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ON ALMOST CONTACT METRIC COMPOUND STRUCTURE

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Introduction. K. Yano and U.-H. Ki [8] have recently introduced the notion of $(f, g, u, v, w, \lambda, \mu, \nu)$-structure in an odd-dimensional manifold M, which is an abstraction of the induced structure in a submanifold of codimension 3 in an almost Hermitian manifold, and studied conditions for such a structure to define an almost contact structure in M and properties of pseudo-umbilical submanifold of codimension 3 satisfying the conditions in a Euclidean space of even-dimension.

In the present paper, we shall introduce in § 1 the notion of metric compound structure in a manifold M of dimension m, which is a generalization of $(f, g, u, v, w, \lambda, \mu, \nu)$ and naturally induced in M if M is a submanifold in an almost Hermitian manifold $\tilde{M}$ of dimension n. In § 2, we shall seek for conditions in order that a metric compound structure defines an almost contact metric structure in M. After the definition of normality in § 3, we shall consider in § 4 submanifolds having a normal contact metric compound structure in a Kaehlerian manifold. In § 5, we shall disscuss properties and give geometrical characterization of pseudo-umbilical submanifolds in a Euclidean space. In § 6, we shall show that a metric compound structure possessing another property gives an almost contact metric structure.

Throughout this paper, we put $l=n-m$ and indices run the following ranges respectively:

$$\kappa, \lambda, \mu, \nu, \cdots = 1, 2, \cdots\cdots, n;$$

$$h, i, j, k, \cdots = 1, 2, \cdots, m;$$

$$p, q, r, s, \cdots = m+1, m+2, \cdots, n;$$

$$A, B, C, D, \cdots = 1, 2, \cdots, m, m+1, \cdots\cdots, n.$$

§ 1. Metric compound structure

Let $\tilde{M}$ be an n-dimensional almost Hermitian manifold and $(G, \tilde{F})$ the almost Hermitian structure, where G is the almost Hermitian metric and $\tilde{F}$ the almost complex structure of $\tilde{M}$. We denote by $G_{\lambda\mu}$ and $\tilde{F}_\lambda{}^\kappa$ components of G and $\tilde{F}$ with respect to a local coordinate system (x^κ). If $I=(\delta_\lambda{}^\kappa)$ indicates the identity

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tensor, then the structure satisfy the equations

$$\tilde{F}^2=-I\,;\ \tilde{F}_\mu{}^\lambda \tilde{F}_\lambda{}^\kappa=-\delta_\mu{}^\kappa \tag{1.1}$$

and

$$^t\tilde{F}G\tilde{F}=G\,;\ \tilde{F}_\nu{}^\lambda \tilde{F}_\mu{}^\kappa G_{\lambda\kappa}=G_{\nu\mu}\,. \tag{1.2}$$

If we put the covariant components of $\tilde{F}$ as

$$\tilde{F}^*=G\tilde{F}\,;\ \tilde{F}_{\mu\lambda}=\tilde{F}_\mu{}^\kappa G_{\lambda\kappa}\,, \tag{1.3}$$

then $\tilde{F}_{\mu\lambda}$ is skew-symmetric in λ and μ.

Let M be an m-dimensional Riemannian manifold and suppose now that it is immersed isometrically in $\tilde{M}$ by the parametric equations

$$x^\kappa=x^\kappa(y^h) \tag{1.4}$$

by use of a local coordinate system (y^h) of M.

We put

$$B_\iota{}^\kappa=\partial_i x^\kappa \tag{1.5}$$

and denote by $C_q{}^\kappa$ l mutually orthogonal unit normal vector fields of M. Then the n vectors $B_\iota{}^\kappa$ and $C_q{}^\kappa$ span the tangent space $T(\tilde{M})$ of $\tilde{M}$ at every point of M and the matrix

$$B=(B_B{}^\kappa)=(B_\iota{}^\kappa,\ C_q{}^\kappa)$$

is regular. The metric tensor g of M is related with G of $\tilde{M}$ by

$$g_{ji}=G_{\mu\lambda}B_j{}^\mu B_\iota{}^\lambda\,. \tag{1.6}$$

Denoting the contravariant components of g by g^{ih}, we put

$$B^h{}_\lambda=g^{ih}G_{\lambda\kappa}B_\iota{}^\kappa\,,$$

$$C_{q\lambda}=G_{\lambda\kappa}C_q{}^\kappa\,.$$

Then the inverse matrix B^{-1} of B is given by

$$B^{-1}=(B_{A\lambda})=\begin{pmatrix} B^h{}_\lambda \\ C_{p\lambda} \end{pmatrix}.$$

Now we put

$$F=B^{-1}\tilde{F}B\,;\ (F_B{}^A)=(B_B{}^\lambda \tilde{F}_\lambda{}^\kappa B^A{}_\kappa)=\begin{pmatrix} f_\iota{}^h & -v_q{}^h \\ v_{p\iota} & f_{qp} \end{pmatrix}. \tag{1.7}$$

Then the components of four kinds of F are given by

$$f_\iota{}^h=B_\iota{}^\lambda \tilde{F}_\lambda{}^\kappa B^h{}_\kappa\,,\qquad v_q{}^h=-C_q{}^\lambda \tilde{F}_\lambda{}^\kappa B^h{}_\kappa\,,$$

$$v_{pi}=B_i{}^\lambda \tilde{F}_\lambda{}^\kappa C_{p\kappa}\,, \qquad f_{qp}=C_q{}^\lambda \tilde{F}_\lambda{}^\kappa C_{p\kappa}\,.$$

Since $\tilde{F}^*=(\tilde{F}_{\mu\lambda})$ is skew-symmetric, we have the relations

$$v_{pi}=v_p{}^h g_{ih} \tag{1.8}$$

and see that

$$f_{ji}=B_j{}^\lambda B_i{}^\kappa \tilde{F}_{\lambda\kappa} \tag{1.9}$$

is skew-symmetric in i and j, and

$$f_{qp}=C_q{}^\lambda C_p{}^\kappa \tilde{F}_{\lambda\kappa} \tag{1.10}$$

is also skew-symmetric in p and q. Thus the sets $f=(f_i{}^h)$, $v=(v_q{}^h)$ and $f^\perp=(f_{qp})$ compose a (1, 1)-tensor, m vector fields and $l(l-1)/2$ scalar fields on M respectively.

