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AN EXTREMAL PROBLEM ASSOCIATED WITH THE SPREAD RELATION

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0. Introduction. The notion of spread was introduced and investigated by Edrei [6], [7], who also conjectured the spread relation. This relation has now been proved by Baernstein [2] whose remarkable analysis rests on the introduction of a new function $T^*(z)$ $(z=re^{i\theta})$, closely related to Nevanlinna characteristic $T(r, f)$.

Let f be meromorphic and nonconstant. Suppose $\delta(\infty, f)>0$. Then it is suggested by Nevanlinna's theory that $|f(z)|$ must be "large" on a substantial portion of each circle $|z|=r$ when r is large. The spread relation provides a quantitative form of this statement.

To state this relation we require some notations. Let f be a meromorphic function of finite lower order μ. Fix a sequence $\{r_m\}$ of Pólya peaks of order μ of $f(z)$. Let $\Lambda(r)$ be a positive function with $\Lambda(r)=o(T(r, f))$ $(r\to\infty)$. Define the set of argument

$$E_\Lambda(r)=\{\theta : \log|f(re^{i\theta})|>\Lambda(r)\},$$

and let

$$\sigma_\Lambda(\infty)=\varliminf_{m\to\infty} \text{meas } E_\Lambda(r_m).$$

Then the spread of ∞ is defined by

$$\sigma(\infty)=\inf_\Lambda \sigma_\Lambda(\infty),$$

where the "inf" is taken over all functions Λ satisfying $\Lambda(r)=o(T(r, f))$.

Spread relation:

$$\sigma(\infty)\geqq\min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\sqrt{\frac{\delta(\infty, f)}{2}}\right\}. \tag{1}$$

(This inequality is best possible.) This makes it possible to solve the deficiency problem for functions with $1/2<\mu\leqq1$. (See [8].)

Baernstein's proof of the spread relation (1) is based on the properties of the function

$$T^*(re^{i\theta})=m^*(re^{i\theta})+N(r, f) \qquad (r>0,\ 0\leqq\theta\leqq\pi), \tag{2}$$

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where

$$m^*(re^{i\theta})=\sup_E \frac{1}{2\pi}\int_E \log|f(re^{i\varphi})|\,d\varphi\,;$$

the "sup" is taken over all measurable sets E of measure $|E|=2\theta$. Baernstein [2] showed that $T^*(re^{i\theta})$ is a subharmonic function in $0<r<\infty$, $0<\theta<\pi$.

In [9], Edrei and Fuchs introduced the notions of the hypotheses ES and the extremal spread.

Hypotheses ES. Let $f(z)$ be a meromorphic function of lower order μ $(0<\mu<\infty)$, and let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, f)$. Assume that

(i) $\delta(\infty, f)>0$ and, if $0<\mu\leqq 1/2$, assume in addition that $\delta(\infty, f)<1-\cos\pi\mu$ holds;

(ii) the sequence $\{r_m\}$ satisfies for some Λ

$$\lim_{m\to\infty} \text{meas}\, E_\Lambda(r_m)=\frac{4}{\mu}\sin^{-1}\sqrt{\frac{\delta(\infty, f)}{2}}\equiv 2\beta\,.$$

Extremal spread. If $f(z)$ satisfies the hypotheses ES, we say that it has extremal spread (of ∞).

Edrei and Fuchs [9], [10] considered all the meromorphic functions characterized by the hypotheses ES. One of their results is the following Theorem A.

THEOREM A. *Let $f(z)$ be meromorphic of lower order μ $(0<\mu<\infty)$ and let $f(z)$ have extremal spread of ∞. Consider the intervals*

$$I_m(s)=\{r\,;\, e^{-s}r_m<r\leqq e^s r_m\} \qquad (s>0,\ m=1,\ 2,\ \cdots)\,.$$

Then, for every $s>0$,

$$\frac{T(r, f)/r^\mu}{T(r_m, f)/r_m{}^\mu}\to 1 \quad (r\in I_m(s))\,, \qquad \frac{N(r, f)}{T(r, f)}\to\cos\beta\mu \quad \Big(r\in\bigcup_{m=1}^{\infty} I_m(s)\Big)\,.$$

Further, there exists a sequence $\{\eta_m\}$, $\eta_m\to 0$, independent of r and θ, such that

$$|T^*(re^{i\theta})-T(r, f)\cos\mu(\beta-\theta)|<\eta_m T(r, f) \qquad (0\leqq\theta\leqq\beta)\,,$$

provided $r\in I_m(s)$.

