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ON THE INVARIANT SUBMANIFOLD OF A *CR*-MANIFOLD

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§0. Introduction.

Differential geometry of Kaehler submanifolds has been studied from many points of view (see K. Ogiue [4], B. Smyth [7] etc.). On the other hand, *CR*-structures are recently developed by several authors (see Chern-Moser [1], Tanaka [8], [9], Webster [10], [11], S. Ishihara [2], Sakamoto-Takemura [5], [6] and so on). In [5], [6], the authors gave a change of canonical connections associated with almost contact structures belonging to a *CR*-structure and also derived the generalized Bochner curvature as a curvature invariant. The purpose of the present paper is to study invariant submanifolds of a *CR*-manifold and to prove some theorems similar to the Kaehler case.

In §1 we shall recall definitions and results given in [5], [6]. §2 well be devoted to the study of an invariant submanifold of a *CR*-manifold. In §3 we shall obtain some theorems similar to the Kaehler case. The author wishs to express his heartly thanks to Professors S. Ishihara and K. Sakamoto for their constant encouragement and valuable suggestions.

§1. Preliminaries.

In this section, we shall recall definitions and some properties of *CR*-structures for later use. Let $\mathcal{M}$ be a connected orientable C^∞-manifold of dimension $2n+1$ $(n\geqq 1)$ and $(\mathcal{D}, J)$ a pair of a hyperdistribution $\mathcal{D}$ and a complex structure J on $\mathcal{D}$. The pair $(\mathcal{D}, J)$ is called a *CR-structure* if the following two conditions hold:

$$[JX, JY]-[X, Y]\in\Gamma(\mathcal{D}), \tag{1.1}$$

$$[JX, JY]-[X, Y]-J([X, JY]+[JX, Y])=0 \tag{1.2}$$

for every $X, Y\in\Gamma(\mathcal{D})$ where $\Gamma(\mathcal{D})$ denotes the set of all vector fields contained in $\mathcal{D}$. Let θ be a local 1-form annihilating the hyperdistribution $\mathcal{D}$. If the restriction of the 2-form $d\theta$ to $\mathcal{D}$ is nondegenerate, then the *CR*-structure $(\mathcal{D}, J)$ is called to be *nondegenerate*. In the sequel $(\mathcal{D}, J)$ will be always a nondegenerate *CR*-structure.

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Now let the manifold $\mathcal{M}$ admit a CR-structure $(\mathcal{D}, J)$, then $\mathcal{M}$ is call a *CR-manifold.* An almost contact structure (ϕ, ξ, θ) is a triplet of (1, 1) tensor field ϕ, a vector field ξ, and an 1-form θ defined on $\mathcal{M}$ satisfying

$$\theta(\xi)=1, \qquad \phi\xi=0, \qquad \theta\circ\phi=0, \tag{1.3}$$

$$\phi^2=-I+\theta\otimes\xi, \qquad \operatorname{rank}\phi=2n.$$

If the 1-form θ annihilates $\mathcal{D}$ and the restriction of ϕ to $\mathcal{D}$ coincides with J, then we say that the almost contact structure (ϕ, ξ, θ) *belongs* to the CR-structure $(\mathcal{D}, J)$. Define a 2-form ω by

$$\omega=-2d\theta. \tag{1.4}$$

Then ω satisfies

$$\omega(JX, JY)=\omega(X, Y) \tag{1.5}$$

for every $X, Y\in\Gamma(\mathcal{D})$ because of (1.1). Moreover define $g;\mathcal{D}\times\mathcal{D}\to R$ by

$$g(X, Y)=\omega(JX, Y), \tag{1.6}$$

which is called *Levi-metric* and satisfies the equations

$$g(X, Y)=g(Y, X), \tag{1.7}$$

$$g(JX, JY)=g(X, Y). \tag{1.8}$$

Given almost contact structure belonging to $(\mathcal{D}, J)$, we can always construct from $(\mathcal{D}, J)$ an almost contact structure (ϕ, ξ, θ) belonging to $(\mathcal{D}, J)$ and satisfying the following condition