The transforms of the tangent vectors $B_i{}^\kappa$ and the normal vectors $C_q{}^\kappa$ to M by $\tilde{F}$ are expressed in the form

$$\tilde{F}_\lambda{}^\kappa B_i{}^\lambda=f_i{}^h B_h{}^\kappa+v_{pi}C_p{}^\kappa \tag{1.11}$$

and

$$\tilde{F}_\lambda{}^\kappa C_q{}^\lambda=-v_q{}^h B_h{}^\kappa+f_{qp}C_p{}^\kappa\,, \tag{1.12}$$

where and in the sequel summation convention is also applied to repeated lower indices $p, q, r, \cdots$ on their own range $m+1, m+2, \cdots, n$. Since the matrix (1.7) satisfies the equation

$$F^2=-I\,,$$

the quantities f, v and $f^\perp$ are in the relation

$$f_j{}^i f_i{}^h=-\delta_j^h+v_{qj}v_q{}^h\,, \tag{1.13}$$

$$f_j{}^i v_{pi}=-v_{qj}f_{qp}=f_{pq}v_{qj}\,, \tag{1.14}$$

$$v_r{}^i f_i{}^h=-f_{rq}v_q{}^h\,, \tag{1.15}$$

$$f_{rq}f_{qp}=-\delta_{rp}+v_r{}^i v_{pi}\,. \tag{1.16}$$

The relation (1.6) is equivalent to

$$f_j{}^k f_i{}^h g_{kh}+v_{qj}v_{qi}=g_{ji}\,. \tag{1.17}$$

Now removing the almost Hermitian ambient manifold $\tilde{M}$, we consider an m-dimensional Riemannian manifold M admitting a metric tensor g, a (1, 1)-tensor field f, m vector fields v_q and $l(l-1)/2$ scalar fields f_{qp} such that they satisfy the relations (1.13), (1.14), (1.15), (1.16) and (1.17), and call the totality $(f, g, v, f^\perp)$ of these quantities a *metric compound structure* on M.

If we put

(1.18) $$\tilde{F}=\begin{pmatrix} f_i{}^h & -v_q{}^h \\ v_{pi} & f_{qp} \end{pmatrix} \quad\text{and}\quad G=\begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{qp} \end{pmatrix},$$

then the set $(\tilde{F}, G)$ defines an almost Hermitian structure in the product space $M\times R^l$ of the manifold M with an l-dimensional Euclidean space R^l.

§ 2. Almost contact metric compound structure

We shall suppose that the tensor field f together with the metric tensor g, a contravariant vector field $\xi=(\xi^h)$ and a covariant vector field $\eta=(\eta_i)$ compose an almost contact metric structure on M. Then we have

(2.1) $$f_j{}^i f_i{}^h=-\delta_j{}^h+\eta_j\xi^h ,$$

(2.2) $$f_i{}^h\xi^i=0 , \qquad f_i{}^h\eta_h=0 ,$$

(2.3) $$\xi^i\eta_i=1$$

and

(2.4) $$f_j{}^k f_i{}^h g_{kh}+\eta_j\eta_i=g_{ji} .$$

In this case we know that the dimension m of M is odd and the rank of $f=(f_j{}^i)$ is equal to $m-1$.

Comparing (1.17) with (2.4), we have

(2.5) $$v_{qj}v_{qi}=\eta_j\eta_i .$$

This equation shows that the product of the matrix (v_{qi}) with the transpose is of rank 1 and consequently that the matrix (v_{qi}) by itself is of rank 1. Therefore we may put

(2.6) $$v_{qi}=\nu_q\eta_i ,$$

where ν_q are proportional factors. Since $v_{qi}v_q{}^i=\eta_i\xi^i=1$, We have

(2.7) $$\nu_q\nu_q=1$$

and the equations (1.15) and (1.16) are reduced to

(2.8) $$f_{qp}\nu_p=0$$

and

(2.9) $$f_{rq}f_{qp}=-\delta_{rp}+\nu_r\nu_p$$

respectively. The equations (2.7), (2.8) and (2.9) mean that the set $(f^\perp, g^\perp, \nu)$ forms an almost contact metric structure on R^l at every point of M, where $g^\perp=(\delta_{qp})$, and we see that the dimension l of R^l is odd.

Conversely, starting from the almost contact metric structure $(f^{\perp}, g^{\perp}, \nu)$ on R^l at every point of M, we can prove that the metric compound structure $(f, g, v, f^{\perp})$ introduces an almost contact metric structure (f, g, ξ, η) on M. Thus we have

THEOREM 1. *Let $(f, g, v, f^{\perp})$ be a metric compound structure on M. In order that f and g constitute an almost contact metric structure (f, g, ξ, η) on M, it is necessary and sufficient that $f^{\perp}$ and $g^{\perp}$ constitute an almost contact metric structure $(f^{\perp}, g^{\perp}, \nu)$ on R^l at every point of M.*

A metric compound structure satisfying the condition in the above theorem is called an *almost contact metric compound structure* on M. In the following we shall confine ourselves to such structures. From the above discussions we can state the following

THEOREM 2. *In order that a metric compound structure $(f, g, v, f^{\perp})$ is almost contact, it is necessary and sufficient that the matrix $(v_q{}^i)$ is of rank 1, that is, the l vector fields v_q are all parallel to each other.*