Also they have satisfactorily determined the asymptotic behavior of $\log|f(z)|$ and of the arguments of almost all the zeros and poles in the annuli $|z|\in I_m(s)$ $(m=1, 2, \cdots)$.

On the other hand, Baernstein [4] also considered extremal problems associated with the spread relation. To describe his result we introduce some notations and terminology. Let u be a δ-subharmonic function which can be represented as

(2) $$u(z)=u_1(z)-u_2(z),$$

where u_1 and u_2 are subharmonic in the plane. For a δ-subharmonic function (2) we put

$$N(r, u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}u(re^{i\theta})d\theta,$$

and the Nevanlinna characteristic of u is defined by

$$T(r, u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}u^{+}(re^{i\theta})d\theta+N(r, u).$$

Further the Baernstein characteristic of u is defined by

$$u^{\#}(re^{i\theta})=\sup_{E}\frac{1}{2\pi}\int_{E}u(re^{i\varphi})d\varphi+N(r, u_2) \qquad (0<r<\infty,\ 0\leqq\theta\leqq\pi),$$

where the "sup" is taken over all the measurable sets E of measure $|E|=2\theta$.

Suppose next that $G\subset(0, \infty)$ is a set which is unbounded above, and that $L(r)$ is a positive function. We say that L varies slowly on G (in the sense of Karamata) if

(3) $$\lim_{\substack{r\to\infty\\ r\in G}}\frac{L(kr)}{L(r)}=1$$

holds uniformly for k in any interval $A^{-1}\leqq k\leqq A$, $A>1$. Further, we say that the set G is very long if

(a) G has logarithmic density one, i. e.

$$\frac{1}{\log r}\int_{G\cap[1,r]}\frac{dt}{t}\longrightarrow 1 \qquad (r\to\infty)$$

and

(b) $$G=\bigcup_{n=1}^{\infty}[a_n, b_n]$$

where $a_n\to\infty$ and $b_n/a_n\to\infty$ as $n\to\infty$.

One of Baernstein's results in [4] is the following Theorem B.

THEOREM B. *Suppose $u=u_1-u_2$ be δ-subharmonic and suppose u has order $\rho\in(0, \infty)$. Let $\Lambda(r)$ be a nonnegative function satisfying $\Lambda(r)=o(T(r, u))$ $(r\to\infty)$. Then, if*

$$\delta(\infty, u)=1-\varlimsup_{r\to\infty}\frac{N(r, u_2)}{T(r, u)}>0$$

and

$$\varlimsup_{r\to\infty}\text{meas}\,\{\theta : u(re^{i\theta})>\Lambda(r)\}\leqq\frac{4}{\rho}\sin^{-1}\sqrt{\frac{\delta(\infty, u)}{2}}\equiv2\beta<2\pi,$$

there exist a very long set G and a function $L(r)$ varying slowly on G such that

$$T(r, u)=r^{\rho}L(r).$$

Moreover, if $\delta(\infty, u)<1$, *then*

$$N(r, u_2)\sim(1-\delta(\infty, u))T(r, u) \qquad (r\to\infty,\ r\in G).$$

In Theorem B, the exceptional set $F\equiv(0, \infty)-G$ on which (3) may fail can actually occur. Baernstein [4] showed this fact by applying Corollary 1 of [1] to the function constructed by Hayman [11, Theorem 3]. In order to see this fact more directly, we can use the notion of the flexible proximate order which was introduced by Drasin [5].

Let ρ and ρ_1 be any positive numbers such that

$$1/2<\rho<\rho_1<\infty.$$

Take for γ ($\leqq 1$) a positive number satisfying

$$\rho_1'\equiv\rho_1\gamma<1,$$

and with this γ we set

$$\rho'=\rho\gamma.$$

Then it is clear that

$$0<\rho'<\rho_1'<1.$$

Let $\lambda(r)$ $(r>0)$ be a continuous, nonnegative function which is continuously differentiable off a discrete set D, such that

$$r\lambda'(r)\longrightarrow 0 \qquad (r\to\infty,\ r\notin D).$$

Let E and E_1 be sets of the form

$$E=\bigcup_{n=1}^{\infty}[a_n, b_n], \qquad E_1=\bigcup_{n=1}^{\infty}[k_n^{-1}a_n, k_nb_n],$$

where

$$(1<)k_n\uparrow\infty \quad (n\to\infty), \qquad [k_n^{-1}a_n, k_nb_n]\cap[k_m^{-1}a_m, k_mb_m]=\phi \quad (m\neq n),$$

