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RIEMANNIAN METRICS ON PRINCIPAL CIRCLE BUNDLES OVER LOCALLY SYMMETRIC KÄHLERIAN MANIFOLDS

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0. Introduction.

In Riemannian Geometry, one of the most interesting problems is to find all Einsteinian manifolds. A. Besse has suggested the research for Einsteinian manifolds satisfying the condition (see [3], p. 165)

$$(*) \qquad R_{ipqr}R_j{}^{pqr}=\text{constant } g_{ij},$$

where $g=(g_{ij})$ is the Riemannian metric and $R=(R^i{}_{jkl})$ is the curvature tensor. Its typical examples are a locally symmetric spaces and a harmonic Riemannian manifold (cf. [13]). But the author [17] has recently shown that $Sp(2)/SU(2)$ of Berger (cf. [2]) is an Einsteinian manifold satisfying (*).

On the other hand, J. E. D'Atri and H. K. Nickerson has initiated a study of the Riemannian manifold whose local geodesic symmetries are volume-preserving (up to sign). In this paper, we call such a manifold a (locally) volume symmetric space. It has been studied by J. E. D'Atri and H. K. Nickerson ([5]. [6]), K. Sekigawa ([15]) and Y. Watanabe ([16]). This class of manifolds obviously includes harmonic Riemannian manifolds and locally symmetric spaces. Then we are interested in Einsteinian manifolds (especially Einsteinian manifolds satisfying (*)), which are volume symmetric.

In this paper, we consider the Riemannian metric $\tilde{g}(t)$ given on a principal circle bundle P over a Kählerian manifold M (cf. § 2). We show that if M is locally symmetric, then $(P, \tilde{g}(t))$ is locally homogeneous and volume symmetric. Especially we also remark that an Einsteinian metric is given on P in the case that M is Einsteinian.

In § 2, we define the Riemannian metric on a principal circle bundle over an n-dimensional Riemannian manifold and give the fundamental formulas. In § 3, we calculate the covariant derivative of the Riemannian curvature tensor by using the structure equations obtained in § 2. In § 4, we define a tensor field T of type (1, 2) on P and state its properties for later use. In § 5, M is assumed to be a locally symmetric Kählerian manifold. Then by using the equations obtained in sections 3 and 4, we show the main results. In the last section, an

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example is stated in detail, because in the opinion of the author the method in this paper should have other applications.

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1. A principal circle bundle.

Let $\tilde{M}$ and M be two Riemannian manifolds of dimensions m and n respectively, where $m-n>0$. A Riemannian submersions, which shall call simply submersion, is a differentiable map $\pi:\tilde{M}\rightarrow M$, which is onto, and for which at each point $x\in M$ the differential map π_* acts as an orthogonal projection of $T_{\tilde{x}}\tilde{M}$ onto T_xM, where $x=\pi(\tilde{x})$. Then for each point $x\in M$ its inverse image $\pi^{-1}(x)$ is a submanifold of M of dimension $m-n$, which is called the fibre over x.

A vector field on $\tilde{M}$ is called vertical if it is always tangent to fibres and holizontal if always orthogonal to fibres; we use corresponding terminology for individual tangent vectors. We denote the Riemannian connections by $\tilde{\nabla}$ on $\tilde{M}$, ∇ on M and $\bar{\nabla}$ on the fibres, defined by the Riemannian metrics $\tilde{g}$ on $\tilde{M}$, g on M and $\bar{g}$ on the fibres respectively. For a submersion $\pi:\tilde{M}\rightarrow M$, let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of the tangent space of $\tilde{M}$ onto the subspaces of horizontal and vertical vectors, respectively. We define a vector field $\tilde{X}$ on $\tilde{M}$ to be basic provided $\tilde{X}$ is horizontal and π-related to a vector field X on M. Every vector field X on M has a unique horizontal lift $\tilde{X}$ to $\tilde{M}$, and $\tilde{X}$ is basic. Thus $\tilde{X}\leftrightarrow X$ is a one-to-one correspondence between basic vector fields on $\tilde{M}$ and arbitrary vector fields on M. This correspondence preserves brackets, inner products and covariant derivatives to the following (see [12]).

LEMMA 2.1. *If X and Y are basic vector fields on M, then*
(1) $\langle\tilde{X}, \tilde{Y}\rangle=\langle X, Y\rangle\circ\pi$,
(2) $[\tilde{X}, \tilde{Y}]$ *is the basic vector field corresponding to* $[X, Y]$,
(3) $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ *is the basic vector field corresponding to* ∇_XY,
where $\langle\tilde{X}, \tilde{Y}\rangle$ (*resp.* $\langle X, Y\rangle$) *is the inner product* $\tilde{g}(\tilde{X}, \tilde{Y})$ (*resp.* $g(X, Y)$) *of vectors* $\tilde{X}$ *and* $\tilde{Y}$ *of* $\tilde{M}$ (*resp.* X *and* Y *of* M).