§ 3. The Nijenhuis tensor

Denoting $\partial_j = \partial/\partial y^j$ and regarding ∂_q as null operators, we define the Nijenhuis tensor of the metric compound structure (1.18) in $M \times R^l$ by

$$S_{CB}{}^A = \tilde{F}_C{}^E(\partial_E \tilde{F}_B{}^A - \partial_B \tilde{F}_E{}^A) - \tilde{F}_B{}^E(\partial_E \tilde{F}_C{}^A - \partial_C \tilde{F}_E{}^A).$$

Using (1.18), we can write down $S_{CB}{}^A$ as the followings;

$$\begin{aligned}
&S_{ji}{}^h = f_j{}^l(\partial_l f_i{}^h - \partial_i f_l{}^h) - f_i{}^l(\partial_l f_j{}^h - \partial_j f_l{}^h) + v_{js}\partial_i v_s{}^h - v_{is}\partial_j v_s{}^h, \\
&S_{jip} = f_j{}^l(\partial_l v_{pi} - \partial_i v_{pl}) - f_i{}^l(\partial_l v_{pj} - \partial_j v_{pl}) - v_{js}\partial_i f_{sp} + v_{is}\partial_j f_{sp}, \\
&S_{jq}{}^h = -f_j{}^l\partial_l v_q{}^h + v_q{}^l(\partial_l f_j{}^h - \partial_j f_l{}^h) + f_{qs}\partial_j v_s{}^h, \\
&S_{jqp} = f_j{}^l\partial_l f_{qp} + v_q{}^l(\partial_l v_{pj} - \partial_j v_{pl}) + f_{qs}\partial_j f_{sp}, \\
&S_{rq}{}^h = v_r{}^l\partial_l v_q{}^h - v_q{}^l\partial_l v_r{}^h, \\
&S_{rqp} = -v_r{}^l\partial_l f_{qp} + v_q{}^l\partial_l f_{rp}.
\end{aligned} \tag{3.1}$$

If the metric compound structure $(f, g, v, f^{\perp})$ gives an almost contact metric structures (f, g, ξ, η) on M and $(f^{\perp}, g^{\perp}, \nu)$ on R^l, then the above expressions are reduced to

$$\begin{aligned}
S_{ji}{}^h &= f_j{}^l(\partial_l f_i{}^h - \partial_i f_l{}^h) - f_i{}^l(\partial_l f_j{}^h - \partial_j f_l{}^h) + \eta_j\partial_i\xi^h - \eta_i\partial_j\xi^h, \\
S_{jip} &= [f_j{}^l(\partial_l\eta_i - \partial_i\eta_l) - f_i{}^l(\partial_l\eta_j - \partial_j\eta_l)]\nu_p \\
&\quad + (f_j{}^l\eta_i - f_i{}^l\eta_j)\partial_l\nu_p + (\eta_j\partial_i\nu_s - \eta_i\partial_j\nu_s)f_{sp},
\end{aligned}$$

$$S_{jq}{}^h=[\xi^l(\partial_l f_j{}^h-\partial_j f_l{}^h)-f_j{}^l\partial_l\xi^h]\nu_q-(f_j{}^l\partial_l\nu_q+f_{qs}\partial_j\nu_s)\xi^h\,, \tag{3.2}$$
$$S_{jqp}=(\xi^l\partial_l\eta_j-\xi^l\partial_j\eta_l)\nu_q\nu_p+(\eta_j\xi^l\partial_l\nu_p-\partial_j\nu_p)\nu_q+f_j{}^l\partial_l f_{qp}+f_{qs}\partial_j f_{sp}\,,$$
$$S_{rq}{}^h=(\nu_r\xi^l\partial_l\nu_q-\nu_q\xi^l\partial_l\nu_r)\xi^h\,,$$
$$S_{rqp}=-\nu_r\xi^l\partial_l f_{qp}+\nu_q\xi^l\partial_l f_{rp}\,,$$

because $\nu_q\nu_q=1$ and $\nu_q\partial_j\nu_q=0$.

On the other hand, the Nijenhuis tensors of the almost contact metric structure (f, g, ξ, η) are given by ([4])

$$\begin{aligned}
N_{ji}{}^h&=f_j{}^l(\partial_l f_i{}^h-\partial_i f_l{}^h)-f_i{}^l(\partial_l f_j{}^h-\partial_j f_l{}^h)+\eta_j\partial_i\xi^h-\eta_i\partial_j\xi^h\,,\\
N_{ji}&=f_j{}^l(\partial_l\eta_i-\partial_i\eta_l)-f_i{}^l(\partial_l\eta_j-\partial_j\eta_l)\,,\\
N_j{}^h&=\xi^l(\partial_l f_j{}^h-\partial_j f_l{}^h)-f_j{}^l\partial_l\xi^h\,,\\
N_j&=\xi^l\partial_l\eta_j-\xi^l\partial_j\eta_l\,.
\end{aligned} \tag{3.3}$$

Comparing (3.2) with (3.3), we have the equations

$$\begin{aligned}
N_{ji}{}^h&=S_{ji}{}^h\,, \qquad N_{ji}=S_{jip}\nu_p\,,\\
N_j{}^h&=S_{jq}{}^h\nu_q\,, \qquad N_j=S_{jqp}\nu_q\nu_p\,.
\end{aligned} \tag{3.4}$$

Therefore we obtain, from (3.4), the following

THEOREM 3. *Let $(f, g, v, f^\perp)$ be an almost contact metric compound structure on M. In order for the almost contact metric structure (f, g, ξ, η) on M to be normal, it is necessary and sufficient that $S_{ji}{}^h=0$.*

