Y. KUBOTA
KODAI MATH. J.
5 (1982), 218—224

AN EXTREMAL PROBLEM ON THE CLASSICAL CARTAN DOMAINS, II

BY YOSHIHISA KUBOTA

1. Let D be a bounded domain in C^n and denote by $\mathcal{F}(D)$ the family of holomorphic mappings from D into the unit hyperball B_n in C^n. For a mapping f in $\mathcal{F}(D)$ we denote by $(\partial f/\partial z)$ the Jacobian matrix of f:

$$\left(\frac{\partial f}{\partial z}\right)=\begin{pmatrix} \dfrac{\partial f_1}{\partial z_1}\cdots\dfrac{\partial f_1}{\partial z_n} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial f_n}{\partial z_1}\cdots\dfrac{\partial f_n}{\partial z_n}\end{pmatrix}, \qquad f=(f_1, \cdots, f_n).$$

In [3] we were concerned with the problem of maximizing $|\det(\partial f/\partial z)_{z=z_0}|$ for $f\in\mathcal{F}(D)$, where z_0 is a fixed point in D, and we found the precise value

$$M(0, D)=\sup_{f\in\mathcal{F}(D)}\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|$$

for classical Cartan domains. In this paper we shall find the value $M(0, D)$ for products of classical Cartan domains.

By a classical Cartan domain we understand a domain of one of the following four types:

$$R_{\mathrm{I}}(r, s)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0,\ \text{where } Z \text{ is an } r\times s \text{ matrix}\},\ (r\leqq s),$$

$$R_{\mathrm{II}}(p)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0,\ \text{where } Z \text{ is a symmetric matrix of order } p\},$$

$$R_{\mathrm{III}}(q)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0,\ \text{where } Z \text{ is a skew-symmetric matrix of order } q\},$$

$$R_{\mathrm{IV}}(m)=\{z=(z_{11}, \cdots, z_{1m}):\ 1+|zz'|^2-2z\bar{z}'>0,\ 1-|zz'|>0\}.$$

Instead of $R_{\mathrm{II}}(p)$ we consider the following modified domain:

$$\hat{R}_{\mathrm{II}}(p)=\{Z=(z_{ij}):\ z_{ij}=\sqrt{2}\,x_{ij}\ (i\neq j),\ z_{ii}=x_{ii},\ \text{where } X=(x_{ij})\in R_{\mathrm{II}}(p)\}.$$

Received November 19, 1980

Let D be a classical Cartan domain. We set

$$n_D=\begin{cases} rs , & \text{if } D=R_{\mathrm{I}}(r, s), \\ \dfrac{p(p+1)}{2}, & \text{if } D=\hat{R}_{\mathrm{II}}(p), \\ \dfrac{q(q-1)}{2}, & \text{if } D=R_{\mathrm{III}}(q), \\ m , & \text{if } D=R_{\mathrm{IV}}(m), \end{cases}$$

and

$$\lambda_D=\begin{cases} \sqrt{s} , & \text{if } D=R_{\mathrm{I}}(r, s), \\ \sqrt{\dfrac{p+1}{2}}, & \text{if } D=\hat{R}_{\mathrm{II}}(p), \\ \sqrt{q-1} , & \text{if } D=R_{\mathrm{III}}(q) \text{ and } q \text{ is even}, \\ \sqrt{q} , & \text{if } D=R_{\mathrm{III}}(q) \text{ and } q \text{ is odd}, \\ \sqrt{m} , & \text{if } D=R_{\mathrm{IV}}(m). \end{cases}$$

Further we denote by λD the set $\{\lambda z : z\in D\}$.

We shall prove the following theorem.

THEOREM. *If $D_1, \cdots, D_N$ are classical Cartan domains and $D=\lambda_{D_1}D_1\times\cdots\times\lambda_{D_N}D_N$, then*

$$M(0, D)=n^{-n/2},$$

where $n=n_{D_1}+\cdots+n_{D_N}$.