Let P be a principal circle bundle over an n-dimensional manifold with projection π, g a Riemannian metric on M and η a connection form on P defining a connection in the bundle P. Functions on M such as components of tensor fields on M are considered sometimes as functions on P in a natural way without any change of notations. We shall also agree on that indices i, j, k and l run from 1 to n and α, β, γ and δ run from 0 to n. Since the structure group S^1 of P is abelian, the structure equation is given by

$$d\eta=\Omega\,, \tag{2.1}$$

where Ω is the curvature form of η. Then $\tilde{g}=\pi^*g+t^2\eta\otimes\eta$ for any $t>0$, is a

Riemannian metric on P with respect to which π becomes a Riemannian submersion ([7], [12]). Explicitly if $\tilde{x}\in P$, $\tilde{X}$, $\tilde{Y}\in T_{\tilde{x}}P$, then

$$\tilde{g}(\tilde{X}, \tilde{Y})=g(\pi_*\tilde{X}, \pi_*\tilde{Y})+t^2\eta(\tilde{X})\eta(\tilde{Y}). \tag{2.2}$$

The action of S^1 on P allows us to take a unit fundamental vector field ξ such that

$$\tilde{g}(\tilde{X}, \xi)=t\eta(\tilde{X}), \tag{2.3}$$

for any vector field $\tilde{X}$ on P. If $\tilde{X}$ is basic, then $[\tilde{X}, \xi]$ is vertical. Thus since $[\tilde{X}, \xi]$ is horizontal (cf. [11]), we have the following

LEMMA 2.2 (cf. [7]). *If $\tilde{X}$ is a basic vector field on M, then*

$$[\tilde{X}, \xi]=0. \tag{2.4}$$

3. Structure equations of the fibring

To calculate the Riemannian curvature tensor $\tilde{R}$ of $\tilde{g}$ at any fixed point $\tilde{x}\in P$, we shall take a special orthonormal frame field on a neighborhood of $x=\pi(\tilde{x})$ such that $\nabla_{X_i}X_j=0$ at x for any i and j. The basic vector fields $\tilde{X}_i$ corresponding to X_i are orthonormal vector fields such that $\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j$ is vertical on the fibre passing through $\tilde{x}$ for any i and j. Therefore we have a local orthonormal frame field $\{\xi, \tilde{X}_1, \cdots, \tilde{X}_n\}$ around $\tilde{x}$. Using the standard formula

$$2\langle\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}\rangle=\tilde{X}\langle\tilde{Y}, \tilde{Z}\rangle+\tilde{Y}\langle\tilde{Z}, \tilde{Y}\rangle-\tilde{Z}\langle\tilde{X}, \tilde{Y}\rangle-\langle\tilde{X}, [\tilde{Y}, \tilde{Z}]\rangle$$
$$+\langle\tilde{Y}, [\tilde{Z}, \tilde{X}]\rangle+\langle\tilde{Z}, [\tilde{X}, \tilde{Y}]\rangle,$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ and the definition (2.2), we can prove the following lemma (cf. [14]).

LEMMA 3.1. *The components of the connection $\tilde{\nabla}$ with respect to the orthonormal frame field taken above are given by*

(1) $\tilde{\nabla}_\xi\xi=0$,

(2) $\tilde{\nabla}_\xi\tilde{X}_i=\nabla_{\tilde{X}_i}\xi=t\Sigma\Omega_{ij}\tilde{X}_j$,

(3) $\mathcal{V}(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j)=(1/2)\mathcal{V}[\tilde{X}_i, \tilde{X}_j]=-t\Omega_{ij}\xi$,

(4) $\mathcal{H}(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j)=\Sigma S_{ijk}\tilde{X}_k$,

(5) $(\tilde{\nabla}_\xi\Omega)(\tilde{X}_i, \tilde{X}_j)=0$,

in a neighborhood of $\tilde{x}$, where $\Omega_{ij}=\Omega(\tilde{X}_i, \tilde{X}_j)$ and $S_{ijk}=\langle\nabla_{X_i}X_j, X_k\rangle\circ\pi$.

Let $\tilde{R}$ (resp. R) be the curvature tensor of P (resp. M). Then by repeated use of Lemma 3.1, we can obtain the following

LEMMA 3.2. (cf. [10]). *The components of the curvature tensor R with respect to the orthonormal frame* $\{\xi, \tilde{X}_1, \cdots, \tilde{X}_n\}$ *are given by*

(1) $\tilde{R}_{i0j0}=t^2\Sigma\Omega_{jl}\Omega_{li}$,

(2) $\tilde{R}_{i0jk}=\tilde{\nabla}_i\Omega_{jk}$,

(3) $\tilde{R}_{ijkl}=R_{ijkl}-t^2(\Omega_{il}\Omega_{jk}-\Omega_{ik}\Omega_{jl}-2\Omega_{ij}\Omega_{kl})$,

where $\tilde{R}_{i0j0}=\langle\tilde{R}(\tilde{X}_i, \xi)\tilde{X}_j, \xi\rangle$, $\tilde{R}_{i0jk}=\langle\tilde{R}(\tilde{X}_i, \xi)\tilde{X}_j, \tilde{X}_k\rangle$, $\tilde{R}_{ijkl}=\langle\tilde{R}(\tilde{X}_i, \tilde{X}_j)\tilde{X}_k, \tilde{X}_l\rangle$, $R_{ijkl}=\langle R(X_i, X_j)X_k, X_l\rangle$ *and* $\tilde{\nabla}_i\Omega_{jk}=(\tilde{\nabla}_{\tilde{X}_i}\Omega)(\tilde{X}_j, \tilde{X}_k)$.