§ 4. Submanifolds of codimension l of an almost Hermitian manifold

In this section we assume that M is an m-dimensional submanifold of codimension l of an almost Hermitian manifold $\tilde{M}$ and $C_p=(C_p{}^\lambda)$ are mutually orthogonal unit vector normal to M in $\tilde{M}$, that is,

$$G_{\mu\lambda}C_q{}^\mu B_i{}^\lambda=0\,, \qquad G_{\mu\lambda}C_q{}^\mu C_p{}^\lambda=g_{qp}=\delta_{qp}\,, \tag{4.1}$$

and that the induced metric compound structure $(f, g, v, f^\perp)$ on M from the almost Hermitian structure $(G, \tilde{F})$ on $\tilde{M}$ defines an almost contact structure. The vector field N^λ defined by

$$N^\lambda=\nu_p C_p{}^\lambda \tag{4.2}$$

is unit normal to M in $\tilde{M}$ because $G_{\mu\lambda}N^\mu N^\lambda=1$. The transforms of the tangent vectors $B_i{}^\lambda$ and the normal vectors $C_p{}^\lambda$ by $\tilde{F}$ is given by

$$\tilde{F}_\mu{}^\lambda B_i{}^\mu=f_i{}^h B_h{}^\lambda+\eta_i N^\lambda \tag{4.3}$$

and

$$\tilde{F}_\mu{}^\lambda C_q{}^\mu = -\nu_q \xi^h B_h{}^\lambda + f_{qp} C_p{}^\lambda \tag{4.4}$$

respectively.

It is well-known that the submanifold M of an almost Hermitian manifold satisfying (4.3) is semi-invariant with respect to N^λ and we call N^λ the *distinguished normal* to M [6].

From (4.2) and (4.4) we have

$$\tilde{F}_\mu{}^\lambda N^\mu = -\xi^h B_h{}^\lambda , \tag{4.5}$$

and hence the transform of the distinguished normal N^λ by the almost complex structure $\tilde{F}$ of $\tilde{M}$ is tangent to M.

Conversely suppose that the submanifold M of codimension l of the almost Hermitian manifold $\tilde{M}$ is semi-invariant with respect to a unit normal N^λ whose transform by $\tilde{F}$ is tangent to M, then we have (4.3) and (4.5) for a vector ξ^h and a 1-form $\eta_i = g_{ih}\xi^h$ of M. Applying $\tilde{F}$ to (4.3) and (4.5), we obtain

$$f_j{}^h f_h{}^i = -\delta_j^i + \eta_j \xi^i , \qquad \eta_j f_i{}^j = 0 ,$$
$$f_j{}^i \xi^j = 0 , \qquad \eta_i \xi^i = 1 ,$$

We also have, from (1.2), (1.6) and (4.3),

$$f_j{}^k f_i{}^h g_{kh} = g_{ji} - \eta_j \eta_i .$$

Therefore we see that the set (f, g, ξ, η) defines an almost contact metric structure. As we have seen in § 2, the induced set $(f^\perp, g^\perp, \nu)$ also defines an almost contact metric structure. Then we have

THEOREM 4. *In order for an induced metric compound structure* $(f, g, v, f^\perp)$ *on a submanifold M of codimensional l of an almost Hermitian manifold $\tilde{M}$ to be an almost contact, it is necessary and sufficient that the submanifold M is semi-invariant with respect to a unit normal vector field whose transform by $\tilde{F}$ is tangent to the submanifold.*

Now denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji}, we have the Gauss equation for M in $\tilde{M}$

$$\nabla_j B_i{}^\lambda = h_{jip} C_p{}^\lambda , \tag{4.6}$$

where h_{jip} is the second fundamental tensor with respect to the normal $C_p{}^\lambda$. The mean curvature vector is defined by

$$H^\lambda = (1/m) g^{ji} \nabla_j B_i{}^\lambda = (1/m) h_l{}^l{}_p C_p{}^\lambda , \tag{4.7}$$

where $h_l{}^l{}_p = g^{ji} h_{jip}$. The Weingarten equation is given by

$$\nabla_j C_p{}^\lambda = -h_{jp}{}^i B_i{}^\lambda + l_{jpq} C_q{}^\lambda , \tag{4.8}$$

where l_{jpq} is the third fundamental tensor. Differentiating (4.1) covariantly and making use of (4.6) and (4.8), we have

$$h_{jq}{}^l g_{li} = h_{jiq} , \tag{4.9}$$

$$l_{jqp} = -l_{jpq} . \tag{4.10}$$

We put $h_{jip}\nu_p = h_{jp}{}^h g_{ih}\nu_p = h_{ji}$ and call h_{ji} the *intrinsic second fundamental tensor* of M. Differentiating N^λ covariantly and using (4.8), we find

$$\nabla_j N^\lambda = -h_j{}^i B_i{}^\lambda + (\nabla_j \nu_p + \nu_q l_{jqp}) C_p{}^\lambda . \tag{4.11}$$

Now we assume that the ambient manifold $\tilde{M}$ is Kaehlerian. Differentiating (4.3) covariantly and taking account of (4.4), (4.6), (4.8) and (4.11), we have

$$\begin{aligned} h_{jiq}(-\nu_q \xi^h B_h{}^\lambda + f_{qp} C_p{}^\lambda) &= (\nabla_j f_i{}^h) B_h{}^\lambda + f_i{}^h h_{jhp} C_p{}^\lambda \\ &\quad + (\nabla_j \eta_i) N^\lambda + \eta_i [-h_j{}^h B_h{}^\lambda + (\nabla_j \nu_p + \nu_q l_{jqp}) C_p{}^\lambda] , \end{aligned}$$

from which follow the equations

$$\nabla_j f_i{}^h = -h_{ji} \xi^h + \eta_i h_j{}^h , \tag{4.12}$$

$$(\nabla_j \eta_i)\nu_p + \eta_i (\nabla_j \nu_p) = h_{jiq} f_{qp} - f_i{}^l h_{jlp} - \eta_i \nu_q l_{jqp} . \tag{4.13}$$

