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KODAI MATH. J.
5 (1982), 122—131

EXISTENCE OF QUASICONFORMAL MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

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Introduction.

In 1960 the first named author [8] proved that two Riemann surfaces are quasiconformally equivalent if and only if their Royden algebras are isomorphic. This result was extended to higher dimensions: to higher dimensional Euclidean domains by L. G. Lewis [6] and to Riemannian manifolds by J. Lelong-Ferrand [5]. These results show that if two Riemannian manifolds M and N are quasiconformally equivalent, then their Royden compactifications M^* and N^* are homeomorphic. The question aries whether the converse is true, that is, whether a homeomorphism from M^* to N^* can always be raised to a quasiconformal mapping from M to N.

In this paper we shall prove that the question is true in a neighborhood of ideal boundary of M, that is, if there is a homeomorphism f of M^* onto N^*, then there exists a compact subset K of M such that the restriction of f to each component of $M-K$ is quasiconformal. Furthermore, for Riemann surfaces, we can find a quasiconformal mapping from M to N. However we do not know whether this is valid for higher dimensional cases.

Notation and terminology

We denote by R^n the n-dimensional Euclidean space whose points x are n-tuple $x=(x_1, x_2, \cdots, x_n)$ of real numbers ($n \geqq 1$). The distance between $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ is denoted by

$$|x-y|=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{1/2}.$$

We denote by ω_{n-1} the $(n-1)$-dimensional Lebesgue measure of the unit sphere $\{x \in R^n ; |x|=1\}$.

1. Riemannian manifolds

Let M be a connected separable, orientable n-idmensional ($n \geqq 2$) differentiable manifold of class C^1 with fundamental metric tensor

Received October 15, 1980.

$$G=\begin{pmatrix} g_{11} \cdots g_{1n} \\ \cdots\cdots \\ g_{n1} \cdots g_{nn} \end{pmatrix}$$

satisfying the following conditions:

In each parametric ball or cube $B=(B, \phi)$ with local parameter $\phi(p)=(x^1, \cdots, x^n)$ $(p\in B)$, the local expressions $g_{ij}(x)$ of g_{ij} $(i, j=1, \cdots, n)$ are continuous functions of $x=(x^1, \cdots, x^n)$ in $\phi(B)$ and there exists a finite constant $k_B\geqq 1$ such that

$$(1)\qquad k_B{}^{-1}\cdot \sum_{i=1}^{n}(\xi^i)^2\leqq \sum_{i,j=1}^{n} g_{ij}(x)\xi^i\xi^j\leqq k_B\cdot \sum_{i=1}^{n}(\xi^i)^2$$

for every x in $\phi(B)$ and for every n-tuple $(\xi^1, \cdots, \xi^n)$ of real numbers. We can, therefore, consider the inverse matrix G^{-1} of G. We set

$$G^{-1}=\begin{pmatrix} g^{11} \cdots g^{1n} \\ \cdots\cdots \\ g^{n1} \cdots g^{nn} \end{pmatrix}, \qquad g=\det G\,.$$

Then it is known that

$$(2)\qquad k_B{}^{-1}\cdot \sum_{i=1}^{n}(\eta_i)^2\leqq \sum_{i,j=1}^{n} g^{ij}(x)\eta_i\eta_j\leqq k_B\cdot \sum_{i=1}^{n}(\eta_i)^2$$

for every n-tuple $(\eta_1, \cdots, \eta_n)$ of real numbers and that

$$(3)\qquad k_B{}^{-n}\leqq g\leqq k_B{}^{n}\,.$$

In terms of local parameter $x=(x^1, \cdots, x^n)$, the line element ds on M is given by $ds^2=\sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j$, and since $g_{ij}(x)$ are continuous, the line integral $\int_\gamma ds$ along a rectifiable curve γ in M can be defined. Therefore the natural distance $d_M(p, q)$ of two points p and q in M is given by

$$d_M(p, q)=\inf \int_\gamma ds$$

where the infinimum is taken with respect to all rectifiable curves γ in M joining p and q.