LEMMA 3.3. *The curvature tensor* $\tilde{R}$ *is expressed in the terms of* R_{ijkl} *and* Ω_{ij} *as follows*:

(1) $\tilde{R}(\xi, \tilde{X}_i)\xi=t^2\Sigma\Omega_{jk}\Omega_{ki}\tilde{X}_j$,

(2) $\tilde{R}(\xi, \tilde{X}_i)\tilde{X}_j=-t^2\Sigma(\Omega_{ik}\Omega_{kj}\xi-t\tilde{\nabla}_i\Omega_{jk}\tilde{X}_k)$,

(3) $\tilde{R}(\tilde{X}_i, \tilde{X}_j)\xi=t\Sigma\tilde{\nabla}_k\Omega_{ij}\tilde{X}_k$,

(4) $\tilde{R}(\tilde{X}_i, \tilde{X}_j)\tilde{X}_k=t\tilde{\nabla}_k\Omega_{ij}\xi+\Sigma[R_{ijkl}-t^2(\Omega_{il}\Omega_{jk}-\Omega_{ik}\Omega_{jl}-2\Omega_{ij}\Omega_{kl})]X_l$.

We shall calculate the covariant derivative $\tilde{\nabla}\tilde{R}$ of the curvature tensor $\tilde{R}$ by using Lemmas 3.1, 3.2 and 3.3.

LEMMA 3.4.

(1) $(\tilde{\nabla}_\xi\tilde{R})(\xi, \tilde{X}_i)\xi=0$,

(2) $(\tilde{\nabla}_{\tilde{X}_i}\tilde{R})(\xi, \tilde{X}_j)\xi=t^2\Sigma[\Omega_{kl}\Omega_{lj}(S_{ika}+S_{iak})$
$+S_{ilk}(\Omega_{kj}\Omega_{al}+\Omega_{ak}\Omega_{lj})+\Omega_{lj}\tilde{\nabla}_i\Omega_{al}+\Omega_{al}\tilde{\nabla}_i\Omega_{lj}]\tilde{X}_a$,

(3) $(\tilde{\nabla}_\xi\tilde{R})(\xi, \tilde{X}_i)\tilde{X}_j=-t\Sigma[\xi(\nabla_i\Omega_{jk})-t\Omega_{lk}\tilde{\nabla}_i\Omega_{jl}+\Omega_{il}\tilde{\nabla}_l\Omega_{jk}+\Omega_{jl}\tilde{\nabla}_i\Omega_{lk}]\tilde{X}_k$,

(4) $(\tilde{\nabla}_{\tilde{X}_k}\tilde{R})(\xi, \tilde{X}_i)\tilde{X}_j=-\Sigma[t^2\{(\tilde{\nabla}_k\Omega_{jl})\Omega_{li}+\Omega_{jl}(\tilde{\nabla}_k\Omega_{li})\}\xi$
$-t\Omega_{kl}R(\tilde{X}_l, \tilde{X}_i)\tilde{X}_j-S_{kil}\tilde{R}(\xi, \tilde{X}_l)\tilde{X}_j-S_{kjl}\tilde{R}(\xi, \tilde{X}_i)\tilde{X}_l]$,

(5) $(\tilde{\nabla}_{\tilde{X}_k}\tilde{R})(\tilde{X}_i, \tilde{X}_j)\xi=\Sigma[t\tilde{X}_k(\tilde{\nabla}_l\Omega_{ij})\tilde{X}_l+tS_{kla}\tilde{\nabla}_l\Omega_{ij}\tilde{X}_a-t\Omega_{kl}\tilde{R}(\tilde{X}_i, \tilde{X}_j)\tilde{X}_l$
$-S_{kil}\tilde{R}(\tilde{X}_l, \tilde{X}_j)\xi-tS_{kjl}\tilde{\nabla}_a\Omega_{il}\tilde{X}_a+t^3\Omega_{al}(\Omega_{ki}\Omega_{lj}-\Omega_{kj}\Omega_{li})\tilde{X}_a+t\tilde{\nabla}_k\Omega_{ij}\xi$,