Transvecting (4.13) with ν_p and ξ^i, we obtain

$$\nabla_j \eta_i = -h_{jl} f_i{}^l , \tag{4.14}$$

$$\nabla_j \nu_p = \xi^l h_{jlq} f_{qp} - \nu_q l_{jqp} . \tag{4.15}$$

Also, differentiating (4.4) covariantly and taking account of (4.3), (4.4), (4.6) and (4.8), we have

$$\begin{aligned} &-h_{jq}{}^i (f_i{}^h B_h{}^\lambda + \eta_i \nu_p C_p{}^\lambda) + l_{jqr}(-\nu_r \xi^h B_h{}^\lambda + f_{rp} C_p{}^\lambda) \\ &= -(\nabla_j(\nu_q \xi^h)) B_h{}^\lambda - \nu_q \xi^l h_{jlp} C_p{}^\lambda + (\nabla_j f_{qp}) C_p{}^\lambda + f_{qr}(-h_{jr}{}^h B_h{}^\lambda + l_{jrp} C_p{}^\lambda) , \end{aligned}$$

from which follow the equations

$$(\nabla_j \nu_q)\xi^h + \nu_q (\nabla_j \xi^h) = h_{jq}{}^i f_i{}^h + l_{jqr} \nu_r \xi^h - h_{jr}{}^h f_{qr} , \tag{4.16}$$

$$\nabla_j f_{qp} = \nu_q \xi^l h_{jlp} - \nu_p \xi^l h_{jlq} + l_{jqr} f_{rp} - l_{jpr} f_{rq} . \tag{4.17}$$

Suppose that the almost contact metric structure (f, g, ξ, η) on M is normal, that is,

$$f_j{}^l(\nabla_l f_i{}^h - \nabla_i f_l{}^h) - f_i{}^l(\nabla_l f_j{}^h - \nabla_j f_l{}^h) + \eta_j \nabla_i \xi^h - \eta_i \nabla_j \xi^h = 0 .$$

Then, substituting (4.12) and (4.14) into this equation, we have the equation

$$(f_j{}^l h_l{}^h - h_j{}^l f_l{}^h)\eta_i = (f_i{}^l h_l{}^h - h_i{}^l f_l{}^h)\eta_j$$

and, transvecting this equation with ξ^i,

$$f_j{}^l h_l{}^h - h_j{}^l f_l{}^h = -\xi^i h_i{}^l f_l{}^h \eta_j . \tag{4.18}$$

Transvecting (4.18) with $f_k{}^j$ and with $f_h{}^i$ successively, we have the equations

$$-h_k{}^h + \eta_k \xi^l h_l{}^h = f_k{}^j h_j{}^l f_l{}^h$$

and

$$f_k{}^l h_l{}^i - h_k{}^l f_l{}^i = \eta_l h_j{}^l f_k{}^j \xi^i - \eta_k \xi^l h_l{}^h f_h{}^i . \tag{4.19}$$

Comparing (4.19) with (4.18), we find $\eta_l h_j{}^l f_k{}^j \xi^i = 0$ or equivalently $\xi^i h_i{}^l f_l{}^j = 0$. Moreover, substituting this equation into (4.18), we have

$$f_j{}^l h_l{}^h = h_j{}^l f_l{}^h . \tag{4.20}$$

Thus we have

THEOREM 5. *Suppose that the submanifold M of codimension l of a Kaehlerian manifold $\tilde{M}$ admits an almost contact metric compound structure $(f, g, v, f^{\perp})$. Then, in order for the almost contact metric structure (f, g, ξ, η) on M to be normal, it is necessary and sufficient that the intrinsic second fundamental tensor h and f commute.*

Suppose that the almost contact metric structure (f, g, ξ, η) on M is normal contact, that is, it satisfies (4.20) and

$$\nabla_j \eta_i - \nabla_i \eta_j = 2 f_{ji} . \tag{4.21}$$

Then, substituting (4.14) into the equation (4.21), we have

$$-h_{jl} f_i{}^l + h_{il} f_j{}^l = 2 f_{ji} ,$$

from which follows the equation

$$h_j{}^l f_l{}^h + f_j{}^l h_l{}^h = 2 f_j{}^h .$$

Substituting (4.20) into this equation, we have

$$h_j{}^l f_l{}^h = f_j{}^h , \tag{4.22}$$

and, transvecting with ξ^j,

$$\xi^j h_j{}^l f_l{}^h = 0 .$$

Transvecting this equation with $f_h{}^i$, we obtain $\xi^j h_j{}^i = \alpha \xi^i$, where we have put

$$\alpha = \xi^j \xi^i h_{ji} . \tag{4.23}$$

Transvecting (4.22) with $f_h{}^i$, we have

$$h_j{}^l(-\delta_l{}^i+\eta_l\xi^i)=-\delta_j{}^i+\eta_j\xi^i$$

or equivalently

$$h_{ji}=g_{ji}+(\alpha-1)\eta_j\eta_i\,. \tag{4.24}$$

In this case we say that the submanifold M is *η-umbilical* with respect to the distinguished normal N^λ.

Conversely if the submanifold M is η-umbilical, we can easily obtain the equations (4.20) and (4.21) by the transvection of (4.24) with f.

In particular, if the distinguished normal N^λ to M is concurrent, that is, $\nabla_j N^\lambda=-\tau B_j{}^\lambda$ for some function τ, then we have form (4.11)

$$\tau\delta_j{}^h=h_j{}^h\,,\qquad \nabla_j\nu_p+\nu_q l_{jqp}=0\,.$$

Since the first of these equations is expressed as

$$h_{ji}=\tau g_{ji}\,, \tag{4.25}$$

then, from (4.23), we find $\alpha=\tau$. Substituting (4.25) and $\alpha=\tau$ into (4.24), we have

$$(\tau-1)(g_{ji}-\eta_j\eta_i)=0\,,$$

which implies $\tau=1$. Consequently we have $h_{ji}=g_{ji}$. Thus we have

THEOREM 6. *Suppose that the submanifold M of codimension l of a Kaehlerian manifold $\tilde{M}$ admits an almost contact metric compound structure $(f, g, v, f^\perp)$. In order for the almost contact metric structure (f, g, ξ, η) on M to be normal contact, that is, Sasakian, it is necessary and sufficient that M is η-umbilical with respect to the distinguished normal N^λ. In addition, if the distinguished normal N^λ to M is concurrent, then M is umbilical with respect to N^λ.*