$$[\xi, \Gamma(\mathcal{D})]\in\Gamma(\mathcal{D}) \tag{1.9}$$

(c. f. [5]). This condition is equivalent to

$$\mathcal{L}_\xi\theta=0 \tag{1.10}$$

or

$$\omega(X, \xi)=0, \tag{1.11}$$

where $\mathcal{L}$ denotes the Lie differentiation with respect to ξ. Such an almost contact structure will be denoted by $(\phi, \xi, \theta)^*$ and we shall restrict our attention to the family of almost contact structures with condition (1.9) which belong to the CR-structure $(\mathcal{D}, J)$. We proved in [5]

LEMMA 1.1. *If $(\phi, \xi, \theta)^*$ and $(\phi', \xi', \theta')^*$ belong to $(\mathcal{D}, J)$, then they are related by*

$$\theta'=\varepsilon e^{2\mu}\theta, \qquad \xi'=\varepsilon e^{-2\mu}(\xi-2Q), \qquad \phi'=\phi-2\theta\otimes P, \tag{1.12}$$

where $\varepsilon=\pm 1$, μ *is a* C^∞-*function,* $P\in\Gamma(\mathcal{D})$ *is defined by* $g(P, X)=d\mu(X)$ *for* $X\in\Gamma(\mathcal{D})$ *and* $Q=JP$.

Next we shall explain canonical connections associated to almost contact structures with condition (1.9) and find a quantity invariant under their changes. Before mentionning the existence of canonical connections, we prepare the notations. For $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$, their always exists a linear connection ∇ such that $\nabla\phi=0$, $\nabla\xi=0$, and $\nabla\theta=0$. Let D denote the induced connection on the hyperdistribution $\mathcal{D}$. Then D satisfies $DJ=0$. Since the equation $\nabla\phi=0$ implies that the parallel displacement with respect to ∇ preserves $\mathcal{D}$, the torsion tensor field T of ∇ satisfies

$$T(X, Y)=T_{\mathcal{D}}(X, Y)-\omega(X, Y)\xi\,, \tag{1.13}$$

$$T_{\mathcal{D}}(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y]|_{\mathcal{D}} \tag{1.14}$$

for every $X, Y\in\Gamma(\mathcal{D})$ where $[X, Y]|_{\mathcal{D}}$ denotes the $\mathcal{D}$-component of $[X, Y]$ and we note that $T_{\mathcal{D}}(X, Y)$ is the $\mathcal{D}$-component of $T(X, Y)$. Let F be a tensor field of type (1, 1) defined by

$$FX=T(\xi, X)\,, \qquad X\in\Gamma(\mathcal{D})\,. \tag{1.15}$$

Tanaka [9] proved (cf. [5])

LEMMA 1.2. *Let* $(\phi, \xi, \theta)^*$ *be an almost contact structure satisfying the condition* (1.9) *and belonging to* $(\mathcal{D}, J)$. *Then there exists uniquely a linear connection* ∇ *such that* $\nabla\phi=0$, $\nabla\xi=0$, $\nabla\theta=0$, $Dg=0$, $T_{\mathcal{D}}=0$, *and* $F=-(1/2)\phi\mathcal{L}_\xi\phi$.

The linear connection stated in the above lemma is called a *canonical connection* associated with $(\phi, \xi, \theta)^*$. We proved in [5], [6]

LEMMA 1.3. *Let* $(\phi, \xi, \theta)^*$ *and* $(\phi', \xi', \phi')^*$ *be two almost contact structures which belong to the CR-structure* $(\mathcal{D}, J)$. *Let* ∇ *and* ∇' *be canonical connections associated with* $(\phi, \xi, \theta)^*$ *and* $(\phi', \xi', \phi')^*$ *respectively. Define the difference* H *between* ∇ *and* ∇' *by*

$$H(X, Y)=\nabla'_X Y-\nabla_X Y \qquad X, Y\in\Gamma(T\mathcal{M})\,. \tag{1.16}$$

Then we have

$$\begin{aligned} H(X, Y)=&\,p(X)Y+p(Y)X-g(X, Y)P \\ &+q(X)JY+q(Y)JX-g(JX, Y)Q\,, \end{aligned} \tag{1.17}$$

$$\begin{aligned} H(\xi, X)=&\,\nabla_{JX}P+\nabla_X Q-2q(X)P \\ &+2p(X)Q+2g(P, P)JX\,, \qquad X, Y\in\Gamma(\mathcal{D})\,. \end{aligned} \tag{1.18}$$

where $p=d\mu$ *and* $q=-p\circ\phi$.