(6) $(\tilde{\nabla}_{\tilde{X}_h}\tilde{R})(\tilde{X}_i, \tilde{X}_j)\tilde{X}_k=\Sigma[\{\langle(\nabla_{X_h}R)(X_i, X_j)X_k, X_l\rangle-t^2(\Omega_{jk}\tilde{\nabla}_h\Omega_{il}$
$+\Omega_{il}\tilde{\nabla}_h\Omega_{jk}-\Omega_{jl}\tilde{\nabla}_h\Omega_{ik}-\Omega_{ik}\tilde{\nabla}_h\Omega_{jl}-2\Omega_{kl}\tilde{\nabla}_h\Omega_{ij}-2\Omega_{ij}\tilde{\nabla}_h\Omega_{kl})\}X_l$
$-t\Omega_{hl}\tilde{R}_{ijkl}\xi+S_{hla}\tilde{R}_{ijkl}\tilde{X}_a-S_{hil}\tilde{R}(\tilde{X}_l, \tilde{X}_j)\tilde{X}_k-S_{hjl}\tilde{R}(X_i, \tilde{X}_l)\tilde{X}_k$
$-S_{ikl}\tilde{R}(\tilde{X}_i, \tilde{X}_j)\tilde{X}_l+t^2(-t\Omega_{kl}\Omega_{lj}\xi+\tilde{\nabla}_i\Omega_{hl}\tilde{X}_l)\Omega_{hi}+t^2\Omega_{hk}(\tilde{\nabla}_l\Omega_{ij})\tilde{X}_l$
$+t^2(t\Omega_{kl}\Omega_{li}\xi+\tilde{\nabla}_i\Omega_{kl}\tilde{X}_l)\Omega_{hj}]$.

Proof. Using the standard formula

$$(\tilde{\nabla}_{\tilde{X}}\tilde{R})(\tilde{Y}, \tilde{Z})\tilde{W}=\tilde{\nabla}_{\tilde{X}}(\tilde{R}(\tilde{Y}, \tilde{Z})\tilde{W})-\tilde{R}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z})\tilde{W}$$
$$-\tilde{R}(\tilde{Y}, \tilde{\nabla}_{\tilde{X}}\tilde{Z})\tilde{W}-\tilde{R}(\tilde{Y}, \tilde{Z})(\tilde{\nabla}_{\tilde{X}}\tilde{W}),$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$, we shall prove this lemma. By using the above formula for $X=\xi$, $Y=\xi$, $Z=X_i$ and $W=\xi$, we have, from (1) and (2) of Lemma 3.1 and (1) of Lemma 3.2,

$$(\tilde{\nabla}_\xi\tilde{R})(\xi, \tilde{X}_i)\xi=\tilde{\nabla}_\xi(\tilde{R}(\xi, \tilde{X}_i)\xi)-\tilde{R}(\tilde{\nabla}_\xi\xi, \tilde{X}_i)\xi-\tilde{R}(\xi, \tilde{\nabla}_\xi\tilde{X}_i)\xi-\tilde{R}(\xi, \tilde{X}_i)(\tilde{\nabla}_\xi\xi)$$
$$=\Sigma[t^2\tilde{\nabla}_\xi(\Omega_{jl}\Omega_{li}\tilde{X}_j)-t\Omega_{il}\tilde{R}(\xi, \tilde{X}_l)\xi]$$
$$=\Sigma[t^2(\Omega_{jl}\Omega_{li})\tilde{\nabla}_\xi\tilde{X}_j-t^2\Omega_{il}\Omega_{jk}\Omega_{kl}\tilde{X}_j]=0\,.$$

Thus we obtain the formula (1). Similarly, we can obtain the formulas (2), (3), (4), (5) and (6) by Lemmas 3.1 and 3.3.

4. The tensor field T of type $(1,2)$ on P.

We shall give a tensor field T of type $(1,2)$, which plays an important role in our assertion, and study its properties. Let us define T by

$$T(\tilde{X}, \tilde{Y})=t[\Omega(\tilde{X}, \tilde{Y})\xi+t(\eta(\tilde{X})\phi\tilde{Y}-\eta(\tilde{Y})\phi\tilde{X})]\,, \tag{4.1}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ on P, where ϕ is defined by $\tilde{g}(\phi\tilde{X}, \tilde{Y})=\Omega(\tilde{X}, \tilde{Y})$, then T is a tensor field of type $(1,2)$ on P such that $T(\tilde{X}, \tilde{Y})=-T(\tilde{Y}, \tilde{X})$. With respect to the orthonormal frame field taken in section 3, we have

$$T(\xi, \tilde{X}_i)=t\Sigma\Omega_{il}\tilde{X}_l\,, \tag{4.2}$$

and

$$T(\tilde{X}_i, \tilde{X}_j)=t\Omega_{ij}\xi\,. \tag{4.3}$$

Hence we obtain

LEMMA 4.1.