2. Proof of Theorem.

We represent the points z in D in the form of vectors in $\boldsymbol{C}^n$

$$z=(z^{(1)}, \cdots, z^{(N)}), \qquad z^{(\nu)}\in\lambda_{D_\nu}D_\nu \quad (\nu=1, \cdots, N),$$

where

$$z^{(\nu)}=\begin{cases} (z_{11}^{(\nu)}, \cdots, z_{1s}^{(\nu)}, \cdots, z_{r1}^{(\nu)}, \cdots, z_{rs}^{(\nu)}) , & \text{if } D_\nu=R_{\mathrm{I}}(r, s), \\ (z_{11}^{(\nu)}, \cdots, z_{1p}^{(\nu)}, z_{22}^{(\nu)}, \cdots, z_{2p}^{(\nu)}, \cdots, z_{pp}^{(\nu)}) , & \text{if } D_\nu=\hat{R}_{\mathrm{II}}(p), \\ (z_{12}^{(\nu)}, \cdots, z_{1q}^{(\nu)}, z_{23}^{(\nu)}, \cdots, z_{2q}^{(\nu)}, \cdots, z_{q-1\,q}^{(\nu)}), & \text{if } D_\nu=R_{\mathrm{III}}(q), \\ (z_{11}^{(\nu)}, \cdots, z_{1m}^{(\nu)}) , & \text{if } D_\nu=R_{\mathrm{IV}}(m). \end{cases}$$

Let f be a mapping in $\mathcal{F}(D)$. We set

$$f=(f^{(1)}, \cdots, f^{(N)}),$$

where

$$f^{(\mu)}=\begin{cases}(f_{11}^{(\mu)}, \cdots, f_{1s}^{(\mu)}, \cdots, f_{r1}^{(\mu)}, \cdots, f_{rs}^{(\mu)}) & , \text{ if } D_\mu=R_{\mathrm{I}}(r, s), \\ (f_{11}^{(\mu)}, \cdots, f_{1p}^{(\mu)}, f_{22}^{(\mu)}, \cdots, f_{2p}^{(\mu)}, \cdots, f_{pp}^{(\mu)}) & , \text{ if } D_\mu=\hat{R}_{\mathrm{II}}(p), \\ (f_{12}^{(\mu)}, \cdots, f_{1q}^{(\mu)}, f_{23}^{(\mu)}, \cdots, f_{2q}^{(\mu)}, \cdots, f_{q-1q}^{(\mu)}) & , \text{ if } D_\mu=R_{\mathrm{III}}(q), \\ (f_{11}^{(\mu)}, \cdots, f_{1m}^{(\mu)}) & , \text{ if } D_\mu=R_{\mathrm{IV}}(m).\end{cases}$$

As we noted in [3], we may assume that $f(0)=0$. Then $f_{kl}^{(\mu)}$ has the expansion

$$f_{kl}^{(\mu)}(z)=\sum_{\nu, i, j} a_{ij}^{(\nu)}(\mu; k, l)z_{ij}^{(\nu)}+(\text{higher powers})$$

in a neighborhood of the origin $z=0$. Further we may assume that $(\partial f/\partial z)_{z=0}$ is a triangular matrix of order n, namely, $a_{ij}^{(\nu)}(\mu; k, l)=0$, if $\nu<\mu$, if $\nu=\mu$ and $i<k$, or if $\nu=\mu$, $i=k$ and $j<l$. Hence we have

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=\prod_{\nu, i, j}|a_{ij}^{(\nu)}(\nu; i, j)|.$$

In the following we prove the inequality

$$\sum_{\nu, i, j}|a_{ij}^{(\nu)}(\nu; i, j)|\leqq\sqrt{n}. \tag{1}$$

Firstly we consider the modified mapping $\tilde{f}=(\tilde{f}^{(1)}, \cdots, \tilde{f}^{(N)})$ which is defined in the following mannner:

If $D_\nu=R_{\mathrm{I}}(r, s)$, then we define $\tilde{f}^{(\nu)}=f^{(\nu)}$.