We can find a covering $\{B\}$ of M consisting of local parametric balls or cubes B and a constant $\tau_M\in(1, \infty)$ such that

$$1\leqq k_B\leqq \tau_M$$

for every B of the covering. Thus we fix such a covering $\{B\}$ of a manifold M and a constant τ_M once for all.

By the aid of (1) we have the following lemma.

LEMMA 1 (cf. [5]). *If $B=(B, \phi)$ is a parametric ball on M, then*

$$(k_B)^{-1/2}|\phi(q)-\phi(p)|\leqq d_M(p, q)\leqq (k_B)^{1/2}|\phi(q)-\phi(p)|$$

for all p and q in B.

In particular, if $\phi(p)=0$, then, for sufficiently small $r, t>0$, we have

(i) $d_M(p, q)=r$ *implies* $k_B^{-1/2}r \leqq |\phi(q)| \leqq k_B^{1/2}r$,

(ii) $|x|=t$ *implies* $k_B^{-1/2}t \leqq d_M(p, \phi^{-1}(x)) \leqq k_B^{1/2}t$.

2. *ACL* functions and Dirichlet integrals

A continuous function f defined on a cube $D: a^i<x^i<b^i$ $(i=1, \cdots, n)$ in R^n is said to be absolutely continuous on lines (abbreviated as ACL) if it is absolutely continuous on almost all line segments parallel to coordinate axes; explicitly, if we denote by D_i the face of D given by $x^i=a^i$, then the function $f(\xi+\eta e_i)$, $e_i=(\delta^{i1}, \cdots, \delta^{in})$, is absolutely continuous in $\eta\in(a^i, b^i)$ for almost all $\xi\in D_i$ with respect to the $(n-1)$-dimensional Lebesgue measure $(i=1, \cdots, n)$. Let G be a domain in R^n and f be a funciton defined on G. Then f is said to be ACL if the restriction $f|D$ of f to D is ACL for all cubes D contained in G.

A function f defined on a parametric ball $B=(B, \phi)$ on M is said to be ACL if $f\circ\phi^{-1}$ is ACL in $\phi(B)$. Furthermore a function f defined on M is said to be ACL if the restriction $f|B$ of f to B is ACL for all parametric balls B on M. For such a function f on M the Dirichlet integral $D_M(f)$ of f is defined by

$$D_M(f)=\int\cdots\int_M\Big(\sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial f}{\partial x^i}(x)\frac{\partial f}{\partial x^j}(x)\Big)^{n/2}\sqrt{g(x)}\,dx^1\cdots dx^n. \tag{4}$$

It may or may not be finite.

For an ACL function f defined on a domain G in R^n we define the Euclidean Dirichlet integral of f by

$$\|f\|_G=\int\cdots\int_G\Big(\sum_{i=1}^{n}\Big(\frac{\partial f}{\partial x^i}(x)\Big)^2\Big)^{n/2}dx^1\cdots dx^n.$$

Then we have the following lemma by the aid of (2) and (4).

LEMMA 2. *Let f be an ACL function defined on a parametric ball $B=(B,\phi)$ in M. Then we have*

$$k_B^{-n}\|f\circ\phi^{-1}\|_{\phi(B)}\leqq D_B(f)\leqq k_B^{n}\|f\circ\phi^{-1}\|_{\phi(B)}.$$

3. Conformal capacity of a ring

A non-empty open subset A of a Riemannian manifold M is called a generalized ring if the complement of A consists of two non-empty closed subsets C_0 and C_1 of M with $C_0\cap C_1=\emptyset$. In this case we write $A=R(C_0, C_1; M)$. In particular if A is relatively compact domain on M, both C_0 and C_1 are connected and C_1 is compact, then we say that A is a ring. Then following C. Loewner [7] (cf. [2], [3]) we define the (conformal) capacity of a generalized ring.

DEFINITION. For a generalized ring $A=R(C_0, C_1; M)$, we define

$$C_M(A)=C_M[R(C_0, C_1; M)]=\inf D_M(f),$$

where f run over all ACL functions f on M such that $f=i$ on C_i $(i=0, 1)$. If there is no such a funciton f, then we define $C_M(A)=\infty$. Furthermore, in the case $M=R^n$, we write $C_M(A)=C(A)$.