$$\int_{E_1\cap[1, r]}t^{-1}dt=o(\log r) \qquad (r\to\infty).$$

Now, suppose that $\lambda(r)$ satisfies

$$0<\rho'\leqq\lambda(r)\leqq\rho_1'<1,$$

$$\lambda(r)=\begin{cases}\rho' & (r\in E_1^c),\\ \rho_1' & (r\in E),\end{cases}$$

and let $\lambda(r)$ be extended to E_1-E so that it is continuous and

$$t\lambda'(t)=\begin{cases} -(\rho_1'-\rho)/\log k_n & t\in(k_n^{-1}a_n,\ a_n)\,, \\ (\rho_1'-\rho)/\log k_n & t\in(b_n,\ k_nb_n)\,. \end{cases}$$

Then it is clear that

$$(\log r)^{-1}\int_1^r \lambda(t)t^{-1}dt \longrightarrow \rho' \qquad (r\to\infty)\,.$$

Let $f(z)$ be a canonical product with negative zeros with counting function

$$n(r)=\left[\exp\left(\int_1^r \lambda(t)t^{-1}dt\right)\right].$$

Then $f(z)$ is of order ρ' (<1) and so, for a suitable branch of $\log f(z)$

$$\log f(z)=z\int_0^\infty \frac{n(t)}{t(t+z)}dt \qquad (|\arg z|<\pi)\,.$$

Using the reasoning of the proof of Proposition in [5, p. 133], we have

$$\log f(z)=\left\{\frac{\pi}{\sin \pi\lambda(r)}e^{i\lambda(r)\theta}+o(1)\right\}n(r)\,,$$

where the $o(1)$ tends to zero uniformly as $z\to\infty$ in any sector: $|\theta|\leqq\pi-\eta$.

Here, we define $u(z)$ as follows:

$$u(z)=\begin{cases} \max\{\log|f(z^{1/\gamma})|,\ 0\} & \left(|\theta|<\beta\equiv\dfrac{\pi}{2\rho}\right), \\ 0 \quad (\beta\leqq|\theta|\leqq\pi)\,. \end{cases}$$

It is easily verified that u is subharmonic in the plane, has order $\rho'/\gamma=\rho$, and satisfies

$$\overline{\lim_{r\to\infty}}\ \text{meas}\{\theta : u(re^{i\theta})>0\}=\pi/\rho=2\beta(<2\pi)\,,$$

$$T(r,\ u)=(1+o(1))\frac{\gamma n(r^{1/\gamma})}{\lambda(r^{1/\gamma})\sin \pi\lambda(r^{1/\gamma})} \qquad (r\to\infty)\,.$$

However, since $r\lambda'(r)\to 0$ $(r\to\infty,\ r\notin D)$ implies $\lambda(kr)=\lambda(r)+o(1)$ $(r\to\infty)$ for fixed k (>0), we have $n(kr)\sim k^{\lambda(r)}n(r)$. Hence

$$\frac{T(kr,\ u)}{T(r,\ u)}=(1+o(1))k^{\lambda(r)/\gamma} \qquad (r\to\infty)\,.$$

This illustrates the existence of the exceptional set F.

Now, comparing Theorem B with Theorem A, the following problem is naturally raised.

Problem. Do the assumptions of Theorem B imply the existence of some very long set G and slowly varying function $L(r)$ on G such that

$$T(r, u)=r^{\rho}L(r) \quad (0<r<\infty), \qquad \frac{u^{\#}(re^{i\theta})}{T(r, u)} \longrightarrow \cos\rho(\beta-\theta) \quad (r\to\infty,\ r\in G)$$

uniformly for $\theta\in[0, \beta]$?

For example, $u(z)$ constructed above satisfies the conclusion of *Problem* with $G=\{r : r^{1/\gamma}\in E_1{}^c\}$ and

$$L(r)=(1+o(1))\frac{n(r^{1/\gamma})r^{-\rho}}{\rho\sin(\pi\rho/\gamma)} \qquad (r\in G).$$

However, I have been unable to solve this problem. In this note, I prove the following result.

THEOREM. *Let the assumptions and notations of Theorem B be unchanged. Further, suppose that $T(r, u)$ satisfies the following growth condition*:

$$\lim_{r\to\infty}\frac{T(kr, u)}{T(r, u)}=k^{\rho}$$

(*uniformly for k in any interval $A^{-1}\leqq k\leqq A$, $A>1$*). *Then, there exist a very long set G and a function $L(r)$ varying slowly on $(0, \infty)$ such that*

$$T(r, u)=r^{\rho}L(r) \quad (0<r<\infty), \qquad \frac{u^{\#}(re^{i\theta})}{T(r, u)} \longrightarrow \cos\rho(\beta-\theta) \quad (r\to\infty, r\in G)$$

uniformly for $\theta\in[0, \beta]$.