Define B_0, $B_1 \in \Gamma(\mathcal{D}^{*3} \otimes \mathcal{D})$ by

$$B_0(X, Y)Z = R(X, Y)Z + l(Y, Z)X - l(X, Z)Y + m(Y, Z)JX - m(X, Z)JY + g(Y, Z)LX - g(X, Z)LY + g(JY, Z)MX - g(JX, Z)MY - 2\{m(X, Y)JZ + g(JX, Y)MZ\}, \tag{1.19}$$

$$B_1(X, Y)Z = \frac{1}{2}\{R(JX, JY)Z - R(X, Y)Z\}, \tag{1.20}$$

where R is the curvature tensor field of the canonical connection associated with $(\phi, \xi, \theta)^*$ and l, m, L, M are following:

$$l(X, Y) = -\frac{1}{2(n+1)} k(X, Y) + \frac{1}{8(n+1)(n+2)} \rho g(X, Y), \tag{1.21}$$

$$\underline{m}(X, Y) = -\frac{1}{2(n+1)} k(JX, Y) + \frac{1}{8(n+1)(n+2)} \rho g(JX, Y), \tag{1.22}$$

$$k(X, Y) = \frac{1}{2} \operatorname{trace}(\phi R(X, \phi Y)), \tag{1.23}$$

$$g(LX, Y) = l(X, Y) \qquad g(MX, Y) = m(X, Y). \tag{1.24}$$

The function ρ appearing in (1.21) and (1.22) is the trace of $S \in \Gamma(\mathcal{D}^* \otimes \mathcal{D})$ which is defined as follows.

$$g(SX, Y) = s(X, Y) = \operatorname{trace}(V \to R(V, X)Y). \tag{1.25}$$

LEMMA 1.4. *$B = B_0 + B_1$ is invariant under the change (1.12), i.e., $B = B'$ holds.*

Now we state a remark concerning to the normality of the almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$. In the definition (1.20), B_1 is also written as follows ([6]).

$$B_1(X, Y)Z = -\frac{1}{2}\{f(JY, Z)X - f(JX, Z)Y + f(Y, Z) - f(X, Z)JY + g(Y, Z)FJX - g(X, Z)FJY + g(JY, Z)FX - g(JX, Z)FY\}, \tag{1.26}$$

where f is defined by $g(FX, Y) = f(X, Y)$. We have already seen that the normality of the almost contact structure belonging to $(\mathcal{D}, J)$ is equivalent to the fact $F=0$ ([5]). In this case B_1 vanishes by (1.26), i.e., $R(X, Y)$ is hybrid. Conversely we assume $B_1(X, Y)Z=0$ for arbitrary $X, Y, Z \in \Gamma(\mathcal{D})$. From (1.26),

$$\operatorname{trace}\{X \to f(JY, Z)X - f(JX, Z)Y + f(Y, Z)JX - f(X, Z)JY + g(Y, Z)FJX - g(X, Z)FJY + g(JY, Z)FX - g(JX, Z)FY\} = 0$$

So we have

$$2(n-1)JF=-(\mathrm{trace}\,F)J. \tag{1.27}$$

As J is skewsymmetric and f is symmetric with respect to g [5], we get $F=0$ if $n\neq 1$. Thus we obtain

PROPOSITION 1.5. *Let $\mathcal{M}$ be a CR-manifold of dimension $2n+1$ $(n\neq 1)$. The almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$ is normal if and only if the curvature tensor with respect to the cononical connection is hybrid.*

And moreover from the fact that B_0 is hybrid tensor and B_1 is pure tensor i.e., that $B_0(JX, JY)=B_0(X, Y)$ and $B_1(JX, JY)=-B_1(X, Y)$, we obtain

PROPOSITION 1.6. *Let $\mathcal{M}$ be a CR-manifold of dimension $2n+1$ $(n\neq 1)$. If the curvature invariant B defined in Lemma 1.4 vanishes, then the almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$ is normal.*

§2. Invariant submanifolds of a CR-manifold.