Let $A^i=R(C_0^{(i)}, C_1^{(i)}; M)$ be two generalized rings $(i=1, 2)$. If $C_0^{(2)}\subset C_0^{(1)}$ and $C_1^{(2)}\subset C_1^{(1)}$, then we write $A^1\leqq A^2$.

The following properties are immediate consequences of the definition of capacity and Lemma 2 except for (d).

Properties of a capacity:

(a) $C_M[R(C_0, C_1; M)]=C_M[R(C_1, C_0; M)]$.

(b) If $A\leqq A'$, then $C_M(A)\geqq C_M(A')$.

(c) Let A be a ring on M which is contained in a parametric ball $B=(B, \phi)$. Then

$$k_B^{-n}C(\phi(A))\leqq C_M(A)\leqq k_B^{n}C(\phi(A)).$$

(d) If $0<a<b<\infty$, then

$$C(\{a<|x|<b\})=\frac{\omega_{n-1}}{(\log(b/a))^{n-1}} \qquad \text{(cf. [12])}.$$

LEMMA 3. *Let p be a point in a parametric ball $B=(B, \phi)$. Let a, b be real numbers such that $0<a<b<\infty$ and $A=\{q\in M; a<d_M(p, q)<b\}$ is a ring in B. Then*

$$k_B^{-n}\frac{\omega_{n-1}}{(\log k_B(b/a))^{n-1}}\leqq C_M(A)\leqq k_B^{n}\frac{\omega_{n-1}}{(\log k_B^{-1}(b/a))^{n-1}}.$$

The last inequality is valid if $b/a>k_B$.

Proof. We may assume that $\phi(p)=0$. If $b/a>k_B$, then it follows from (i) in Lemma 1 that

$$\{k_B^{-1/2}a<|x|<k_B^{1/2}b\}\supseteq\phi(A)\supseteq\{k_B^{1/2}a<|x|<k_B^{-1/2}b\}.$$

This completes the proof.

Let $\{A_j=R(C_{0,j}, C_{1,j}; M)\}_{j=1}^{\infty}$ be a family of rings on M. We say that $\{A_j\}_{j=1}^{\infty}$ is a distinguished family of rings on M if

$$(A_j\cup C_{1,j})\cap(A_k\cup C_{1,k})=\emptyset \qquad \text{if} \quad j\neq k.$$

Then we have the following lemma.

LEMMA 4. *Let $\{A_j\}_{j=1}^{\infty}$ be a distinguished family of rings on M. Let $C_0=\bigcap_{j=1}^{\infty}C_{0,j}$ and $C_1=\bigcup_{j=1}^{\infty}C_{1,j}$. If $A=R(C_0, C_1; M)$ is a generalized ring, then*

$$C_M(A)=\sum_{j=1}^{\infty} C_M(A_j)\,.$$

The following theorem is due to J. Väisälä.

THEOREM 1 (cf. Theorem 11. 9 in [12]). *Suppose that $A=R(C_0, C_1; R^n)$ is a ring and that $c\in C_0$ and $a, b\in C_1$. Then*

$$C(A)\geqq \mathcal{H}_n\left(\frac{|c-a|}{|b-a|}\right),$$

where $\mathcal{H}_n(r)$ is a positive constant depending only on $r>0$ and n.

4. Homeomorphism

DEFINITION. Let $f: M\to N$ be a homeomorphism. For $p\in M$ and $r>0$, we set

$$l(p, f, r)=\inf_{d_M(p,q)=r} d_N(f(p), f(q)),$$

$$L(p, f, r)=\sup_{d_M(p,q)=r} d_N(f(p), f(q)),$$

$$A^*(p, r)=\{q'\in N; l(p, f, r)<d_N(f(p), q')<L(p, f, r)\}\,.$$

PROPOSITION. *If $f^{-1}(A^*(p, r))$ is contained in $B=(B, \phi)$, then*

$$C_M[f^{-1}(A^*(p, r))]\geqq k_B^{-n}\mathcal{H}_n(k_B^2)>0\,.$$

In particular, if $l(p, f, r)=L(p, f, r)$, then we set $C_M[f^{-1}(A^*(p, r))]=\infty$.