1. Preliminaries of the proof of Theorem. In order to prove our theorem we need some facts. The fact that we need about very long set is contained in Lemma 1 below.

LEMMA 1. *Let $G_1, \cdots, G_n$ $(2\leqq n<\infty)$ be distinct very long sets. Then, there exists a very long set G such that $G\subset\bigcap_{k=1}^{n}G_k$.*

Proof. We may prove Lemma 1 in case of $n=2$. First, an easy computation shows that

$$\log\operatorname{dens}(G_1\cap G_2)=1. \tag{4}$$

Next, we put $G_1=\bigcup_{n=1}^{\infty}[a_n, b_n]$, $G_2=\bigcup_{n=1}^{\infty}[c_n, d_n]$. Then

$$a_n\longrightarrow\infty,\quad b_n/a_n\longrightarrow\infty,\quad c_n\longrightarrow\infty,\quad d_n/c_n\longrightarrow\infty \qquad (n\to\infty). \tag{5}$$

It is clear that for every n $(=1, 2, \cdots)$ there exist at most finitely many m's such that $[c_m, d_m]\cap[a_n, b_n]\neq\phi$. We denote such m's by $m_n, \cdots, m_n+j_n$ (j_n: a nonnegative integer) (if any). Then

$$G_1 \cap G_2 = \bigcup_{n=1}^{\infty} \{([a_n, b_n] \cap [c_{m_n}, d_{m_n}]) \cup ([a_n, b_n] \cap [c_{m_n+1}, d_{m_n+1}]) \cup \cdots$$

$$\cup ([a_n, b_n] \cap [c_{m_n+J_n}, d_{m_n+J_n}])\} .$$

Now, starting from $G_1 \cap G_2$, we construct a subset G of $G_1 \cap G_2$ as follows: Firstly, let $I(J)$ be a subset $\{n\}$ of positive integers satisfying $a_n > c_{m_n}$ ($b_n < d_{m_n+J_n}$). Secondly, we take

$$a_n' = \lambda_n a_n , \qquad b_n' = b_n / \lambda_n ,$$

where

$$\lambda_n = \min (a_n^{\delta_n}, (b_n/a_n)^{\delta_n})$$

and $\{\delta_n\}$ is a positive sequence satisfying

$$\delta_n \longrightarrow 0 , \qquad a_n^{\delta_n} \longrightarrow \infty , \qquad (b_n/a_n)^{\delta_n} \longrightarrow \infty .$$

And thirdly, making use of $\{a_n'\}$ and $\{b_n'\}$, we define two subsets I', J' of positive integers:

$$I' = \{n \,;\, n \in I,\, d_{m_n} < a_n'\} ,$$

$$J' = \{n \,;\, n \in J,\, c_{m_n+J_n} > b_n'\} .$$

Here we put

$$G = (G_1 \cap G_2) \setminus \{ \bigcup_{n \in I'} ([a_n, b_n] \cap [c_{m_n}, d_{m_n}])\}$$

$$\cup \{ \bigcup_{n \in J'} ([a_n, b_n] \cap [c_{m_n+J_n}, d_{m_n+J_n}])\}$$

$$\equiv \bigcup_{n=1}^{\infty} [e_n, f_n] .$$

Then it follows from (5) and the definitions of I', J' that

$$e_n \longrightarrow \infty , \qquad f_n/e_n \longrightarrow \infty \qquad (n \to \infty) .$$

Finally we prove log dens $G=1$. Noting (4), it is sufficient to prove $\log \tilde{G} = 0$, where

$$\tilde{G} = \{ \bigcup_{n \in I'} ([a_n, b_n] \cap [c_{m_n}, d_{m_n}])\} \cup \{ \bigcup_{n \in J'} ([a_n, b_n] \cap [c_{m_n+J_n}, d_{m_n+J_n}])\} .$$

