\#}G \sum_{\psi\in X_M^-} \tilde{e}(\psi)s_t\delta_{1/M}))$$
$$= \log_p \rho\left((g_{k,M})^{^{\#}G \sum_{\psi\in X_M^-} e(\psi)s}\right)$$

(5.12) Claim. Let $L|N$ and let $\phi\in X_L^-$. Then

$$\sum_{s\in G} \phi^{-1}(s)\log_p \rho\left((g_{k,M})^{^{\#}G \sum_{\psi\in X_M^-} e(\psi)s}\right)$$
$$= \begin{cases} ^{\#}G \sum_{s\in G} \phi^{-1}(s)\log_p \rho((g_{k,M})^s) & \text{if } L=M, \\ 0 & \text{if } L\neq M. \end{cases}$$

In fact, computing in $\bar{\boldsymbol{Q}}_p[G]\otimes\log_p \rho(\mathcal{G}_k^{1-j}) = \bar{\boldsymbol{Q}}_p[G] = \sum_{\phi} e(\phi)\bar{\boldsymbol{Q}}_p$, because $\log_p \rho(\mathcal{G}_k^{1-j}) \subset \boldsymbol{Q}_p$, we have

$$e(\phi)\sum_{s\in G} s^{-1}\log_p \rho\left((g_{k,M})^{^{\#}G \sum_{\psi\in X_M^-} e(\psi)s}\right)$$

$$=e(\phi)^{\#}G\sum_{\phi\in X_M^-}e(\phi)\sum_{s\in G}s^{-1}\log_p\rho((g_{k,M})^s)$$

$$=\begin{cases}e(\phi)^{\#}G\sum_{s\in G}\phi^{-1}(s)\log_p\rho((g_{k,M})^s) & \text{if } L=M,\\ 0 & \text{if } L\neq M.\end{cases}$$

Proof of Theorem (5.9)
Define the map $\log_p: H_0^{1-j}\to(1-j)\boldsymbol{Q}_p[G]$ by

$$\log_p(x)=(1-j)\sum_{\bar{s}\in G/\{1,j\}}\log_p\rho(x^s)s^{-1}\qquad\text{for}\quad x\in H_0^{1-j}.$$

Since $\log_p(\mathcal{G}_k^{1-j})\subset(1-j)\boldsymbol{Q}_p[G]$, we compute $R_p(\mathcal{G}_k^{1-j})$ in $(1-j)\boldsymbol{Q}_p[G]\otimes\bar{\boldsymbol{Q}}_p=(1-j)\bar{\boldsymbol{Q}}_p[G]=\bigoplus_{\phi\in X^-}e(\phi)\bar{\boldsymbol{Q}}_p$. Since $\mathcal{G}_k^{1-j}$ is generated by

$$(g_{k,M}^{1-j})^{\#G\sum_{\phi\in X_L^-}e(\phi)}\qquad\text{for all}\quad M|N,\quad L|N,$$

we have from (5.12)

$$\pm R_p(\mathcal{G}_k^{1-j})=\prod_{M|N}\prod_{\phi\in X_M^-}2^{\#}G\sum_{s\in G}\phi^{-1}(s)\log_p\rho((g_{k,M})^s)$$

$$=(2^{\#}G)^{\#X^-}\prod_{M|N}\prod_{\phi\in X_M^-}N^{\#}H_M L_p'(0;\omega\phi^{-1}).\qquad\text{Q.E.D.}$$

Ferrero and Greenberg [2] have proved that $L_p'(0;\omega\phi^{-1})\neq0$. Then we see that $R_p(\mathcal{G}_k^{1-j})\neq0$. Hence, by Lemma (4.3),

$$R_p(H_0^{1-j})\neq0\qquad(\text{c.f. [3]}).\tag{5.13}$$

§6. Proof of Theorem (2.1).

In this section, we shall prove Theorem (2.1). Let $n\geqq0$. Recall that $\mathbf{D}_n=\{\prod\mathfrak{P}^{m(\mathfrak{P})}|\mathfrak{P}|p\text{ in }k_n,\ m(\mathfrak{P})\in\boldsymbol{Z}\}$. Let $\mathcal{P}_n$ be the principal ideal group of k_n. Put $\mathbf{P}_n=\mathcal{P}_n\cap\mathbf{D}_n$ and $\mathcal{D}_n=\mathbf{D}_n/\mathbf{P}_n$. Then

$$(\mathbf{D}_0^{1-j}:\mathbf{P}_0^{1-j})={}^{\#}\mathcal{D}_0^{1-j}\qquad(\text{up to a 2-factor})\underset{p}{\sim}{}^{\#}D_0^-.$$

From Lemma (4.3) and (5.13), we have

$$\pm\frac{R_p(\mathcal{G}_k^{1-j})}{R_p(H_0^{1-j})}=(\mathbf{P}_0^{1-j}:\mathbf{G}_k^{1-j})=\frac{(\mathbf{D}_0^{1-j}:\mathbf{G}_k^{1-j})}{(\mathbf{D}_0^{1-j}:\mathbf{P}_0^{1-j})}\qquad(\text{up to a 2-factor}).$$