Proof. We may assume that $l(p, f, r)\neq L(p, f, r)$. Then there exist $p_1, p_2 \in B$ such that $d_N(f(p), f(p_1))=l(p, f, r)$ and $d_N(f(p), f(p_2))=L(p, f, r)$. It follows from (c) that

$$C_M[f^{-1}(A^*(p, r))]\geqq k_B^{-n}C[\phi(f^{-1}(A^*(f, r)))]=(*)\,.$$

If we set $\phi(p_i)=x_i$ $(i=1, 2)$, then it follows from (i) in Lemma 1 and Theorem 1 that

$$(*)\geqq k_B^{-n}\mathcal{H}_n\left(\frac{|x_2|}{|x_1|}\right)\geqq k_B^{-n}\mathcal{H}_n(k_B^2)>0\,.$$

5. Quasiconformal mappings on Riemannian manifolds

Let M and N be connected separable, orientable n-dimensional $(n\geqq 2)$ differentiable manifolds of class C^1. The tangent bundle of M is denoted by TM. The derivative of a differentiable mapping $f: M\to N$ is a fibre mapping $Df: TM \to TN$ and the norm of Df is denoted by $\|Df\|$. The Jacobian of f at $p\in M$ is denoted by $J_f(P)=\det Df(p)$.

We say that $f: M \to N$ is an ACL^n-mapping if, for any parametric balls $B=(B, \phi)$ on M and $B'=(B', \psi)$ on N such that $f(B) \subset B'$, $\psi \circ f \circ \phi^{-1}$ is an ACL-mapping and the partial derivatives of $\psi \circ f \circ \phi^{-1}$ are locally L^n-integrable on $\phi(B)$. Then f has a fibre mapping Df almost everywhere on M.

DEFINITION. A homeomorphism $f: M \to N$ is called a quasiconformal mapping if it is an ACL^n-mapping and if there exists a finite constant K $(\geqq 1)$ such that

$$\|Df\|^n \leqq K \cdot |J_f|$$

almost everywhere in M.

For a homeomorphism $f: M \to N$, we set

$$H(p, f) = \overline{\lim_{r \to 0}} \frac{L(p, f, r)}{l(p, f, r)} \qquad (p \in M).$$

Since the theory of quasiconformal mappings between Euclidean domains obviously carries over to Riemannian manifolds, we obtain the following theorem (cf. F. Gehring [3]).

THEOREM 2. *Let $f: M \to N$ be an ACL^n-homeomorphism. Then f is a quasiconformal mapping if and only if $H(p, f)$ is bounded.*

For a homeomorphism $f: M \to N$ we have the following theorem.

THEOREM 3. *f is a quasiconformal mapping if and only if there exists a finite constant $\alpha > 0$ with the following property:*

For every $p \in M$, there is $r(p) > 0$ such that $\{q' \in N; d_N(f(p), q') \leqq r(p)\}$ is compact in N and such that $C_N(A^*(p, r)) \geqq \alpha$ for all $r(0 < r \leqq r(p))$.

Proof. Suppose there is a constant K $(1 \leqq K < \infty)$ such that $H(p, f) \leqq K$ for all $p \in M$. Then, for any $\varepsilon > 0$ and $p \in M$, there exists $r(p) > 0$ such that $F = \{q \in M; d_M(p, q) \leqq r(p)\}$ is compact in M and

$$1 \leqq \frac{L(p, f, r)}{l(p, f, r)} < K + \varepsilon$$

for all r $(0 < r \leqq r(p))$. Then we may assume that $f(F)$ is contained in a parametric ball $B'=(B', \psi)$ in N such that $\psi(f(p))=0$. Then it follows from (c) that

$$C_N(A^*(p, r)) \geqq \tau_N^{-n} C[\psi(A^*(p, r))].$$

On the other hand there exist q_1' and q_2' in B' such that

$$d_N(f(p), q_1') = l(p, f, r) \quad \text{and} \quad d_N(f(p), q_2') = L(p, f, r).$$