If $D_\nu=\hat{R}_{\mathrm{II}}(p)$, then we define $\tilde{f}^{(\nu)}=(\tilde{f}_{11}^{(\nu)}, \cdots, \tilde{f}_{1p}^{(\nu)}, \cdots, \tilde{f}_{p1}^{(\nu)}, \cdots, \tilde{f}_{pp}^{(\nu)})$, where

$$\tilde{f}_{ij}^{(\nu)}=\tilde{f}_{ji}^{(\nu)}=\frac{1}{\sqrt{2}}f_{ij}^{(\nu)} \qquad (i<j), \qquad \tilde{f}_{ii}^{(\nu)}=f_{ii}^{(\nu)}.$$

If $D_\nu=R_{\mathrm{III}}(q)$, then we define $\tilde{f}^{(\nu)}=(\tilde{f}_{11}^{(\nu)}, \cdots, \tilde{f}_{1q}^{(\nu)}, \cdots, \tilde{f}_{q1}^{(\nu)}, \cdots, \tilde{f}_{qq}^{(\nu)})$, where

$$\tilde{f}_{ij}^{(\nu)}=\tilde{f}_{ji}^{(\nu)}=\frac{1}{\sqrt{2}}f_{ij}^{(\nu)} \qquad (i<j), \qquad \tilde{f}_{ii}^{(\nu)}=0.$$

If $D_\nu=R_{\mathrm{IV}}(m)$, then we define $\tilde{f}^{(\nu)}=f^{(\nu)}$.

The mapping $\tilde{f}$ maps D into the unit hyperball $B_{\tilde{n}}$ in $\boldsymbol{C}^{\tilde{n}}$, where $\tilde{n}=\tilde{n}_{D_1}+\cdots+\tilde{n}_{D_N}$,

$$\tilde{n}_{D_\nu}=\begin{cases}rs, & \text{if } D_\nu=R_{\mathrm{I}}(r, s), \\ p^2, & \text{if } D_\nu=\hat{R}_{\mathrm{II}}(p), \\ q^2, & \text{if } D_\nu=R_{\mathrm{III}}(q), \\ m, & \text{if } D_\nu=R_{\mathrm{IV}}(m).\end{cases}$$

Next we consider mappings σ_ν defined as follows:

If $D_\nu=R_{\mathrm{I}}(r, s)$, σ_ν is a one-to-one mapping from $\{1, \cdots, r\}$ into $\{1, \cdots, s\}$.

If $D_\nu=\hat{R}_{\mathrm{II}}(p)$ and p is even, σ_ν is a one-to-one mapping from $\{1, \cdots, p\}$ onto itself such that

$$\sigma_\nu(i)\neq i, \qquad \sigma_\nu\circ\sigma_\nu(i)=i \quad (i=1, \cdots, p),$$

or

$$\sigma_\nu(i)=i \qquad (i=1, \cdots, p).$$

If $D_\nu=\hat{R}_{\mathrm{II}}(p)$ and p is odd, σ_ν is a one-to-one mapping from $\{1, \cdots, p\}$ onto itself such that

$$\sigma_\nu(i_0)=i_0; \quad \sigma_\nu(i)\neq i, \quad \sigma_\nu\circ\sigma_\nu(i)=i \quad (i\neq i_0,\ 1\leqq i\leqq p)$$

for some i_0, or

$$\sigma_\nu(i)=i \qquad (i=1, \cdots, p).$$

If $D_\nu=R_{\mathrm{III}}(q)$ and q is even, σ_ν is a one-to-one mapping from $\{1, \cdots, q\}$ onto itself such that

$$\sigma_\nu(i)\neq i, \qquad \sigma_\nu\circ\sigma_\nu(i)=i \qquad (i=1, \cdots, q).$$

If $D_\nu=R_{\mathrm{III}}(q)$ and q is odd, σ_ν is a one-to-one mapping from $\{1, \cdots, q\}$ onto itself such that

$$\sigma_\nu(i_0)=i_0; \quad \sigma_\nu(i)\neq i, \quad \sigma_\nu\circ\sigma_\nu(i)=i \quad (i\neq i_0,\ 1\leqq i\leqq q)$$

for some i_0.

If $D_\nu=R_{\mathrm{IV}}(m)$, σ_ν is a mapping from $\{1\}$ into $\{1, \cdots, m\}$.

