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ON A CERTAIN HYPERSURFACES OF R^{2m+1}

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Introduction

It is a well-known theorem of Reeb (See [3], p. 25) that if a compact differentiable n-manifold M admits a Morse function with exactly two critical points, then M is a topological sphere.

Recently, Nomizu and Rodriguez [4] showed the following results as their geometric nature: Let M be a connected Riemannian $n(n\geqq 2)$-manifold isometrically immersed in a Euclidean m-space R^m and f its isometric immersion. Put $L_p(x)=(d(f(x), p))^2$ for $p\in R^m$, $x\in M$, where d is the Euclidean distance function. (a) If M is complete, and there exists a dense subset D of R^m such that every function of the form L_p, $p\in D$, has index 0 or n at any of its nondegenerate critical points, then M is totally umbilical in R^m, i.e., M is isometric to a Euclidean n-subspace or a Euclidean n-sphere in R^m. (b) If M is compact, and there exists a dense subset D of R^m such that every function of the form L_p, $p\in D$, has exactly two critical points, then M is isometric to a Euclidean n-sphere.

In the present paper we shall prove the following result.

THEOREM. *Let M be a connected, complete Riemannian $2m$ $(m\geqq 2)$-manifold isometrically immersed in R^{2m+1} with constant mean curvature. If there exists a dense subset D of R^{2m+1} such that every function of the form L_p, $p\in D$, has index 0, m or $2m$ at any of its nondegenerate critical points, then M is isometric to a Euclidean $2m$-subspace R^{2m}, a Euclidean $2m$-sphere S^{2m} in R^{2m+1} or the product $R^m\times S^m$ of an m-subspace R^m of R^{2m+1} and a sphere S^m in the Euclidean subspace perpendicular to R^m.*

When we consider the problem similar to (b) to obtain a result that M is isometric to S^{2m} or $R^m\times S^m$, it seems to be the natural condition that M is complete and there exists a dense subset D of R^{2m+1} such that every function of the form L_p, $p\in D$, has two critical points ([1], pp. 714-715).

1. Preliminaries

Let f be an isometric immersion of a connected Riemannian n-manifold M

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into R^m. Any point of the normal bundle $N(M)$ of M is denoted by $(x, t\xi)$, where $x \in M$, $t \in R^1$ and ξ is a unit vector in $T_x^{\perp}(M)$, the normal space to M at $f(x)$. Let F be a differentiable mapping of $N(M)$ in R^m given by $F(x, t\xi)=f(x)+t\xi$.

DEFINITION. A point $p \in R^m$ is called a *focal point* of M if $p=F(x, t\xi)$, where $(x, t\xi)$ is a point of $N(M)$ where the Jacobian F_* of F is degenerate. In this case, we also say that p is a focal point of (M, x).

Let I denote the identity transformation of the tangent space $T_x(M)$ and A_ξ the symmetric endomorphism of $T_x(M)$ corresponding to the second fundamental form of M at x in the direction of ξ. Then we have

LEMMA 1 (Nomizu and Rodriguez). *A point $p=F(x, t\xi)$, where $(x, t\xi) \in N(M)$, is a focal point of (M, x) if and only if the endomorphism $I-tA_\xi$ on $T_x(M)$ is degenerate.*

Now, let $p \in R^m$ and consider the function $L_p(x)=(d(f(x), p))^2$ on M. Then

LEMMA 2 (Nomizu and Rodriguez). *L_p has a critical point x if and only if p can be expressed as $F(x, t\xi)$, where ξ is a unit vector in $T_x^{\perp}(M)$. In this case, the Hessian H of L_p at x, which is a bilinear symmetric function on $T_x(M)\times T_x(M)$, is given by*

$$H(X, Y)=2\langle(I-tA_\xi)X, Y\rangle, \qquad X, Y \in T_x(M),$$

where $\langle , \rangle$ is the inner product on $T_x(M)$ induced from the Euclidean metric in R^m through f.

Hence we see that x is a degenerate critical point of L_p if and only if p is a focal point of (M, x) and that index of L_p, $p=F(x, t\xi)$, at a non-degenerate critical point x equals the number of eigenvalues of A_ξ which are larger than $1/t$, counting multiplicities.

2. Proof of Theorem

We remark that Theorem is an immediate consequence of the following lemma (See [2], [5]).

LEMMA 3. *Let M be a connected (not necessarily complete) Riemannian $2m(m\geqq 1)$-manifold isometrically immersed in R^{2m+1} (not also necessarily having constant mean curvature). Under the assumption of Theorem, the second fundamental form A of M has at most two distinct eigenvalues at each point.*

Proof. Let $x \in M$ and ξ be a field of unit normal vectors. Suppose $A(=A_\xi)$ has a non-zero eigenvalue, say a. We may assume that $a>0$, because if $a<0$, then $A_{-\xi}$ has $-a>0$ as eigenvalue.

Assuming thus that a is the largest positive eigenvalue of A take $t_1>0$ such that $1/a<t_1<1/b$, where b is the next largest positive eigenvalue if any (if a

is the only positive eigenvalue, just consider $1/a<t_1$). Then $p=F(x, t_1\xi)$ is not a focal point of (M, x) and the function L_p has x as a nondegenerate critical point. The index at x is equal to the multiplicity, say k, of the eigenvalue a. If $p\in D$, $k=m$ or $2m$, since k cannot be 0. Now p may not belong to D. By denseness of D, however, we know that there exists a point $q\in D$ such that L_q has y as a nondegenerate critical point of index k (q and y may be chosen as close to p and x, respectively, as we want). Thus we may conclude that $k=m$ or $2m$.

Now if $k=2m$, then a is an eigenvalue of A with multiplicity $2m$ so that x is umbilic. Suppose then that $k=m$. The following two subcases should be discussed:

(i) There exist positive eigenvalues of A other than a.

(ii) A negative of (i).

(i) Assuming that b is the next largest positive eigenvalue of A, take $t_2>0$ such that $1/a<t_1<1/b<t_2<1/c$, where c is the third largest positive eigenvalue if any (if a and b are the only positive eigenvalues, just consider $1/a<t_1<1/b<t_2$). By the same argument as above, $p_1=F(x, t_2\xi)$ is not a focal point of (M, x) and the function L_{p_1} has x as a nondegenerate critical point of index $2m$. Thus multiplicity of b is m.

(ii) If there exist non-zero eigenvalues of A other than a, then let b be the smallest eigenvalue of A. Noting that $-b$ is the largest positive eigenvalue of $A_{-\xi}$, take $t_2>0$ such that $1/-b<t_2<1/-c$, where c is the next smallest eigenvalue of A if any (if b is the only negative eigenvalue of A, just consider $1/-b<t_2$). By the same argument as above, $p_2=F(x, t_2(-\xi))$ is also not a focal point of (M, x) and the function L_{p_2} has x as a nondegenerate critical point of index m. Thus multiplicity of b is also m.

Therefore A has at most two distinct eigenvalues at each point.

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