(1) $(\tilde{\nabla}_\xi T)(\tilde{X}_i, \tilde{X}_j)=(\tilde{\nabla}_\xi T)(\xi, \tilde{X}_i)=0$,

(2) $(\tilde{\nabla}_{\tilde{X}_i}T)(\xi, \tilde{X}_j)=t\Sigma(\tilde{\nabla}_i\Omega_{jk})\tilde{X}_k$,

(3) $(\tilde{\nabla}_{\tilde{X}_i}T)(\tilde{X}_j, \tilde{X}_k)=t(\tilde{\nabla}_i\Omega_{jk})+\Sigma t^2(\Omega_{jk}\Omega_{il}+\Omega_{ij}\Omega_{kl}-\Omega_{ik}\Omega_{jl})\tilde{X}_l$.

Proof. Using the standard formula

$$(\tilde{\nabla}_{\tilde{X}}T)(\tilde{Y}, \tilde{Z})=\tilde{\nabla}_{\tilde{X}}(T(\tilde{Y}, \tilde{Z}))-T(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z})-T(\tilde{Y}, \tilde{\nabla}_{\tilde{X}}\tilde{Z})\,, \tag{4.4}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$, we shall prove this lemma. Using the above formula for $\tilde{X}=\tilde{X}_i$, $\tilde{Y}=\tilde{X}_j$ and $\tilde{Z}=\tilde{X}_k$, we verify the third assertion as follows: By (2) and (3) of Lemma 3.1 and the definition (4.1) of T, we have

$$\begin{aligned}(\tilde{\nabla}_{\tilde{X}_i}T)(\tilde{X}_i, \tilde{X}_k) &= \tilde{\nabla}_{\tilde{X}_i}(T(\tilde{X}_j, \tilde{X}_k)) - T(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j, \tilde{X}_k) - T(\tilde{X}_j, \tilde{\nabla}_{\tilde{X}_i}\tilde{X}_k)\\ &= \tilde{\nabla}_{\tilde{X}_i}[t\Omega(\tilde{X}_j, \tilde{X}_k)\xi + t^2(\eta(\tilde{X}_j)\phi\tilde{X}_k - \eta(\tilde{X}_k)\phi\tilde{X}_j)]\\ &\quad -[t\Omega(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j, \tilde{X}_k)\xi + t^2(\eta(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j)\phi\tilde{X}_k - \eta(\tilde{X}_k)\phi\tilde{X}_j)]\\ &\quad -[t\Omega(\tilde{X}_j, \tilde{\nabla}_{\tilde{X}_i}\tilde{X}_k)\xi + t^2(\eta(\tilde{X}_j)\phi(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_k) - \eta(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j)\phi\tilde{X}_j]\\ &= t(\tilde{\nabla}_i\Omega_{jk})\xi + t^2\Sigma(\Omega_{jk}\Omega_{il} + \Omega_{ij}\Omega_{kl} - \Omega_{ik}\Omega_{jl})\tilde{X}_l\,.\end{aligned}$$

The assertions (1) and (2) follow similarly from (4.1), (4.2), (4.3) and Lemma 3.1.

Here if we put

$$(T_{\tilde{X}}\cdot T)(\tilde{Y}, \tilde{Z}) = T(\tilde{X}, T(\tilde{Y}, \tilde{Z})) - T(T(\tilde{X}, \tilde{Y}), \tilde{Z}) - T(\tilde{Y}, T(\tilde{X}, \tilde{Z}))\,, \tag{4.5}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$, the $T_{\tilde{X}}\cdot T$ is a tensor field of type (1, 2) on P and satisfies the following

LEMMA 4.2.

(1) $(T_{\tilde{X}_i}\cdot T)(\tilde{X}_i, \tilde{X}_j) = -t^2\Sigma(\Omega_{jk}\Omega_{il} + \Omega_{ij}\Omega_{kl} - \Omega_{ik}\Omega_{jl})\tilde{X}_l\,,$

(2) $(T_{\tilde{X}_i}\cdot T)(\xi, \tilde{X}_j) = (T_\xi\cdot T)(\tilde{X}_i, \tilde{X}_j) = 0\,.$

Proof. By (4.1), (4.2), (4.3) and Lemma 3.1, we have

$$\begin{aligned}&T(\tilde{X}_i, T(\tilde{X}_j, \tilde{X}_k)) - T(T(\tilde{X}_i, \tilde{X}_j), \tilde{X}_k) - T(\tilde{X}_j, T(\tilde{X}_i, \tilde{X}_k))\\ &\quad = t[\Omega_{jk}T(\tilde{X}_i, \xi) - \Omega_{ij}T(\xi, \tilde{X}_k) - \Omega_{ik}T(\tilde{X}_j, \xi)]\\ &\quad = -t^2\Sigma(\Omega_{jk}\Omega_{il} + \Omega_{ij}\Omega_{kl} - \Omega_{ik}\Omega_{jl})\tilde{X}_l\,.\end{aligned}$$

Similarly we can verify the assertion (2).