Further we consider a unitary matrix $V=(v_{\alpha\beta})$ of order $\tilde{n}$ such that $v_{1\beta}=\rho_\mu\varepsilon_k^{(\mu)}$ for $\beta=\sum_{\nu=1}^{\mu-1}\tilde{n}_{D_\nu}+(k-1)\eta_\mu+\sigma_\mu(k)$ and otherwise $v_{1\beta}=0$, where $\varepsilon_k^{(\mu)}=0$, if $D_\mu=R_{\mathrm{III}}(q)$ (q is odd) and $\sigma_\mu(k)=k$, and otherwise $|\varepsilon_k^{(\mu)}|=1$, and ρ_μ, η_μ are numbers defined as follows:

$$\rho_\mu=\begin{cases} \sqrt{\dfrac{s}{n}}, & \text{if } D_\mu=R_{\mathrm{I}}(r, s), \\ \sqrt{\dfrac{p+1}{2n}}, & \text{if } D_\mu=\hat{R}_{\mathrm{II}}(p), \\ \sqrt{\dfrac{q-1}{2n}}, & \text{if } D_\mu=R_{\mathrm{III}}(q) \text{ and } q \text{ is even}, \\ \sqrt{\dfrac{q}{2n}}, & \text{if } D_\mu=R_{\mathrm{III}}(q) \text{ and } q \text{ is odd}, \\ \sqrt{\dfrac{m}{n}}, & \text{if } D_\mu=R_{\mathrm{IV}}(m), \end{cases}$$

$$\eta_\mu=\begin{cases} s, & \text{if } D_\mu=R_{\mathrm{I}}(r, s), \\ p, & \text{if } D_\mu=\hat{R}_{\mathrm{II}}(p), \\ q, & \text{if } D_\mu=R_{\mathrm{III}}(q), \\ m, & \text{if } D_\mu=R_{\mathrm{IV}}(m). \end{cases}$$

Denote by ϕ the automorphism of $B_{\tilde{n}}$ defined by $\phi(w)=wV'$, $w\in \boldsymbol{C}^{\tilde{n}}$. The mapping

$$g=\phi\circ\tilde{f}=(g_1, \cdots, g_{\tilde{n}})$$

maps D into $B_{\tilde{n}}$. Since

$$g_1=\sum_{\mu, k}\rho_\mu\varepsilon_k^{(\mu)}\tilde{f}_{k\sigma_\mu(k)}^{(\mu)},$$

we have the expansion

$$g_1(z)=\sum_{\nu, i, j}b_{ij}^{(\nu)}z_{ij}^{(\nu)}+(\text{higher powers}),$$

$$b_{ij}^{(\nu)}=\sum_{\mu, k}\rho_\mu\varepsilon_k^{(\mu)}\tilde{a}_{ij}^{(\nu)}(\mu;\, k,\, \sigma_\mu(k)) \tag{2}$$

$$=\sum_{\mu=1}^{\nu}\rho_\mu\{\sum_k\varepsilon_k^{(\mu)}\tilde{a}_{ij}^{(\nu)}(\mu;\, k,\, \sigma_\mu(k))\},$$

where

$$\tilde{f}_{kl}^{(\mu)}(z)=\sum_{\nu, i, j}\tilde{a}_{ij}^{(\nu)}(\mu;\, k,\, l)z_{ij}^{(\nu)}+(\text{higher powers}).$$

Let ζ be a complex number with $|\zeta|<1$. We take a point $z=(z^{(1)}, \cdots, z^{(N)})$ in $\boldsymbol{C}^n$ such that $z_{ij}^{(\nu)}=\delta_i^{(\nu)}\tau(D_\nu, i)\zeta$ for $j=\sigma_\nu(i)$ and otherwise $z_{ij}^{(\nu)}=0$, where $|\delta_i^{(\nu)}|=1$ and $\tau(D_\nu, i)<\sqrt{2}\,\lambda_{D_\nu}$, if $D_\nu=\hat{R}_{\mathrm{II}}(p)$ and $i<\sigma_\nu(i)$, and otherwise $\tau(D_\nu, i)=\lambda_{D_\nu}$. Here we note that we consider only i such that $i\leqq\sigma_\nu(i)$ in the case $D_\nu=\hat{R}_{\mathrm{II}}(p)$ and $i<\sigma_\nu(i)$ in the case $D_\nu=R_{\mathrm{III}}(q)$. Obviously z belongs to D. Hence the function