Thus, from Proposition (5.8) and Theorem (5.9), we have

$$\pm\frac{\prod_{\phi\in X^-}L_p'(0;\omega\phi^{-1})}{R_p(H_0^{1-j})}=\frac{\prod_{\phi\in X^-}L(0;\phi^{-1})}{{}^{\#}\mathcal{D}_0^{1-j}}\qquad(\text{up to a 2-factor})\tag{6.1}$$

(6.2) LEMMA. *For* $m\geqq0$,

$$\#(H_0^{1-j}/(H_0^{1-j}\cap N_{m,0}(k_m^\times)^{1-j}))\underset{p}{\sim}\frac{p^{(m+1)g}}{R_p(H_0^{1-j})}\,.$$

Proof. Recall the map $\log_p$ in § 5. Since k_m/k is totally ramified at $\mathfrak{p}$, by Hasse's norm theorem following [3], we have

$$x\in N_{m,0}(k_m^\times)^{1-j}\Leftrightarrow \log_p(x)\in(1-j)p^{m+1}\boldsymbol{Z}_p[G]\,,\qquad \text{for}\quad x\in H_0^{1-j}\,.\qquad \text{Q.E.D.}$$

(6.3) For $n\geqq 0$ and for a sufficiently large $m\geqq n$,

$$\boldsymbol{Z}_p\otimes(H_0^{1-j}\cap N_{m,0}(k_m^\times)^{1-j})=\boldsymbol{Z}_p\otimes(N_{n,0}(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j})\,.$$

In fact, by Lemma (6.2), there exists a sufficiently large $m\geqq n$, such that

$$\boldsymbol{Z}_p\otimes N_{n,0}(H_n^{1-j})\subset\boldsymbol{Z}_p\otimes(H_0^{1-j}\cap N_{m,0}(k_m^\times)^{1-j})\,.$$

(6.4) For $m\geqq n\geqq 0$, we have

$$(\boldsymbol{Z}_p\otimes H_0^{1-j}:\boldsymbol{Z}_p\otimes(H_0^{1-j}\cap N_{m,0}(k_m^\times)^{1-j}))$$
$$=(\boldsymbol{Z}_p\otimes\mathbf{P}_0^{1-j}:\boldsymbol{Z}_p\otimes(\mathbf{P}_0^{1-j}\cap\mathrm{N}_{m,0}(\mathcal{P}_m^{1-j})))$$

and

$$(\boldsymbol{Z}_p\otimes(H_n^{1-j}\cap N_{m,n}(k_m^\times)^{1-j}:\boldsymbol{Z}_p\otimes N_{m,n}(H_m^{1-j}))$$
$$=(\boldsymbol{Z}_p\otimes(\mathbf{P}_n^{1-j}\cap\mathrm{N}_{m,n}(\mathcal{P}_m^{1-j})):\boldsymbol{Z}_p\otimes\mathrm{N}_{m,n}(\mathbf{P}_m^{1-j}))\,.$$

For any $n\geqq 0$, let m be an integer satisfying (6.3). Since the norm maps $\mathrm{N}_{m,n}:\mathbf{D}_m^{1-j}\to\mathbf{D}_n^{1-j}$ and $\mathrm{N}_{n,0}:\mathbf{D}_n^{1-j}\to\mathbf{D}_0^{1-j}$ are bijective, we have the following diagram:

$$\begin{array}{lcc}
\mathbf{D}_m^{1-j} & \supset & \mathbf{P}_m^{1-j} \\
\wr\downarrow N_{m,n} & & \wr\downarrow \\
\mathbf{D}_n^{1-j} & \supset\mathbf{P}_n^{1-j}\cap\mathrm{N}_{m,n}(\mathcal{P}_m^{1-j})\supset & \mathrm{N}_{m,n}(\mathbf{P}_m^{1-j}) \\
\wr\downarrow \mathrm{N}_{n,0} & \wr\downarrow & \wr\downarrow \\
\mathbf{D}_0^{1-j}\supset\mathbf{P}_0^{1-j}\supset & \mathbf{P}_0^{1-j}\cap\mathrm{N}_{m,0}(\mathcal{P}_m^{1-j})\supset & \mathrm{N}_{m,0}(\mathbf{P}_m^{1-j})\,.
\end{array}\tag{6.5}$$