Now for later use, we define a tensor field $T_{\tilde{X}}\cdot\tilde{R}$ of type (1, 3) on P by

$$\begin{aligned}(T_{\tilde{W}}\cdot\tilde{R})(\tilde{X}, \tilde{Y})\tilde{Z} &= T(\tilde{W}, \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}) - \tilde{R}(T(\tilde{W}, \tilde{X}), \tilde{Y})\tilde{Z}\\ &\quad - \tilde{R}(\tilde{X}, T(\tilde{W}, \tilde{Y}))\tilde{Z} - \tilde{R}(\tilde{X}, \tilde{Y})(T(\tilde{W}, \tilde{Z}))\,,\end{aligned} \tag{4.6}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$. Then we have the following

LEMMA 4.3.

(1) $(T_\xi\cdot\tilde{R})(\xi, \tilde{X}_i) = 0\,,$

(2) $(T_{\tilde{X}_i}\cdot\tilde{R})(\xi, \tilde{X}_j)\xi = -t^2\Sigma\Omega_{ik}(\tilde{\nabla}_j\Omega_{kl})\tilde{X}_l\,,$

(3) $(T_\xi\cdot\tilde{R})(\xi, \tilde{X}_i)\tilde{X}_j = t^2\Sigma[-\Omega_{kl}(\tilde{\nabla}_i\Omega_{jk}) + \Omega_{ik}(\tilde{\nabla}_k\Omega_{jl}) + \Omega_{jk}(\tilde{\nabla}_i\Omega_{kl})]\tilde{X}_l\,,$

(4) $(T_{\tilde{X}_k}\cdot\tilde{R})(\xi,\ \tilde{X}_i)\tilde{X}_j=\Sigma[t^3\Omega_{jl}\Omega_{li}\Omega_{ka}\tilde{X}_a-t^2\Omega_{kl}(\tilde{\nabla}_i\Omega_{jl})\xi$

$+t\Omega_{kl}\tilde{R}(\tilde{X}_l,\ \tilde{X}_i)\tilde{X}_j-t^3\Omega_{kj}\Omega_{al}\Omega_{li}\tilde{X}_a]$,

(5) $(T_{\tilde{X}_h}\cdot\tilde{R})(\tilde{X}_i,\ \tilde{X}_j)\xi=\Sigma[t\Omega_{kl}(\tilde{\nabla}_l\Omega_{ij})\xi-t^3(\Omega_{ki}\Omega_{al}\Omega_{lj})\tilde{X}_a$

$+t^2\Omega_{kj}\Omega_{al}\Omega_{li}\tilde{X}_a+t\Omega_{kl}\tilde{R}(\tilde{X}_i,\ \tilde{X}_j)\tilde{X}_l]$,

(6) $(T_{\tilde{X}_h}\cdot\tilde{R})(\tilde{X}_i,\ \tilde{X}_j)\tilde{X}_k=\Sigma[-t\Omega_{hl}\tilde{\nabla}_k\Omega_{ij}\tilde{X}_l+t\Omega_{kl}\tilde{R}_{ijkl}\xi+t^3\Omega_{hi}\Omega_{kl}\Omega_{lj}\xi$

$+t^2\Omega_{hi}(\tilde{\nabla}_j\Omega_{kl})\tilde{X}_l-t^3\Omega_{hj}\Omega_{kl}\Omega_{li}\xi-t^2\Omega_{hk}(\tilde{\nabla}_l\Omega_{ij})\tilde{X}_l]$.

Proof. By Lemmas 3.1 and 3.2, (2.4), (4.1), (4.2), (4.3) and (4.6) imply that

$$(T_\xi\cdot\tilde{R})(\xi,\tilde{X}_i)\xi=T(\xi,\tilde{R}(\xi,\tilde{X}_i)\xi)-\tilde{R}(T(\xi,\xi),\tilde{X}_i)\xi-\tilde{R}(\xi,T(\xi,\tilde{X}_i))\xi-\tilde{R}(\xi,\tilde{X}_i)T(\xi,\xi)$$

$$=\Sigma[t^2\Omega_{jl}\Omega_{li}T(\xi,\ \tilde{X}_j)-t\Omega_{il}\tilde{R}(\xi,\ \tilde{X}_l)\xi]=0\,.$$

Similarly we can obtain the equations (2), (3), (4), (5) and (6).

5. Main results.

Let M be a locally symmetric Kählerian manifold with a Kählerian metric g and the fundamental 2-form Θ. Let P be a principal circle bundle over M, $\tilde{g}$ a Riemannian metric on P given by (2.2) and η a connection form on P such that $d\eta=\pi^*\Theta$. Let τ be a local cross-section of P defined on a neighborhood U of x such that $\tau(x)=\tilde{x}$ and the differential map of τ maps the tangent space of M at x onto the horizontal space of P at $\tilde{x}$, and J_{ij} the component of the almost complex structure J with respect to the orthonormal frame field $\{X_1, \cdots, X_n\}$ taken in section 3. Noting that $\mathcal{H}\tau_*X_j=X_j$ on U for each j, we obtain

$$\Omega(\tilde{X}_j,\ \tilde{X}_k)_{\tau(y)}=\Omega(\tau_*X_j,\ \tau_*X_k)_{\tau(y)}=(\tau^*\Omega)(\tilde{X}_j,\ \tilde{X}_k)=-\langle JX_j,\ X_k\rangle_y=-J_{jk}\,, \tag{5.1}$$

for any y of U. Hence we see that

$$\tilde{\nabla}_i\Omega_{jk}=\tilde{\nabla}_i\tilde{\nabla}_j\Omega_{kh}=0\,. \tag{5.2}$$