Since

$$|\psi(q_2')| / |\psi(q_1')| \leqq \tau_N \frac{L(p, f, r)}{l(p, f, r)} < \tau_N(K + \varepsilon),$$

it follows from Theorem 1 that

$$C[\psi(A^*(p, r))] \geqq \mathcal{H}_n(|\psi(q_2')|/|\psi(q_1')|) \geqq \mathcal{H}_n(\tau_N(K+\varepsilon))>0\,.$$

Hence we can choose $\tau_N{}^{-n}\mathcal{H}_n(\tau_N(K+\varepsilon))$ as α.

Conversely suppose there exists a finite constant $\alpha>0$ with the property in the theorem. First we assume that $H(p, f)>\tau_N$. Then there exists a decreasing sequence of real numbers $\{r_j\}_{j=1}^{\infty}$ such that $r_j \to 0$ as $j\to\infty$, $L(p,f,r_j)/l(p,f,r_j)$ $\to H(p, f)$ as $j\to\infty$ and $L(p, f, r_j)/l(p, f, r_j)>\tau_N$ for all j. Then it follows from Lemma 3 that

$$0<\alpha \leqq C_N(A^*(p, r_j)) \leqq \tau_N{}^n \frac{\omega_{n-1}}{\left(\log \tau_N{}^{-1}\dfrac{L(p, f, r_j)}{l(p, f, r_j)}\right)^{n-1}}\,.$$

This implies that

$$\frac{L(p, f, r_j)}{l(p, f, r_j)} \leqq \tau_N \exp\left\{\left(\frac{\tau_N{}^n \cdot \omega_{n-1}}{\alpha}\right)^{1/(n-1)}\right\}.$$

By letting $j\to\infty$, we have

$$H(p, f) \leqq \tau_N \exp\left\{\left(\frac{\tau_N{}^n \cdot \omega_{n-1}}{\alpha}\right)^{1/(n-1)}\right\}.$$

Hence we always have the same inequality. This completes the proof.

Remark. Theorem 3 is a generalization of Theorem 1 in [1] to the case of Riemannian manifolds.

6. Main result

For a non-compact Riemannian manifold M we denote by $R(M)$ the class of all bounded ACL funcitons f on M which have finite Dirichlet integral $D_M(f)$ $<\infty$. Then $R(M)$ constitutes an algebra over the field of real numbers in a usual way and is called the Royden algebra associated with M. The Royden compactification of M is denoted by M^* (cf. [5], [6], [8], [10]).

Let M and N be two Riemannian manifolds of dimension n ($n \geqq 2$). Let $f: M \to N$ be a homeomorphism. Then we have the following lemma.

LEMMA 5. *The following conditions are equivalent.*

(i) *f can be extended to a homeomorphism of M^* onto N^*.*

(ii) *Let X and Y be any subsets of M. Then $\bar{X} \cap \bar{Y} = \emptyset$ in M^* if and only if $\overline{f(X)} \cap \overline{f(Y)} = \emptyset$ in N^*.*

(iii) *Let A be any generalized ring in M. Then $C_M(A)<\infty$ if and only if $C_M(f(A))<\infty$.*

Remark. Any homeomorphism f: $M^* \to N^*$ induces a homeomorphism $f|M$: $M \to N$ satisfying (iii) which is called a Royden's map in [9, 11].

THEOREM 4. *Let M and N be two Riemannian manifolds of dimension n $(\geqq 2)$. If there exists a homeomorphism f of M^* onto N^*, then there exists a compact subset K of M such that the restriction of f to each component of $M-K$ is quasiconformal.*

Proof. Suppose the theorem were not the case. Then we could find a compact exhaustion $\{K_j\}_{j=1}^{\infty}$ of M such that $\sup_{p\in M-K_j} H(p, f)=\infty$ for every j. Hence there exsits a sequence $\{p_j\}_{j=1}^{\infty}$ of points in M such that

$$H(p_j, f)>\tau_N \exp(j^{2/(n-1)}) \qquad (>\tau_N).$$

We may assume that $\{p_j\}_{j=1}^{\infty}$ is a discrete set.