For each $r \geqq a_1$, we can uniquely determine $n = n(r)$ such that $a_n \leqq r < a_{n+1}$, and it is clear that $n(r) \to \infty$ as $r \to \infty$. By the definitions of I' and J', we have

$$\frac{1}{\log r} \int_{\tilde{G} \cap [1, r]} \frac{dt}{t} \leqq \frac{1}{\log r} \Big\{ \Big(\sum_{\substack{n \in I' \\ n < n(r)}} + \sum_{\substack{n \in J' \\ n < n(r)}} \Big) \delta_n \log \Big(\frac{b_n}{a_n} \Big)$$

(6)

$$+ \log \Big(\frac{\min [a'_{n(r)}, r]}{a_{n(r)}} \Big) + \log^+ \Big(\frac{\min [b_{n(r)}, r]}{b'_{n(r)}} \Big) \Big\} .$$

However, since $\delta_n \to 0$ as $n \to \infty$ and

$$\frac{1}{\log r}\left\{\sum_{n<n(r)} \log\left(\frac{b_n}{a_n}\right)+\log\left(\frac{\min[b_{n(r)}, r]}{a_{n(r)}}\right)\right\} \longrightarrow 1 \qquad (r\to\infty),$$

the right hand side of (6)$\to 0$ as $r\to\infty$. Hence $\log\operatorname{dens}\tilde{G}=0$. This completes the proof of Lemma 1.

Our second lemma is concerned with the estimate of $u^{\#}(re^{i\theta})$ $(0\leqq\theta\leqq\beta)$ from above under the assumptions of Theorem. For the proof, the following two propositions are essential.

PROPOSITION 1. ([3, Theorem A′, pp. 144-148]) *Suppose $u=u_1-u_2$ be δ-subharmonic. Then $u^{\#}$ is subharmonic in $\{re^{i\theta};\ 0<r<\infty,\ 0<\theta<\pi\}$ and is continuous on $\{re^{i\theta};\ 0<r<\infty;\ 0\leqq\theta\leqq\pi\}$.*

PROPOSITION 2. (cf. [2, p. 430]) *Suppose that a function h is harmonic in the half-disk $D_R=\{z=re^{i\theta};\ 0<r<R,\ 0<\theta<\pi\}$ and continuous on the closure. Then, for $z\in D_R$*

$$h(re^{i\theta})=\int_{-R}^{R} h(t)A(t, r, \theta, R)dt+\int_0^{\pi} h(Re^{i\varphi})B(\varphi, r, \theta, R)d\varphi,$$

where

$$A(t, r, \theta, R)=\frac{1}{\pi}\frac{r\sin\theta}{t^2+r^2-2tr\cos\theta}-\frac{1}{\pi}\frac{R^2 r\sin\theta}{R^4-2rtR^2\cos\theta+r^2t^2},$$

$$B(\varphi, r, \theta, R)=\frac{2Rr\sin\theta}{\pi}\frac{(R^2-r^2)\sin\varphi}{|R^2e^{2i\varphi}-2rRe^{i\varphi}\cos\theta+r^2|^2}.$$

Now we prove

LEMMA 2. *Let the assumptions and notations of Theorem be unchanged. Then there exists a slowly varying function $L(r)$ on $(0, \infty)$ satisfying the following conditions:*

(i) $T(r, u)=r^{\rho}L(r)$ $\quad(0<r<\infty)$,

(ii) *For any $\eta>0$, there exists $r_0=r_0(\eta)>0$ such that $r\geqq r_0$ implies*

$$u^{\#}(re^{i\theta})<[\cos(\beta-\theta)\rho+\eta]r^{\rho}L(r) \qquad (0\leqq\theta\leqq\beta).$$

Proof. First, we consult [4, § 5, pp. 98-100]. Then it is easy to see under our assumptions that

$$u^{\#}(re^{i\beta})\sim T(r, u)=r^{\rho}L(r) \qquad (r\to\infty), \tag{7}$$

where $L(r)$ is a slowly varying function on $(0, \infty)$.

Choose a positive number $\varepsilon=\varepsilon(\eta)$ satisfying

$$\varepsilon+(2\varepsilon+\varepsilon^2)\varepsilon<\eta. \tag{8}$$

Further, let A ($\geqq 2$) be a number such that

$$\varepsilon+(2\varepsilon+\varepsilon^2)\Big\{\varepsilon+\frac{2\cdot A^{1-\gamma\rho}}{\pi(A-1)^2}+\frac{32}{A^{1-\gamma\rho}}\Big\}<\eta\,, \tag{9}$$

where $\gamma=\beta/\pi$ ($\gamma\rho\leqq 1/2$). By the definition of $\delta(\infty, u)$, we have

$$u^{\#}(r)=N(r, u_2)<(1-\delta(\infty, u)+\varepsilon)T(r, u)=(\cos\beta\rho+\varepsilon)r^{\rho}L(r)\quad (r>t_1=t_1(\varepsilon))\,. \tag{10}$$