Now we define a linear connection D on P,

$$D_{\tilde{X}}\tilde{Y}=\tilde{\nabla}_{\tilde{X}}\tilde{Y}+T(\tilde{X},\ \tilde{Y})\,, \tag{5.3}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$. Then by (2.3) and (4.1), we have

$$(D_{\tilde{X}}\tilde{g})(\tilde{Y},\ \tilde{Z})=0\,. \tag{5.4}$$

This means that D is a metric connection. Since M is locally symmetric Kählerian manifold, after a long calculation Lemmas 3.4, 4.1, 4.2, 4.3, 5.2 and 5.3 imply that for any vector field $\tilde{X}$,

$$D_{\tilde{X}}\tilde{R}=0 \quad \text{and} \quad D_{\tilde{X}}T=0\,.$$

By a result of W. Ambrose and I. M. Singer (see [1], p. 656), we find that P is locally homogeneous. Thus we have

PROPOSITION 5.1. *Let M be a locally symmetric Kählerian manifold and P a principal circle bundle over M with a Riemannian metric given by* (2.2) *and a connection form η such that $d\eta=\pi^*\Theta$, where Θ is the fundamental 2-form of M. Then P is locally homogeneous.*

Furthermore we find from the definition of T and (5.5) that

$$(\tilde{\nabla}_{\tilde{X}}\tilde{R})(\tilde{Y}, \tilde{X})\tilde{X}=-T(\tilde{X}, \tilde{R}(\tilde{Y}, \tilde{X})\tilde{X})+\tilde{R}(T(\tilde{X}, \tilde{Y}), \tilde{X})\tilde{X} \tag{5.6}$$

and

$$(\tilde{\nabla}_{\tilde{X}}T)(\tilde{X}, \tilde{Y})=0\,, \tag{5.7}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$. Thus by a result J. E. D'Atri and H. K. Nickerson (see [6], p. 252), we have

THEOREM 5.2. *Let M be a locally symmetric Kählerian manifold and P a principal circle bundle over M with a Riemannian metric given by* (2.2) *and a connection form η such that $d\eta=\pi^*\Theta$. Then P is a locally homogeneous and volume symmetric space.*

Now let $\widetilde{\mathrm{Ric}}$ and Ric be the Ricci tensor of $\tilde{g}$ and g respectively. Then by Lemma 3.2, we have the following

LEMMA 5.3 (cf. [10]).

(1) $\widetilde{\mathrm{Ric}}(\tilde{X}_i, \tilde{X}_j)=\mathrm{Ric}(X_i, X_j)-2t^2\delta_{ij}\,,$

(2) $\widetilde{\mathrm{Ric}}(\tilde{X}_i, \xi)=0\,,$

(3) $\widetilde{\mathrm{Ric}}(\xi, \xi)=nt^2\,.$

By this lemma, S. Kobayashi [10] showed that if M is Einsteinian, $\mathrm{Ric}(X_i, X_j)=(S/n)\delta_{ij}$, then putting $t^2=S/n(n+2)$, we have an Einsteinian metric on P. Then he proved the following

PROPOSITION 5.4. *If M is a complete Einstein-Kählerian manifold with positive scalar curvature, then we can construct a principal circle bundle P over M and specially an Einsteinian metric with positive scalar curvature.*

Combining Theorem 5.2 and Proposition 5.4, we have the following

THEOREM 5.5. *If M is a complete locally symmetric Einstein-Kählerian manifold with positive scalar curvature. Then we can construct a principal circle bundle P over M, which is locally homogeneous and volume symmetric* (*especially even Einsteinian*).

Now assume that the Einsteinian metric obtained in Proposition 5.4 satisfies the condition (*). Then we first note that

$$\Sigma\tilde{R}_{o\alpha\beta\gamma}\tilde{R}_{o\alpha\beta\gamma}=2\Sigma\tilde{R}_{oioj}\tilde{R}_{oioj}=2S^2/n(n+2)^2\,,$$

taking (1) of Lemma 3.2. Hence we have

$$\begin{aligned}\Sigma\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}_{\alpha\beta\gamma\delta}&=\Sigma\tilde{R}_{o\beta\gamma\delta}\tilde{R}_{o\beta\gamma\delta}+\Sigma\tilde{R}_{i\beta\gamma\delta}\tilde{R}_{i\beta\gamma\delta}\\&=(n+1)\Sigma\tilde{R}_{o\beta\gamma\delta}\tilde{R}_{o\beta\gamma\delta}=2\tilde{S}^2/n(n+1)\,.\end{aligned}$$

This concludes that P is of constant curvature. Thus we have

PROPOSITION 5.6. *A Riemannian metric on P obtained in Proposition 5.4 is an Einsteinian manifold satisfying (*) if and only if it is of constant curvature.*

6. An example.

Consider the complex quadric of complex dimension 2, $Q^2(C)=CP^1\times CP^1$. Then $Q^2(C)$ admits the natural product Einstein-Kählerian metric, which is locally symmetric. Therefore by Theorem 5.5, we can construct a principal circle bundle over $Q^2(C)$, which is homogeneous and volume symmetric (especially Einsteinian).