§ 5. Submanifolds of codimension l of an even-dimensional Euclidean space

In this section we assume that M is a submanifold of codimension l of an even-dimensional Euclidean space E^n and an almost contact metric compound structure $(f, g, v, f^\perp)$ is induced on M. Then the Gauss, Codazzi and Ricci equations are given by

$$K_{kjih}=h_{khp}h_{jip}-h_{jhp}h_{kip}\,, \tag{5.1}$$

$$\nabla_k h_{jiq}-\nabla_j h_{kiq}=-l_{jqp}h_{khp}+l_{kqp}h_{jip}\,, \tag{5.2}$$

$$\nabla_k l_{jqp}-\nabla_j l_{kqp}=h_j{}^l{}_q h_{klp}-h_k{}^l{}_q h_{jlp}+l_{kqr}l_{jrp}-l_{jqr}l_{krp} \tag{5.3}$$

respectively, where $K_{kji}{}^h=g^{hl}K_{kjil}$ is the curvature tensor of M.

Now we shall prove

THEOREM 7. *Let M be a submanifold of dimension $m>3$ in an even-dimensional Euclidean space E^n and assume that the induced metric compound structure $(f, g, v, f^{\perp})$ is almost contact. Then, in order for the submanifold M to be umbilical with respect to the distinguished normal N^{λ} and N^{λ} parallel to the mean curvature vector of M in E^n, it is necessary and sufficient that the distinguished normal N^{λ} is concurrent. In this case the mean curvature of M is constant.*

Proof. If the submanifold M is umbilical with respect to the distinguished normal N^{λ} and N^{λ} is parallel to the mean curvature vector H^{λ} of M, we have

$$h_{ji}=\rho g_{ji}\,, \tag{5.4}$$

$$h_l{}^l{}_p=h_l{}^l\nu_p=m\rho\nu_p \tag{5.5}$$

for a certain scalar function ρ. By means of (5.4) the equations of (4.14) and (4.12) have the following expressions

$$\nabla_j\eta_i=\rho f_{ji}\,, \tag{5.6}$$

$$\nabla_k f_{ji}=\rho(\eta_j g_{ki}-\eta_i g_{kj}) \tag{5.7}$$

respectively.

Substituting (4.15) and (5.6) into (4.13), we have

$$\rho f_{ji}\nu_p+\eta_i\xi^l h_{jlq}f_{qp}=h_{jiq}f_{qp}-f_i{}^l h_{jlp}\,, \tag{5.8}$$

and, transvecting this equation with g_{ji},

$$\xi^j\xi^i h_{jiq}f_{qp}=h_l{}^l{}_q f_{qp}=0\,.$$

This equation implies

$$\xi^j\xi^i h_{jiq}=A\nu_q\,,$$

where $A=\xi^j\xi^i h_{ji}=\rho$ and consequently

$$\xi^j\xi^i h_{jiq}=\rho\nu_q\,. \tag{5.9}$$

If we transvect (5.2) with ν_q and make use of (4.15), we have

$$\nabla_k h_{ji}-\nabla_j h_{ki}=\xi^l h_{klq}h_{jip}f_{qp}-\xi^l h_{jlq}h_{kip}f_{qp} \tag{5.10}$$

or, by means of (5.4),

$$(\nabla_k\rho)g_{ji}-(\nabla_j\rho)g_{ki}=\xi^l h_{klq}h_{jip}f_{qp}-\xi^l h_{jlq}h_{kip}f_{qp}\,. \tag{5.11}$$

Differentiating (5.6) covariantly and using (5.7), we have

$$\nabla_k\nabla_j\eta_i=(\nabla_k\rho)f_{ji}+\rho^2(\eta_j g_{ki}-\eta_i g_{kj})\,,$$

from which, using the Ricci identity,

$$-K_{kji}{}^h\eta_h=(\nabla_k\rho)f_{ji}-(\nabla_j\rho)f_{ki}+\rho^2(\eta_j g_{ki}-\eta_k g_{ji})\,.$$

From this, by the Bianchi identity, we obtain

$$(\nabla_k \rho) f_{ji} + (\nabla_j \rho) f_{ik} + (\nabla_i \rho) f_{kj} = 0 . \tag{5.12}$$

Transvecting (5.12) with f^{ji}, we get

$$(m-3)\nabla_k \rho + 2\xi^l (\nabla_l \rho) \eta_k = 0 .$$

Moreover the transvection of (5.12) with $\xi^i f^{kj}$ yields $\xi^l \nabla_l \rho = 0$. Therefore we see that ρ is constant for $m>3$.

From (5.11) and the above result we have

$$\xi^l h_{klq} h_{jip} f_{qp} = \xi^l h_{jlq} h_{kip} f_{qp} ,$$

and, transvecting with ξ^j and using (5.9),

$$\xi^l h_{klq} \xi^j h_{jip} f_{qp} = 0 . \tag{5.13}$$

Transvecting (5.8) with $\xi^j \xi^k h_{khp}$ and using (5.13), we have

$$f_i{}^l \xi^j h_{jlp} \xi^k h_{khp} = 0 ,$$

and, transvecting with $f_m{}^i$ and using (5.9),

$$\xi^l h_{ljp} \xi^k h_{kip} = \rho^2 \eta_j \eta_i . \tag{5.14}$$

Let H be the matrix $(\xi^l h_{ljp})$. Then (5.14) means that ${}^tHH = \rho^2(\eta_j \eta_i)$, where ${}^tH$ is the transpose of H. Since the rank of matrix $(\eta_j \eta_i)$ is 1, then the rank of H is also 1. Therefore we may put

$$\xi^l h_{lip} = \rho \eta'_i \nu'_p . \tag{5.15}$$

Comparing the transvection of (5.15) with ξ^i and (5.9), we see that $\nu_p = A\nu'_p$, where $A = \xi^l \eta'_l$. Hence we have

$$\xi^l h_{lip} f_{pq} = 0 \tag{5.16}$$

or equivalently, from (4.15),

$$\nabla_j \nu_q + \nu_p l_{jpq} = 0 . \tag{5.17}$$

Finally we see, from (5.6) and (5.17), that the distinguished normal N^λ is concurrent.