Since $L(r)$ is a slowly varying function on $(0, \infty)$, we have

$$\Big|\frac{L(kr)}{L(r)}-1\Big|>\varepsilon\quad \Big(\frac{1}{A^{\gamma}}\leqq k\leqq A^{\gamma},\ r\geqq t_2,\ t_2=t_2(A, \varepsilon, \gamma)\Big). \tag{11}$$

Here we put

$$L_1(r)=L(r^{\gamma})\,. \tag{12}$$

Then we can rewrite (7), (10) and (11) as follows:

$$|u^{\#}(r^{\gamma}e^{i\beta})-r^{\gamma\rho}L_1(r)|<\varepsilon r^{\gamma\rho}L_1(r)\qquad (r\geqq t_0^{1/\gamma},\ t_0=t_0(\varepsilon))\,, \tag{7$'$}$$

$$u^{\#}(r^{\gamma})<(\cos\pi\gamma\rho+\varepsilon)r^{\gamma\rho}L_1(r)\qquad (r\geqq t_1^{1/\gamma})\,, \tag{10$'$}$$

$$\Big|\frac{L_1(kr)}{L_1(r)}-1\Big|<\varepsilon\quad \Big(\frac{1}{A}\leqq k\leqq A,\ r\geqq t_2^{1/\gamma}\Big)\,. \tag{11$'$}$$

Now, we define

$$v(z)=u^{\#}(z^{\gamma})\qquad (0<|z|<\infty,\ 0\leqq \arg z\leqq\pi)\,. \tag{13}$$

Then it follows from Propositions 1 and 2 that for $z=re^{i\theta}\in D_R$

$$v(re^{i\theta})\leqq\int_0^R v(te^{i\pi})A(t, r, \pi-\theta, R)dt$$

$$+\int_0^R v(t)A(t, r, \theta, R)dt+\int_0^{\pi} v(Re^{i\varphi})B(\varphi, r, \theta, R)d\varphi\,.$$

Some elementary computations show that for $0<r<R/2$, $0<\theta<\pi$

$$\begin{aligned} v(re^{i\theta})\leqq&\frac{1}{\pi}\int_0^R v(te^{i\pi})\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}dt\\ &+\frac{1}{\pi}\int_0^R v(t)\frac{r\sin\theta}{t^2+r^2-2tr\cos\theta}dt+\frac{32r}{R}T(R^{\gamma}, u)\,. \end{aligned} \tag{14}$$

Fix $r>T_0\equiv\max(At_0^{1/\gamma}, At_1^{1/\gamma}, t_2^{1/\gamma})$ and put $R=Ar$. From (12), (7)$'$ and (11)$'$ it follows that

$$\begin{aligned}(15)\quad &\int_0^{Ar} v(-t)\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}\,dt=\left(\int_0^{r/A}+\int_{r/A}^{Ar}\right)\\ &<v\left(-\frac{r}{A}\right)\frac{r}{A}\frac{r}{(r-r/A)^2}+\int_{r/A}^{Ar}(1+\varepsilon)t^{\gamma\rho}L_1(t)\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}\,dt\\ &<(1+\varepsilon)^2\frac{r^{\gamma\rho}}{A^{\gamma\rho}}L_1(r)\frac{r}{A}\frac{r}{(r-r/A)^2}+(1+\varepsilon)^2r^{\gamma\rho}L_1(r)\int_{1/A}^{A}u^{\gamma\rho}\frac{\sin\theta}{u^2+1+2u\cos\theta}\,du\\ &<(1+\varepsilon)^2\left\{\frac{A^{1-\gamma\rho}}{(A-1)^2}+\frac{\pi\sin\theta\gamma\rho}{\sin\pi\gamma\rho}\right\}r^{\gamma\rho}L_1(r)\,.\end{aligned}$$

In the same way, we have from (12), (10)′, (11)′

$$\begin{aligned}(16)\quad &\int_0^{Ar} v(t)\frac{r\sin\theta}{t^2+r^2-2tr\cos\theta}\,dt\\ &<(1+\varepsilon)^2\left\{\frac{A^{1-\gamma\rho}}{(A-1)^2}+(\cos\pi\gamma\rho+\varepsilon)\frac{\pi\sin(\pi-\theta)\gamma\rho}{\sin\pi\gamma\rho}\right\}r^{\gamma\rho}L_1(r)\,.\end{aligned}$$