Now we give an explicit construction of this example. We take G/H as follows:

$$G=\left\{\begin{pmatrix} SU(2) & 0 \\ 0 & SU(2)\end{pmatrix}\right\},\qquad H=\left\{\begin{pmatrix} e^{-iu} & & & 0 \\ & e^{iu} & & \\ & & e^{-iv} & \\ 0 & & & e^{-iv}\end{pmatrix},\ u,v\in\boldsymbol{R}\right\}.$$

Then G is a compact connected Lie group with the Lie algebra $\mathfrak{g}$ and H a closed subgroup with the Lie algebra $\mathfrak{h}$. His locally the direct product of two closed normal subgroups H_1 and H_2:

$$H_1=\left\{\begin{pmatrix} e^{-iu} & & & 0 \\ & e^{iu} & & \\ & & e^{-iu} & \\ 0 & & & e^{iu}\end{pmatrix},\ u\in\boldsymbol{R}\right\},\qquad H_2=\left\{\begin{pmatrix} e^{-iv} & & & 0 \\ & e^{iv} & & \\ & & e^{iv} & \\ 0 & & & e^{-iv}\end{pmatrix},\ v\in\boldsymbol{R}\right\}.$$

This means that $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ (direct sum of ideals), where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the Lie algebras of H_1 and H_2, respectively. Then $P=G/H_2$ is a principal circle bundle over G/H with projection π and structure group H_1 where $\pi: P\rightarrow G/H$ is defined by $\pi(gH_2)=gH$ for $g\in G$ (see [9]). The $ad(G)$-invariant positive definite symmetric bilinear form $B: \mathfrak{g}\times\mathfrak{g}\to R$ is taken to be $-(1/2)$ the killing form on $\mathfrak{g}=su(2)+su(2)$. An orthonormal basis for $\mathfrak{g}$ with respect to this form is given by

$$e_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad e_2=\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad e_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_4=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},\quad e_5=1/\sqrt{2}\begin{pmatrix} -i & & & \mathbf{0} \\ & i & & \\ & & -i & \\ \mathbf{0} & & & i \end{pmatrix},\quad e_6=1/\sqrt{2}\begin{pmatrix} -i & & & \mathbf{0} \\ & i & & \\ & & i & \\ \mathbf{0} & & & -i \end{pmatrix}.$$

Now note that $\{e_5, e_6\}$ is a basis for the Lie algebra $\mathfrak{h}$ of H, and e_6 a basis for the Lie algebra $\mathfrak{h}_2$. We set $\mathfrak{m}$=the orthogonal complement of $\mathfrak{h}_2$ in $\mathfrak{g}$ spanned by $\{e_1, e_2, e_3, e_4, e_5\}$ and $\mathfrak{m}'$=the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ spanned by $\{e_1, e_2, e_3, e_4\}$. Let $\langle\,,\rangle$ be the metric induced by restricting B to $\mathfrak{m}'$, $\langle\,,\rangle_{o'}=B_{|\mathfrak{m}'}$ and $J_{o'}=-ad_{\mathfrak{m}'}(e_6)$, where $\langle\,,\rangle_{o'}$ (resp. $J_{o'}$) is the restriction of $\langle\,,\rangle$ (resp. J) to the tangent space $T_{o'}(G/H)$ of G/H at $o'=H$, identifying $\mathfrak{m}'$ with $T_{o'}(G/H)$. Then $(J, \langle\,,\rangle)$ is the natural (product) Kählerian structure on G/H. Furthermore let $\widetilde{\langle\,,\rangle}$ be the metric induced by restricting B to $\mathfrak{m}$, $\widetilde{\langle\,,\rangle}_o=B_{|\mathfrak{m}}$, where $\widetilde{\langle\,,\rangle}_o$ is the restriction of $\widetilde{\langle\,,\rangle}$ to tangent space $T_o(G/H_2)$ of G/H_2 at $o=\{H_2\}\in G/H_2$, identifying $\mathfrak{m}$ with $T_o(G/H_2)$. Setting $\eta_o(X)=\langle X, e_5\rangle_o$ for $X\in\mathfrak{m}$, one can see that η is a connection form on G/H_2 and $d\eta=\pi^*\Theta$ where Θ is the fundamental 2-form of the Kählerian structure $(J, \langle\,,\rangle)$.

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