Let $\bar{\mathcal{M}}$ be a connected orientable CR-manifold of dimension $2(n+\rho)+1$, $(\bar{\mathcal{D}}, \bar{J})$ be a nondegenerate CR-structure on $\bar{\mathcal{M}}$ with the condition (1.9), and $(\bar{\phi}, \bar{\xi}, \bar{\theta})$ be an almost contact structure belonging to $(\bar{\mathcal{D}}, \bar{J})$. Moreover we assume that the Levi-metric $\bar{g}$ defined on $\bar{\mathcal{D}}$ is positive definite.

Let $\mathcal{M}$ be a submanifold of $\bar{\mathcal{M}}$ of codimention $2p$, and we assume the vector field $\bar{\xi}$ is tangent to $\mathcal{M}$. For arbitrary point x of $\mathcal{M}$, we put

$$\mathcal{D}_x=T_x\mathcal{M}\cap\bar{\mathcal{D}}_x \tag{2.1}$$

and assume that the dimension of $\mathcal{D}_x$ is always $2n$ and that $\mathcal{D}_x$ is invariant by $\bar{J}$. Then we get a hyperdistribution $\mathcal{D}$ of $\mathcal{M}$ and moreover we put J to be the restriction of $\bar{J}$ to $\mathcal{D}$. Then M is called an *invariant submanifold* of $\bar{\mathcal{M}}$.

Next we put $\phi(X)=\bar{\phi}(X)$, $\theta(X)=\bar{\theta}(X)$ for $X\in\Gamma(T\mathcal{M})$ and put $\xi=\bar{\xi}$ on $\mathcal{M}$. And let g be the restriction of $\bar{g}$ on $\mathcal{D}$, i.e., the induced metric on $\mathcal{D}$. Then g is the Levi-metric on $\mathcal{D}$; because $g(X, Y)=\bar{g}(X, Y)=-2d\bar{\theta}(X, Y)=-2d\theta(X, Y)$ for $X, Y\in\Gamma(\mathcal{D})$. Thus we obtain

LEMMA 2.1. *The pair $(\mathcal{D}, J)$ is a nondegenerate CR-structure on $\mathcal{M}$ and the Levi-metric induced on $\mathcal{D}$ is positive definite.*

LEMMA 2.2. *The triplet (ϕ, ξ, θ) is an almost contact structure belonging to $(\mathcal{D}, J)$ and satisfies the condition* (1.9).

Now we study the connection on $\mathcal{M}$ induced from the canonical connection on $\bar{\mathcal{M}}$. Let $\bar{\nabla}$ be the canonical connection on $\bar{\mathcal{M}}$. And take the normal space as the orthogonal complement of $\mathcal{D}$ with respect to the Levi-metric $\bar{g}$. We write

(2.2) $$\bar{\nabla}_X Y=\nabla_X Y+\sigma(X, Y) \quad \text{for } X, Y\in\Gamma(T\mathcal{M}),$$

where $\nabla_X Y$ is the tangential component of $\bar{\nabla}_X Y$, that is, ∇ is the induced connection on $\mathcal{M}$ and σ is the second fundamental form of the submanifold $\mathcal{M}$. We note that $\sigma(X, Y)$ is contained in $\Gamma(\bar{\mathcal{D}})$ as $\bar{\xi}$ is tangent to $\mathcal{M}$. We study the properties of ∇ and σ.

First we obtain the next lemma concerning ξ, because $\nabla_X\xi=0$ for $X\in\Gamma(T\mathcal{M})$.