Further, from (7) and (11)′ it follows that

$$\begin{aligned}(17)\quad &\frac{32r}{Ar}T(A^\gamma r^\gamma,\,u)=\frac{32}{A}A^{\gamma\rho}r^{\gamma\rho}L_1(Ar)\\ &<\frac{32}{A}A^{\gamma\rho}r^{\gamma\rho}(1+\varepsilon)L_1(r)=\frac{32}{A^{1-\gamma\rho}}(1+\varepsilon)r^{\gamma\rho}L_1(r)\,.\end{aligned}$$

Substituting (15), (16) and (17) into (14) with $R=Ar$, we deduce

$$\begin{aligned}(18)\quad v(re^{i\theta})&<(1+\varepsilon)^2\left\{\frac{\sin\theta\gamma\rho}{\sin\pi\gamma\rho}+\frac{\sin(\pi-\theta)\gamma\rho\cdot\cos\pi\gamma\rho}{\sin\pi\gamma\rho}+\varepsilon\frac{\sin(\pi-\theta)\gamma\rho}{\sin\pi\gamma\rho}\right.\\ &\qquad\left.+\frac{2A^{1-\gamma\rho}}{\pi(A-1)^2}+\frac{32}{A^{1-\gamma\rho}}\right\}r^{\gamma\rho}L_1(r)\\ &<(1+\varepsilon)^2\left\{\cos(\pi-\theta)\gamma\rho+\varepsilon+\frac{2A^{1-\gamma\rho}}{\pi(A-1)^2}+\frac{32}{A^{1-\gamma\rho}}\right\}r^{\gamma\rho}L_1(r)\\ &<\left\{\cos(\pi-\theta)\gamma\rho+\varepsilon+(\varepsilon^2+2\varepsilon)\left(\varepsilon+\frac{2A^{1-\gamma\rho}}{\pi(A-1)^2}+\frac{32}{A^{1-\gamma\rho}}\right)\right\}r^{\gamma\rho}L_1(r)\,.\end{aligned}$$

Using (9) into (18) we have

$$v(re^{i\theta})<[\cos(\pi-\theta)\gamma\rho+\eta]r^{\gamma\rho}L_1(r)\qquad(r>T_0,\ 0\leqq\theta\leqq\pi)\,.$$

Therefore, in view of (13)

$$u^{\#}(r^\gamma e^{i\gamma\theta})<[\cos(\pi\gamma-\theta\gamma)\rho+\eta]r^{\gamma\rho}L_1(r)\qquad(r>T_0,\ 0\leqq\theta\leqq\pi)\,.$$

Hence it follows from (12) that

$$u^{\#}(re^{i\theta})<[\cos(\beta-\theta)\rho+\eta]r^{\rho}L(r) \qquad (r>T_0^{\gamma}\equiv r_0).$$

This completes the proof of Lemma 2.

Combining Lemma 2 with Baernstein's method in [4, §5, pp. 98-110], we can prove the following Lemma 3.

LEMMA 3. *Let the assumptions and notations of Theorem be unchanged. Let* α $(0<\alpha<\beta)$ *be given. Then, there exists a very long set* G_α *and a slowly varying function* $L(r)$ *on* $(0, \infty)$ *such that*

$$T(r, u)=r^{\rho}L(r) \quad (0<r<\infty), \qquad u^{\#}(re^{i\alpha})\sim\cos\rho(\beta-\alpha)\cdot r^{\rho}L(r) \quad (r\to\infty,\ r\in G_\alpha).$$

To see this, we may follow Baernstein's procedure in [4, §5] with $\gamma_1\equiv(\beta-\alpha)/\pi$, $u^{\#}(z^{\gamma_1}e^{i\alpha})$, $u^{\#}(r^{\gamma_1}e^{i\beta})$, $u^{\#}(r^{\gamma_1}e^{i\alpha})$ in place of his γ, $v(z)$, $T_1(r)$, $N_1(r)$, respectively. In fact, by virtue of Lemma 2 his argument there does work in this case.

The following proposition will play an important role in the proof of Theorem.