We have, by (6.5),

$$(\boldsymbol{Z}_p\otimes(\mathbf{P}_n^{1-j}\cap\mathrm{N}_{m,n}(\mathcal{P}_m^{1-j})):\boldsymbol{Z}_p\otimes\mathrm{N}_{m,n}(\mathbf{P}_m^{1-j}))$$
$$=\frac{(\boldsymbol{Z}_p\otimes\mathbf{D}_m^{1-j}:\boldsymbol{Z}_p\otimes\mathbf{P}_m^{1-j})}{(\boldsymbol{Z}_p\otimes\mathbf{D}_0^{1-j}:\boldsymbol{Z}_p\otimes\mathbf{P}_0^{1-j})(\boldsymbol{Z}_p\otimes\mathbf{P}_0^{1-j}:\boldsymbol{Z}_p\otimes(\mathbf{P}_0^{1-j}\cap\mathrm{N}_{m,0}(\mathcal{P}_m^{1-j})))}\,.$$

Hence, by Lemma (3.2) and (6.4), we have

$$\#\mathrm{Im}\,\varphi_n^{(m)}=\frac{\#(D_m^-/D_0^-)}{(\boldsymbol{Z}_p\otimes H_0^{1-j}:\boldsymbol{Z}_p\otimes(H_0^{1-j}\cap N_{m,0}(k_m^\times)^{1-j}))}\,.$$

Thus, by Lemma (6.2) and Lemma (1.3), we have

$$ {}^{\#}\mathrm{Im}\,\varphi_n^{(m)} \underset{p}{\sim} p^{-g} R_p(H_0^{1-j}) . \tag{6.6}$$

Note that

$$ L'_p(0\,;\omega\phi^{-1}) \underset{p}{\sim} p\hat{f}(0\,;\omega\phi^{-1}) . \tag{6.7}$$

Proof of Theorem (2.1)

For a given n, take an integer $m \geqq n$ such that $(A_\infty'^-)^{\Gamma_n}=(A_m'^-)^{\Gamma_n}$ and (6.3) holds. By using (3.3), (6.1), (6.6) and (6.7), we have

$$ {}^{\#}(A_m'^-)^{\Gamma_n} \underset{p}{\sim} \prod_{\phi\in X^-} \prod_{\zeta^{p^n}=1} \tilde{f}(\zeta-1\,;\omega\phi^{-1}) ,$$

because

$$ {}^{\#}(A_n'^-/A_0'^-) \underset{p}{\sim} \prod_{\phi\in X^-} \prod_{\substack{\zeta^{p^n}=1\\ \zeta\neq 1}} \tilde{f}(\zeta-1\,;\omega\phi^{-1}) .$$

Hence, we complete the proof of Theorem (2.1).

§ 7. Examples.

1. Let $p=5$, and k be the unique subfield of $\boldsymbol{Q}(\exp(2\pi i/1949))$ of degree 4 over $\boldsymbol{Q}$. Then 5 splits completely in $k/\boldsymbol{Q}$. There are two imaginary $\boldsymbol{Q}_5$-irreducible characters of $\mathrm{Gal}(k/\boldsymbol{Q})$. We have ${}^{\#}A_0^-=5^3$ and $D_0^- \cong \boldsymbol{Z}/5\boldsymbol{Z} \oplus \boldsymbol{Z}/5\boldsymbol{Z}$, by an easy computation. Hence ${}^{\#}A_0'^-=5$. By a computation (modulo congruence) of the coefficients of $f(T\,;\omega\phi^{-1})$, we have

$$ {}^{\#}\bigoplus_{\Phi} \Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_0)=5^3 .$$

Using Theorem (0.3) we have ${}^{\#}(A_\infty'^-)^{\Gamma_0}=5^3$. Hence $A_0'^- \subsetneqq (A_\infty'^-)^{\Gamma_0}$. Moreover, we see that ${}^{\#}A_1^-=5^9$, ${}^{\#}D_1^-=5^4$, and ${}^{\#}A_1'^-=5^5$. Note that λ^--invariant of k (for $p=5$) is 6.

2. Let $p=5$, and k be the unique subfield of $\boldsymbol{Q}(\exp(2\pi i/2269))$ of degree 4 over $\boldsymbol{Q}$. We have

$$ \bigoplus_{\Phi} \Lambda_\phi/(\hat{f}(T\,;\omega\phi^{-1}),\,\omega_0)=\{0\} .$$

By using Theorem (0.3), we have $(A_\infty'^-)^{\Gamma_0}=\{0\}$. Hence ${}^{\#}D_0^-={}^{\#}A_0^-=5^3$. Note that λ^--invariant of k is 2. We have $A_0'^-=(A_\infty'^-)^{\Gamma_0}=\{0\}$.

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Addendum in proof

We received from Gross the preprint of [8] after we had sent him the preprint of this paper. There is a partial overlap between the content of [8] and that of this paper. Assuming Conjecture (5.3) of the paper of Federer and Gross [7] and combinig their Proposition (3.9) [7], one will get Theorem (0.3) in this paper in the case of $n=0$. In [7], they announced that Conjecture (5.3) of abelian case was proved in [8]. In this paper, without assuming Conjecture (5.3) [7], we prove Theorem (0.3) for all $n\geqq 0$ by using Theorem (5.9).