Conversely if the distinguished normal N^λ is concurrent, that is, $\nabla_j N^\lambda = -\tau B_j{}^\lambda$ for a certain function τ, then we have $h_{ji} = \tau g_{ji}$, which shows that M is umbilical with respect to N^λ, and (5.17). Substituting (4.14) and the above equations into (4.13), we have

$$f_i{}^l h_{jlp} = h_{jiq} f_{qp} - \tau f_{ji} \nu_p ,$$

and, transvecting with g^{ji},

$$h_l{}^l{}_q f_{qp} = 0\,,$$

which implies

$$h_l{}^l{}_q = h_l{}^l \nu_q = m\tau\nu_q\,.$$

Therefore the distinguished normal N^λ is parallel to the mean curvature vector H^λ.

In this case we easily see that the mean curvature of M is constant. This completes the proof.

Now we assume that the mean curvature vector H^λ is parallel to the distinguished normal N^λ of M, that is, $H^\lambda = \rho N^\lambda$ for a certain function ρ. Then we have (5.5).

If the submanifold M is pseudo-umbilical, we have

$$G_{\lambda\mu} h_{ji}{}^\lambda H^\mu = \rho^2 g_{ji} \tag{5.18}$$

because $|\rho|$ is the length of H^λ. From (5.5) and (5.18) we find that $h_{ji} = |\rho| g_{ji}$, which means that M is umbilical with respect to the distinguished normal N^λ.

Conversely if the submanifold M is umbilical with respect to N^λ, we have (5.18) from (5.4) and (5.5). Thus we have

THEOREM 8. *Let M be a submanifold of codimension l with the induced almost contact metric compound structure $(f, g, v, f^\perp)$ of an even-dimensional Euclidean space E^n and the mean curvature vector H^λ of M parallel to the distinguished normal N^λ of M in E^n. Then, in order for the submanifold M to be pseudo-umbilical, it is necessary and sufficient that M is umbilical with respect to the distinguished normal N^λ.*

It is well-known that pseudo-umbilical submanifolds in a Euclidean space with the mean curvature vector parallel in the normal bundle are minimal submanifolds of a hypersphere [7]. From Theorem 7, we see that the mean curvature vector is parallel in the normal bundle. Therefore it follows from Theorems 7 and 8 that the submanifold M of dimension $m>3$ is contained as a minimal submanifold in a hypersphere in E^n.

On the other hand, we see that the direct sum of the tangent space of M and the distinguished normal N^λ is invariant because of (4.3) and (4.5). Therefore M is an intersection of a complex cone with generator N^λ on M and an $(n-1)$-dimensional sphere.

Thus we have the following

THEOREM 9. *Let M be a submanifold of codimension l with the induced almost contact metric compound structure $(f, g, v, f^\perp)$ of an even-dimensional Euclidean space E^n. If the submanifold M satisfies one of the followings;*

(1) *M of dimension $m>3$ is umbilical with respect to the distinguished normal N^λ, and N^λ parallel to the mean curvature vector,*

(2) *M of dimension $m>3$ is pseudo-umbilical submanifold and the distinguished*

normal N^λ *parallel to the mean curvature vector,*

(3) *The distinguished normal* N^λ *is concurrent,*

then M *is the intersection of a complex cone with generator* N^λ *and an* $(n-1)$*-dimensional sphere.*

We now assume that the metric compound structure $(f, g, v, f^\perp)$ induced on a submanifold M of codimension l of an even-dimensional Euclidean space E^n defines a normal almost contact metric structure (f, g, ξ, η) on M and the distinguished normal N^λ is parallel in the normal bundle of M. Then we have the equation (4.20), that is,

$$h_{jl}f_i{}^l + h_{il}f_j{}^l = 0 . \tag{5.19}$$

Transvecting (5.19) with $f_k{}^i$ and taking the skew-symmetric part, we have

$$h_{jl}\xi^l\eta_k = h_{kl}\xi^l\eta_j ,$$

which means that we may put

$$h_{jl}\xi^l = \alpha\eta_j , \tag{5.20}$$

where $\alpha = \xi^j\xi^i h_{ji}$. Differentiating (5.20) covariantly and substituting (4.14) into this equation, we have

$$(\nabla_k h_{jl})\xi^l + h_j{}^l(-h_{ki}f_l{}^i) = (\nabla_k\alpha)\eta_j + \alpha(-h_{kl}f_j{}^l)$$

and, taking the skew-symmetric part and using (5.19), the equation

$$(\nabla_k h_{jl} - \nabla_j h_{kl})\xi^l + 2h_j{}^l h_{li} f_k{}^i = (\nabla_k\alpha)\eta_j - (\nabla_j\alpha)\eta_k + 2\alpha h_{jl}f_k{}^l . \tag{5.21}$$

On the other hand, since N^λ is parallel in the normal bundle of M, we have (5.17) or equivalently (5.16). From (5.10) and (5.16) we find

