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RATIONAL APPROXIMATION AND SWISS CHEESES OF POSITIVE AREA

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Abstract

Let J and K be two compact sets in the complex plane such that $K\backslash J$ has zero planar measure. If $R(J)=C(J)$ then $R(K)=C(K)$. This result is used to produce many Swiss cheeses K of positive area, for which $R(K)=C(K)$.

For any compact set K in the complex plane, let $C(K)$ and $R(K)$ denote, respectively, the algebra of continuous functions on K, and the subalgebra of functions which are uniformly approximable on K by rational functions with poles off K. Hartogs and Rosenthal proved in [2] that if $m_2(K)=0$ (where m_2 denotes planar Lebesgue measure), then $R(K)=C(K)$. We extend this theorem here, and apply it to get new examples of Swiss cheeses K with $R(K)=C(K)$, yet $m_2(K)>0$.

THEOREM. *Let J and K be compact sets such that $m_2(K\backslash J)=0$. If $R(J)=C(J)$ then $R(K)=C(K)$.*

The proof of this result depends on the following. Let μ be a finite measure with compact support in the complex plane. The Cauchy transform of μ is defined by $\mu\hat{}(w)=\int(z-w)^{-1}d\mu(z)$. It is the convolution of μ with the locally integrable function $1/z$. So the integral defining $\mu\hat{}$ converges absolutely except for w belonging to a set of zero planar measure. Clearly, $\mu\hat{}$ is analytic off the closed support of μ. A converse of this statement is true.

PROPOSITION 1. (See [1], Theorem 8.2.) *Let μ be a finite measure of compact support in the plane. Suppose U is an open set, and f is a function analytic on U such that $f=\mu\hat{}$ almost everywhere with respect to m_2 on U. Then $|\mu|(U)=0$.*

Proof of Theorem 1. We show that any measure μ with support in K which is orthogonal to $R(K)$ must be the zero measure. In Proposition 1, set $f\equiv0$ and $U=CJ$. Since $\mu\perp R(K)$, $\mu\hat{}=0$ on CK. Since $m_2(K\backslash J)=0$, we have $\mu\hat{}=f$ almost

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everywhere with respect to m_2 on U because $U=CJ=(K\setminus J)\cup(CJ\cap CK)$. Thus the support of μ is contained in J. Also, $\hat{\mu}=0$ throughout CJ because $CK\cap CJ$ is a dense open subset of CJ. So $\mu\perp R(J)$, and thus $\mu=0$ because $R(J)=C(J)$, and the theorem is proved.

We shall consider some special cases of the theorem. If we take J to be a singleton, then we get the theorem of Hartogs-Rosenthal by a different proof from theirs. In another direction, we will construct a Swiss cheese K with $m_2(K)>0$, yet $R(K)=C(K)$. Such an example, based on different ideas, was given by C. R. Putnam in [3]. Let us be more detailed.

If D_n, $n=1, 2, 3, \cdots$ are open discs contained in the unit disc D, with the D_n having disjoint closures, and with $\cup D_n$ dense in D, then $K=D\setminus\cup D_n$ is called a "Swiss cheese." Certain Swiss cheeses provide the simplest examples of compact sets K with empty interior for which $R(K)\neq C(K)$. These are the ones for which $\Sigma r_n<\infty$, where r_n is the radius of D_n. The Hartogs-Rosenthal theorem implies on the other hand that if $m_2(K)=0$, then $R(K)=C(K)$. Putnam in [3] extended this result to show that if K is a compact set which is "areally disconnected," then $R(K)=C(K)$. A corollary to this result if that is there exists a set of real numbers $\{t\}$ dense on the real line for which each of the vertical lines $\mathrm{Re}(z)=t$ intersects K on a set of zero linear measure, then $R(K)=C(K)$. Using this corollary he constructs a Swiss cheese so that $m_2(K)>0$ and $R(K)=C(K)$.

Here is how we can use our Theorem to produce many other such examples. Let J be any compact subset of D such that $m_2(J)>0$ and $R(J)=C(J)$. For example, J could be an arc of positive area or a Cantor set of positive area, in which cases Mergelyan's Theorem (see [1], Theorem 9.1) shows that $P(J)=C(J)$ so that $R(J)=C(J)$. (Here $P(J)$ is the class of functions uniformly approximable on J by polynomials.) Now just construct the Swiss cheese K so that $K\supseteqq J$ and $m_2(K\setminus J)=0$. This can easily be achieved by proper choice of the D_n. Clearly, $m_2(K)>0$, and yet our Theorem implies that $R(K)=C(K)$.

Remark. By a slight variation of the above proof, one can prove the following. Let $A(K)$ be the algebra of continuous functions on K that are analytic in the interior of K. Suppose now that J and K are compact sets with $m_2(K\setminus J)=0$ and so that $J\setminus K$ has empty interior. If $A(J)=R(J)$, then it follows that $A(K)=R(K)$.

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