PROPOSITION 3. ([9, Lemma 6.1.] *Let* t_1 *and* t_2 *satisfy all the following conditions*:

$$0<R_0=R_0(u)<t_j\leqq R/4 \qquad (j=1, 2),$$

$$(1+\sigma)^{-1}\leqq\frac{t_1}{t_2}\leqq 1+\sigma \qquad (\sigma\geqq 0).$$

Then

$$|u^{\#}(t_1e^{i\theta_1})-u^{\#}(t_2e^{i\theta_2})|$$

$$\leqq A_0T(R, u)\left\{\sigma\left(1+\log^{+}\frac{1}{\sigma}\right)+|\theta_2-\theta_1|\left(1+\log^{+}\frac{1}{|\theta_2-\theta_1|}\right)\right\}$$

$$(0\leqq\theta_1\leqq\pi,\ 0\leqq\theta_2\leqq\pi),$$

where A_0 *is an absolute constant* (>0).

2. Proof of Theorem. Let η $(0<\eta<1)$ be given. Choose σ $(0<\sigma<1)$ such that

$$A_0 4^{\rho}\left(1+\log\frac{1}{\sigma}\right)\sigma+\sigma\rho<\eta/2,$$

where A_0 is the absolute constant (>0) which appears in Proposition 3. Further, take $\varepsilon>0$ so that

$$(1+\varepsilon)A_0 4^{\rho}\left(1+\log\frac{1}{\sigma}\right)\sigma+\sigma\rho<\eta/2. \tag{19}$$

By Theorem B and Lemma 3, for each α $(0\leqq\alpha\leqq\beta)$ there exist a very long set G_α and a function $L(r)$ varying slowly on $(0, \infty)$ such that

$$T(r, u)=r^\rho L(r) \qquad (0<r<\infty), \tag{20}$$

$$|u^\#(re^{i\alpha})-\cos\rho(\beta-\alpha)\cdot r^\rho L(r)|<(\eta/2)r^\rho L(r) \qquad (r\in G_\alpha,\ r\geqq r_\alpha(\eta)). \tag{21}$$

Since L is a slowly varying function on $(0, \infty)$, we have

$$\left|\frac{L(kr)}{L(r)}-1\right|<\varepsilon \qquad \left(\frac{1}{4}\leqq k\leqq 4,\ r\geqq t_1(4, \varepsilon)\right). \tag{22}$$

It follows from Proposition 3 that

$$\begin{gathered}|u^\#(re^{i\theta})-u^\#(re^{i\alpha})|<A_0\sigma\left(1+\log\frac{1}{\sigma}\right)T(4r, u)\\ (|\theta-\alpha|<\sigma,\ \theta\in[0, \beta],\ r>R_0).\end{gathered} \tag{23}$$

Now, we put $R_\alpha\equiv\max\{r_\alpha, t_1, R_0\}$. Then from (23), (20), (21), (22) and (19) it follows that

$$\begin{aligned}&|u^\#(re^{i\theta})-\cos\rho(\beta-\theta)T(r, u)|\\ &\quad\leqq|u^\#(re^{i\theta})-u^\#(re^{i\alpha})|+|u^\#(re^{i\alpha})-\cos\rho(\beta-\alpha)\cdot r^\rho L(r)|\\ &\qquad+|\cos\rho(\beta-\alpha)-\cos\rho(\beta-\theta)|r^\rho L(r)\\ &\quad<\left\{A_0\sigma\left(1+\log\frac{1}{\sigma}\right)(1+\varepsilon)4^\rho+\eta/2+\sigma\rho\right\}r^\rho L(r)<\eta T(r, u)\\ &\qquad(r\in G_\alpha,\ r\geqq R_\alpha,\ |\theta-\alpha|<\sigma,\ \theta\in[0, \beta]).\end{aligned} \tag{24}$$

Since $\{(\alpha-\sigma, \alpha+\sigma)\}_{\alpha\in[0,\beta]}$ is a covering of $[0, \beta]$, there exist $\{\alpha_j\}_{j=1}^m$ $(\alpha_j\in[0, \beta],$ $m<\infty)$ such that

$$[0, \beta]\subset\bigcup_{k=1}^m(\alpha_k-\sigma, \alpha_k+\sigma). \tag{25}$$

Hence, if we put

$$R\equiv\max(R_{\alpha_1}, \cdots, R_{\alpha_m})=R(\eta), \qquad \tilde{G}\equiv\bigcap_{k=1}^m G_{\alpha_k},$$

we deduce from (24) and (25) that

$$\begin{gathered}|u^\#(re^{i\theta})-\cos\rho(\beta-\theta)T(r, u)|<\eta\cdot T(r, u)\\ (r\in\tilde{G},\ r\geqq R(\eta),\ 0\leqq\theta\leqq\beta).\end{gathered} \tag{26}$$

Combining (26) with Lemma 1, we have the desired result.

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