LEMMA 2.3.

(2.3) $$\nabla_X\xi=0,$$

(2.4) $$\sigma(X, \xi)=0 \quad \textit{for } X\in\Gamma(T\mathcal{M}).$$

Secondly by comparing the following two identities:

(2.5) $$\bar{\nabla}_X\bar{\phi}Y=\nabla_X\phi Y+\sigma(X, \phi Y),$$

(2.6) $$\bar{\phi}\bar{\nabla}_X Y=\phi\nabla_X Y+\bar{\phi}\sigma(X, Y) \quad \text{for } X, Y\in\Gamma(T\mathcal{M})$$

and taking account of $\bar{\nabla}\bar{\phi}=0$, we get

LEMMA 2.4.

(2.7) $$\nabla\phi=0,$$

(2.8) $$\sigma(X, \phi Y)=\bar{\phi}\sigma(X, Y) \quad \textit{for } X, Y\in\Gamma(T\mathcal{M}).$$

Because of

$$\begin{aligned}0=(\bar{\nabla}_X\bar{\theta})Y&=\bar{\nabla}_X(\bar{\theta}(Y))-\bar{\theta}(\bar{\nabla}_X Y)\\&=\nabla_X(\theta(Y))-\bar{\theta}(\nabla_X Y+\sigma(X, Y))\\&=\nabla_X(\theta(Y))-\theta(\nabla_X Y)=(\nabla_X\theta)Y\end{aligned}$$

for $X, Y\in\Gamma(T\mathcal{M})$, we get

(2.9) $$\nabla\theta=0.$$

Moreover from the fact that the torsion tensor field $\bar{T}(X, Y)$ of the canonical connection $\bar{\nabla}$ for $X, Y\in\Gamma(\mathcal{D})$ is

(2.10) $$\bar{T}(X, Y)=-\bar{\omega}(X, Y)\xi \quad \text{(cf. [5])},$$

we see that $\nabla_X Y-\nabla_Y X-[X, Y]+\sigma(X, Y)-\sigma(Y, X)=-\bar{\omega}(X, Y)\xi$. So we get

LEMMA 2.5.

(2.11) $$\nabla_X Y-\nabla_Y X-[X, Y]|_{\mathcal{D}}=0,$$

(2.12) $$\sigma(X, Y)=\sigma(Y, X) \quad \textit{for } X, Y\in\Gamma(\mathcal{D}).$$

Next, concerning to the tensor field $\bar{F}$ and F defined as (1.15) on $\bar{\mathcal{M}}$ and $\mathcal{M}$, we get $\bar{F}X=-(1/2)\phi([\xi, \phi X]-\phi[\xi, X])\in\Gamma(\mathcal{D})$ for $X\in\Gamma(\mathcal{D})$. So we obtain

LEMMA 2.6.

$$FX=\bar{F}X=-\frac{1}{2}(\phi\Gamma_{\xi}\phi)(X) \qquad \text{for } X\in\Gamma(\mathcal{D}), \tag{2.13}$$

$$\sigma(\xi, X)=0 \qquad \text{for } X\in\Gamma(T\mathcal{M}). \tag{2.14}$$

And from (2.3), (2.7), (2.9), (2.10) and (2.13), we get

PROPOSITION 2.7. *The induced connection ∇ on $\mathcal{M}$ is the canonical connection on $\mathcal{M}$ associated with $(\phi, \xi, \theta)^*$ and satisfies $F=-(1/2)\phi\mathcal{L}_{\xi}\phi$.*

From (2.13) we also have

PROPOSITION 2.8. *If the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\theta})^*$ belonging to $(\bar{\mathcal{D}}, \bar{J})$ is normal, then the almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathcal{D}, J)$ is also normal.*