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0 . \tag{5.22}$$

Substituting (5.22) into (5.21), we have

$$2h_j{}^l h_{li} f_k{}^i = (\nabla_k\alpha)\eta_j - (\nabla_j\alpha)\eta_k + 2\alpha h_{jl}f_k{}^l , \tag{5.23}$$

and transvecting (5.23) with ξ^i and using (5.20),

$$\nabla_k\alpha = A\eta_k , \tag{5.24}$$

where $A = \xi^l\nabla_l\alpha$. Thus (5.23) implies

$$h_j{}^l h_{li} f_k{}^i = \alpha h_{jl} f_k{}^l .$$

If we transvect this equation with $f_h{}^k$ and make use of (5.20), we obtain

$$h_j{}^l h_{li} = \alpha h_{ji} . \tag{5.25}$$

Differentiating (5.24) covariantly and substituting (4.14) into this equation, we have

$$(\nabla_k A)\eta_j-(\nabla_j A)\eta_k+2Ah_{jl}f_k{}^l=0\,,$$

and, transvecting with ξ^j and using (5.19),

$$\nabla_k A=(\xi^l\nabla_l A)\eta_k\,.$$

The two equations above show that $Ah_{jl}f_k{}^l=0$. Transvecting this equation with $f_i{}^k$ and using (5.20), we have

$$A(h_{ji}-\alpha\eta_j\eta_i)=0\,. \tag{5.26}$$

Now suppose that M is locally irreducible. Then we have $A=0$ from (5.26). In fact, if $A\neq 0$, we have $h_{ji}=\alpha\eta_j\eta_i$. Substituting this equation into (4.14), we find $\nabla_j\eta_i=0$, which means that ξ^h is parallel vector field. This contradicts to the local irreducibility of M. Therefore we see that α is constant from (5.24). Moreover this constant is nonzero. In fact, if $\alpha=0$, we have $h_{ji}=0$ from (5.25) and finally we also have $\nabla_j\eta_i=0$.

Differentiating (5.25) covariantly, we have

$$(\nabla_k h_{jl})h_i{}^l+h_j{}^l(\nabla_k h_{il})=\alpha\nabla_k h_{ji}\,.$$

From this equation, taking the skew-symmetric part with respect to i and k and using (5.22), we have

$$(\nabla_k h_{jl})h_i{}^l-(\nabla_i h_{jl})h_k{}^l=0\,.$$

Since the sum of this equation and one with exchanged j and k is

$$2(\nabla_k h_{jl})h_i{}^l=\alpha\nabla_k h_{ji} \tag{5.27}$$

by means of (5.22), then we have, transvecting (5.27) with $h_h{}^i$ and using (5.25) and $\alpha\neq 0$,

$$(\nabla_k h_{jl})h_i{}^l=0\,. \tag{5.28}$$

Therefore, from (5.27) and (5.28), we have

$$\nabla_k h_{ji}=0\,. \tag{5.29}$$

By the irreducibility of M, it follows from (5.29) that h_{ji} is proportional to g_{ji} and from (5.25) that the proportional factor is equal to α, that is,

$$h_{ji}=\alpha g_{ji}\,. \tag{5.30}$$

Consequently we see, from (5.17) and (5.30), that the distinguished normal N^λ is concurrent.

Thus, from Theorem 9, we have

THEOREM 10. *Let M be a locally irreducible submanifold of codimension l*

with an induced metric compound structure $(f, g, v, f^{\perp})$ *of a Euclidean space* E^n *such that the distinguished normal* N^{λ} *is parallel in the normal bundle. If the metric compound structure* $(f, g, v, f^{\perp})$ *defines a normal almost contact metric structure* (f, g, ξ, η) *on* M, *then* M *is the intersection of a complex cone with generator* N^{λ} *and an* $(n-1)$*-dimensional sphere.*

§ 6. Metric compound structure $(f, g, v, f^{\perp})$ in which $f^{\perp}=0$

Let the set $(f, g, v, f^{\perp})$ be a metric compound structure on M and assume that the tensor field $f^{\perp}$ on R^l vanishes identically. Then, from (1.14), (1.15) and (1.16), we have

$$f_j{}^i v_{pi}=0\,, \qquad v_q{}^i f_i{}^h=0\,, \tag{6.1}$$

$$v_q{}^i v_{pi}=\delta_{qp}\,. \tag{6.2}$$

We assume that M is odd-dimensional and put $l=2a+1$.

We choose one of the l vector fields v_{pi} as η_i, for example,

$$\eta_i=v_{2a+1,\,i} \tag{6.3}$$

and put $\xi^h=g^{ih}\eta_i$. Then, by means of (6.2), we have

$$\xi^i\eta_i=1\,. \tag{6.4}$$

Now we put

$$\phi_i{}^h=f_i{}^h-\Big(\sum_{p=1}^{a} v_{pi}v_{\bar{p}}{}^h-\sum_{p=1}^{a} v_{\bar{p}i}v_p{}^h\Big)\,, \tag{6.5}$$

where $\bar{p}=a+p$. Then, using (6.1) and (6.2), we have

$$\phi_j{}^i\phi_i{}^h=f_j{}^i f_i{}^h-\sum_{p=1}^{2a} v_{pj}v_p{}^h=f_j{}^i f_i{}^h-v_{pj}v_p{}^h+\eta_j\xi^h\,,$$

which implies, from (1.13),

$$\phi_j{}^i\phi_i{}^h=-\delta_j^h+\eta_j\xi^h\,. \tag{6.6}$$

From (6.3) and (6.5) we also have

$$\phi_j{}^i\xi^j=\phi_j{}^i\eta_i=0\,. \tag{6.7}$$

Using (6.1) and (6.2), we also have

$$\phi_j{}^k\phi_i{}^h g_{kh}=f_j{}^k f_i{}^h g_{kh}+v_{pj}v_{pi}-\eta_j\eta_i\,,$$

which implies, from (1.17),

$$\phi_j{}^k\phi_i{}^h g_{kh}=g_{ji}-\eta_j\eta_i\,. \tag{6.8}$$

Thus we have the following

THEOREM 11. *Let* $(f, g, v, f^{\perp})$ *be a metric compound structure on an odd-dimensional manifold M. If the tensor* $f^{\perp}$ *on* R^{l} *vanishes identically, then the manifold M admits an almost contact metric structure* (ϕ, g, ξ, η), *where* η *is one of l vector fields v and* ϕ *is given by* (6.5).

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