$$h(\zeta)=g_1(z)=\{\sum_{\nu=1}^{N}\sum_i b_{i\sigma_\nu(i)}^{(\nu)}\delta_i^{(\nu)}\tau(D_\nu, i)\}\zeta+(\text{higher powers})$$

is holomorphic in $|\zeta|<1$ and satisfies the conditions $|h(\zeta)|<1$, $h(0)=0$. Therefore, by Schwarz lemma we have

$$|\sum_{\nu=1}^{N}\sum_i b_{i\sigma_\nu(i)}^{(\nu)}\delta_i^{(\nu)}\tau(D_\nu, i)|\leqq 1. \tag{3}$$

Since $\varepsilon_k^{(\mu)}$ and $\delta_i^{(\nu)}$ are arbitrary, we obtain, by (2) and (3),

$$\sum_{\nu=1}^{N}A_\nu\leqq\sqrt{n}, \tag{4}$$

where A_ν is the number determined as follows:

If $D_\nu=R_{\mathrm{I}}(r, s)$, then

$$A_\nu=s\sum_{i=1}^{r}|a_{i\sigma_\nu(i)}^{(\nu)}(\nu;\, i,\, \sigma_\nu(i))|\cdot$$

If $D_\nu=\hat{R}_{\mathrm{II}}(p)$ and $\sigma_\nu(i)=i$ $(i=1, \cdots, p)$, then

$$A_\nu=\frac{p+1}{2}\sum_{i=1}^{p}|a_{ii}^{(\nu)}(\nu;\, i,\, i)|.$$

If $D_\nu=\hat{R}_{\rm II}(p)$ and p is even and if $\sigma_\nu(i)\neq i$, $\sigma_\nu\circ\sigma_\nu(i)=i$ $(i=1, \cdots, p)$, then

$$A_\nu=(p+1)\sum_{\alpha=1}^{t}|a^{(\nu)}_{i_\alpha\sigma_\nu(i_\alpha)}(\nu;i_\alpha,\sigma_\nu(i_\alpha))|,$$

where $p=2t$ and $1=i_1<\cdots<i_t<p$, $i_\alpha<\sigma_\nu(i_\alpha)$ $(\alpha=1, \cdots, t)$.

If $D_\nu=\hat{R}_{\rm II}(p)$ and p is odd and if $\sigma_\nu(i_0)=i_0$; $\sigma_\nu(i)\neq i$, $\sigma_\nu\circ\sigma_\nu(i)=i$ $(i\neq i_0, 1\leqq i\leqq p)$, then

$$A_\nu=\frac{p+1}{2}|a^{(\nu)}_{i_0i_0}(\nu;i_0,i_0)|+(p+1)\sum_{\alpha=1}^{t}|a^{(\nu)}_{i_\alpha\sigma_\nu(i_\alpha)}(\nu;i_\alpha,\sigma_\nu(i_\alpha))|,$$

where $p=2t+1$ and $1\leqq i_1<\cdots<i_t<p$, $i_\alpha<\sigma_\nu(i_\alpha)$ $(\alpha=1, \cdots, t)$.

If $D_\nu=R_{\rm III}(q)$ and q is even and if $\sigma_\nu(i)\neq i$, $\sigma_\nu\circ\sigma_\nu(i)=i$ $(i=1, \cdots, q)$, then

$$A_\nu=(q-1)\sum_{\alpha=1}^{t}|a^{(\nu)}_{i_\alpha\sigma_\nu(i_\alpha)}(\nu;i_\alpha,\sigma_\nu(i_\alpha))|,$$

where $q=2t$ and $1=i_1<\cdots<i_t<q$, $i_\alpha<\sigma_\nu(i_\alpha)$ $(\alpha=1, \cdots, t)$.