Now we take orthonormal normal vector fields of the form $\{N_1, N_2, \cdots, N_p, \bar{J}N_1, \cdots, \bar{J}N_p\}$ in $\bar{\mathcal{D}}$ with respect to $\bar{g}$. If we set

$$g(A_{\alpha}X, Y)=\bar{g}(\sigma(X, Y), N_{\alpha}) \qquad \alpha=1, 2, \cdots, p \tag{2.15}$$

for $X, Y\in\Gamma(\mathcal{D})$, then we easily see from (2.8) that

$$A_{\alpha}J=-JA_{\alpha} \qquad \alpha=1, 2, \cdots, p \tag{2.16}$$

and A_{α} and JA_{α} are symmetric with respect to $\bar{g}$. In particular, trace A_{α} on $\mathcal{D}$ is zero, so considering with (2.13), we obtain

PROPOSITION 2.9. *The submanifold $\mathcal{M}$ is a minimal submanifold.*

To conclude of this section, we are going to give the equation of Gauss. The computation of the curvature tensor on the distribution is quite similar to the Kaehler case. However, in the process of the computation, we must notice

$$\begin{aligned}\bar{\nabla}_{[X,Y]}Z&=\bar{\nabla}_{(\nabla_X Y-\nabla_Y X+\omega(X,Y)\xi)}Z\\&=\nabla_{(\nabla_X Y-\nabla_Y X)}Z+\sigma(\nabla_X Y-\nabla_Y X, Z)+\omega(X, Y)\nabla_{\xi}Z\\&=\nabla_{[X,Y]}Z+\sigma(\nabla_X Y-\nabla_Y X, Z) \qquad \text{for any } X, Y, Z\in\Gamma(\mathcal{D}).\end{aligned}$$

After all we have

PROPOSITION 2.10. *For $X, Y, Z, W\in\Gamma(\mathcal{D})$, the following equation of Gauss holds:*

(2.17) $$\bar{g}(\bar{R}(X, Y)Z, W)-\bar{g}(R(X, Y)Z, W)$$
$$=\bar{g}(\sigma(X, Z), \sigma(Y, W))-\bar{g}(\sigma(X, W), \sigma(Y, Z)),$$

where $\bar{R}$ and R are the curvature tensors with respect to the canonical connections $\bar{\nabla}$ and ∇.

§ 3. Some theorems for the invariant submanifold.

Now we state some theorems concerning to the invariant submanifolds of a CR-manifold. To begin with we recall a theorem for a Kaehler submanifold proved by M. Kon ([3]).

THEOREM. *Let $\bar{\mathfrak{N}}$ be a Kaehler manifold of dimension $2m+2p$ with vanishing Bochner curvature tensor, and let $\mathfrak{N}$ be a Kaehler submanifold of $\bar{\mathfrak{N}}$ of codimension $2p$. If $p<(m+1)(m+2)(4m+2)$, then $\mathfrak{N}$ is totally geodesic in $\bar{\mathfrak{N}}$ if and only if the Bochner curvature tensor of $\mathfrak{N}$ vanishes.*

Here as in § 2, let $\bar{\mathcal{M}}$ be a CR-manifold with dimension $2(m+p)+1$ and with positive definite Levi-metric. And let $\mathcal{M}$ be an invariant submanifold of $\bar{\mathcal{M}}$ of codimention $2p$. Let $\bar{B}=\bar{B}_0+\bar{B}_1$ and $B=B_0+B_1$ are respectively the curvature invariant of $\bar{\mathcal{M}}$ and $\mathcal{M}$ with respect to their canonical connections. And we assume that $\bar{B}=0$. Then from Proposition 1.6 and (2.13) in Lemma 2.6, we can prove the following theorem by the similar method to the previous theorem.