For each j, there is a sequence $\{r_\nu\}_{\nu=1}$ of real numbers such that $r_\nu\to 0$ as $\nu\to\infty$ and

$$\frac{L(p_j, f, r_\nu)}{l(p_j, f, r_\nu)}>\tau_N \exp(j^{2/(n-1)}) \qquad (>\tau_N).$$

for all $\nu=1, 2, \cdots$. Then we may assume that $\{q'\in N: d_N(f(p_j), q')\leqq r_\nu\}$ is contained in a parametric ball $B_j'=(B_j', \phi_j')$ for sufficiently large ν. Then it follows from the Proposition that

$$C_M[f^{-1}(A^*(p_j, f, r_\nu))]\geqq \tau_M{}^{-n}\mathscr{H}_n(\tau_M^2)>0$$

for sufficiently large ν. We may assume that $\phi_j(f(p_j))=0$. Since $L(p_j, f, r_\nu)/l(p_j, f, r_\nu)>\tau_N$ for sufficiently large ν, we obtain that

$$C_N(A^*(p_j, f, r_\nu))\leqq \tau_N{}^n C[\phi_j(A^*(p_j, f, r_\nu))]<\frac{\tau_N{}^n\cdot\omega_{n-1}}{j^2}.$$

This shows that, for each j, there exists a sufficiently small r_j such that $\{A^*(p_j, f, r_j)\}_{j=1}^{\infty}$ is a distinguished family in N and

$$C_M[f^{-1}(A^*(p_j, f, r_j))]\geqq \tau_M{}^{-n}\cdot\mathscr{H}_n(\tau_M^2)>0,$$

$$C_N(A^*(p_j, f, r_j))<\frac{\tau_M{}^n\cdot\omega_{n-1}}{j^2}.$$

We set

$$C_{0,j}=\{q'\in N; d_N(f(p_j), q')\geqq L(p_j, f, r_j)\}$$

and

$$C_{1,j}=\{q'\in N; d_N(f(p_j), q')\leqq l(p_j, f, r_j)\}.$$

Furthermore if we set

$$C_0=\bigcap_{j=1}^{\infty} C_{0,j} \quad\text{and}\quad C_1=\bigcup_{j=1}^{\infty} C_{1,j},$$

then $A=R(C_0, C_1; N)$ is a generalized ring. Since $f^{-1}(A)=R(f^{-1}(C_0), f^{-1}(C_1); M)$ is also a generalized ring, it follows from Lemma 4 that

$$C_N(A)=\sum_{j=1}^{\infty} C_N(A^*(p_j, f, r_j))$$

and

$$C_M(f^{-1}(A))=\sum_{j=1}^{\infty} C_M[f^{-1}(A^*(p_j, f, r_j))]=\infty .$$

This contradicts (iii) in Lemma 5. Hence there exists a compact set K in M such that f is quasiconformal on each component of $M-K$.

For Riemann surfaces we can prove the following sharp theorem. However we do not know whether this is valid for higher dimensional cases.

THEOREM 5. *Let M and N be two Riemann surfaces. If there exists a homeomorphism of M^* onto N^*, then there exists a quasiconformal mapping of M onto N.*

Proof. Let f be a homeomorphism of M^* onto N^*. By Theorem 4, there exists a compact set K in M such that the restriction of f to each component of $M-K$ is a quasiconformal mapping. Then we can find a compact bordered surface R of M such that $K\subset\bar{R}\subset M$. If we set $S=f(R)$, then the borders ∂R and ∂S consist of a finite number of disjoint quasiconformal curves (cf. [4, p. 101]). By a slight modication of the proof of Satz 8.2 in [4] we can find a quasiconformal mapping f_1 of R onto S such that $f=f_1$ in a neighborhood of ∂R. By setting $g=f_1$ in R and $=f$ on $M-K$, we have a desired quasiconformal mapping.

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