If $D_\nu=R_{\rm III}(q)$ and q is odd and if $\sigma_\nu(i_0)=i_0$; $\sigma_\nu(i)\neq i$, $\sigma_\nu\circ\sigma_\nu(i)=i$ $(i\neq i_0, 1\leqq i\leqq q)$, then

$$A_\nu=q\sum_{\alpha=1}^{t}|a^{(\nu)}_{i_\alpha\sigma_\nu(i_\alpha)}(\nu;i_\alpha,\sigma_\nu(i_\alpha))|,$$

where $q=2t+1$ and $1\leqq i_1<\cdots<i_t<q$, $i_\alpha<\sigma_\nu(i_\alpha)$ $(\alpha=1, \cdots, t)$.

If $D_\nu=R_{\rm IV}(m)$, then

$$A_\nu=m|a^{(\nu)}_{1\sigma_\nu(1)}(\nu;1,\sigma_\nu(1))|.$$

Now, from this inequality we can prove (1). Indeed, for instance, if $D_1=\hat{R}_{\rm II}(p)$ and p is odd, by taking appropriate p mappings σ from $\{1, \cdots, p\}$ onto itself such that $\sigma(i_0)=i_0$ for some i_0 and $\sigma(i)\neq i$, $\sigma\circ\sigma(i)=i$ for the other i's as σ_1 in (4), we have

$$\frac{p+1}{2}\sum_{i=1}^{p}|a^{(1)}_{ii}(1;i,i)|+(p+1)\sum_{i<j}|a^{(1)}_{ij}(1;i,j)|+p\sum_{\nu=2}^{N}A_\nu\leqq p\sqrt{n}.$$

Further, by taking the identity mapping as σ_1 in (4), we have

$$\frac{p+1}{2}\sum_{i=1}^{p}|a^{(1)}_{ii}(1;i,i)|+\sum_{\nu=2}^{N}A_\nu\leqq\sqrt{n}.$$

Hence, from these two inequalities, we obtain

$$\sum_{i\leqq j}|a^{(1)}_{ij}(1,i,j)|+\sum_{\nu=2}^{N}A_\nu\leqq\sqrt{n}.$$

By a similar argument, in general, we obtain

$$\sum|a^{(1)}_{ij}(1;i,j)|+\sum_{\nu=2}^{N}A_\nu\leqq\sqrt{n}. \tag{5}$$

Similarly, from (5) we have the inequality

$$\sum|a_{ij}^{(1)}(1;i,j)|+\sum|a_{ij}^{(2)}(2;i,j)|+\sum_{\nu=3}^{N}A_\nu\leqq\sqrt{n}\,,$$

and so on we obtain (1). Immediately inequality (1) implies that

$$M(0,D)\leqq n^{-n/2}\,.$$

On the other hand, we showed in [3] that

$$R_{\mathrm{I}}(r,s)\subset\sqrt{r}\,B_k,\quad (k=rs);\quad \hat{R}_{\mathrm{II}}(p)\subset\sqrt{p}\,B_k,\quad \left(k=\frac{p(p+1)}{2}\right);$$

$$R_{\mathrm{III}}(q)\subset\sqrt{\left[\frac{q}{2}\right]}\,B_k,\quad \left(k=\frac{q(q-1)}{2}\right);\quad R_{\mathrm{IV}}(m)\subset B_m\,.$$

Hence

$$D\subset\sqrt{n}\,B_n\,.$$

Thus we complete the proof of theorem.

REFERENCES

[1] CARATHÉODORY, C., Über die Abbildungen, die durch Systeme von analytischen Funktionen von mehreren Veranderlichen erzeugt werden. Math. Z. **34** (1932), 758-792.

[2] HUA, L.K., Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains. Amer. Math. Soc., Providence, Rhode Island, 1963.

[3] KUBOTA, Y., An extremal problem on the classical Cartan domains. Kodai Math. J. **4** (1980), 278-287.

DEPARTMENT OF MATHEMATICS
TOKYO GAKUGEI UNIVERSITY