THEOREM 3.1. *Let $\bar{\mathcal{M}}$ be a CR-manifold of dimension $2(m+p)+1$. Assume that the curvature invariant $\bar{B}$ defined in Lemma 1.4 of $\bar{\mathcal{M}}$ vanishes. And let $\mathcal{M}$ be an invariant submanifold of $\bar{\mathcal{M}}$ of codimension $2p$. If $p<(m+1)(m+2)\cdot(4m+2)$, then $\mathcal{M}$ is totally geodesic in $\bar{\mathcal{M}}$ if and only if the curvature invariant B of $\mathcal{M}$ vanishes.*

Now we moreover assume that the restriction of the Ricci tensor of $\bar{\mathcal{M}}$ to $\bar{\mathcal{D}}$ is proportional to the Levi-metric $\bar{g}$. Then $\bar{\mathcal{M}}$ is said to be $\bar{\mathcal{D}}$-Einstein. So by the definition

(3.1) $$\bar{s}(X, Y)=\frac{\bar{\rho}}{2n}\bar{g}(X, Y) \qquad \text{for } X, Y\in\Gamma(\bar{\mathcal{D}}).$$

In this case if the curvature invariant $\bar{B}$ of $\bar{\mathcal{M}}$ vanishes, then the curvature tensor with respect to the canonical connection is of the following form:

(3.2) $$\bar{R}(X, Y)Z=\frac{\bar{c}}{4}\{\bar{g}(Y, Z)X-\bar{g}(X, Z)Y$$
$$+\bar{g}(JY, Z)JX-\bar{g}(JX, Z)JY+2\bar{g}(X, JY)JZ\}$$
$$\text{for } X, Y, Z\in\Gamma(\bar{\mathcal{D}}),$$

where

$$\bar{c}=\frac{\bar{\rho}}{n(n+1)} \qquad n=m+p\,. \tag{3.3}$$

The curvature tensor $\bar{R}$ has the same form as the constant holomorphic sectional curvature. Then the equation of Gauss (2.17) is written as

$$\begin{aligned} g(R(X, Y)Z, W)=&\bar{g}(\sigma(X, W), \sigma(Y, Z))-\bar{g}(\sigma(X, Z), \sigma(Y, W)) \\ &+\frac{\bar{c}}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W) \\ &+g(JX, W)g(JY, Z)-g(JX, Z)g(JY, W) \\ &+2g(X, JY)g(JZ, W)\} \quad \text{for } X, Y, Z, W\in\Gamma(\mathcal{D})\,. \end{aligned} \tag{3.4}$$

Let s be the Ricci tensor and ρ be the scalar curvature on $\mathcal{D}$ defined in (1.25), then we have

$$s(X, Y)=(m+1)\frac{\bar{c}}{2}g(X, Y)-2\sum_{\alpha}g(A_{\alpha}X, A_{\alpha}Y)\,, \tag{3.5}$$

$$\rho=m(m+1)\bar{c}-\|\sigma\|^{2}\,, \tag{3.6}$$

where $\|\sigma\|^2$ is the length of the second fundamental form σ of the submanifold. In particular if we define $\mathcal{D}$-holomorphic sectional curvature H determined by a unit vector $X\in\Gamma(\mathcal{D})$ in the same way as the holomorphic sectional curvature in a Kaehler manifold, we obtain the next theorem.

THEOREM 3.2. *Let $\bar{\mathcal{M}}$ be a CR-manifold of dimension $2(m+p)+1$ with positive definite Levi-metric. Assume that the curvature invariant defined in Lemma 1.4 of $\bar{\mathcal{M}}$ vanishes. Moreover we assume that $\bar{\mathcal{M}}$ is $\bar{\mathcal{D}}$-Einstein in the sense of (3.1). And let $\mathcal{M}$ be an invariant submanifold of $\bar{\mathcal{M}}$ of codimension $2p$. Then in the notation above*

(a) $s-\dfrac{(m+1)\bar{c}}{2}g$ *is negative semi-definite*,

(b) $\rho\leqq m(m+1)\bar{c}$,

(c) $H\leqq\bar{c}$,

If we say $\mathcal{M}$ is $\mathcal{D}$-locally symmetric if $(\nabla_X R)(Y, Z)W=0$, for $X, Y, Z, W\in\Gamma(\mathcal{D})$, then we can state the following theorem as in [7], when $p=1$.

THEOREM 3.3. *Let $\bar{\mathcal{M}}$ and $\mathcal{M}$ be as above and $p=1$. If $\mathcal{M}$ is $\mathcal{D}$-Einstein, then it is $\mathcal{D}$-locally symmetric.*

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